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A systematic approach to the SILH Lagrangian

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Abstract

We consider the electroweak chiral Lagrangian, including a light scalar boson, in the limit of small $\xi = v^2/f^2$. Here v is the electroweak scale and f is the corresponding scale of the new strong dynamics. We show how the conventional SILH Lagrangian, defined as the effective theory of a strongly-interacting light Higgs (SILH) to first order in ξ , can be obtained as a limiting case of the complete electroweak chiral Lagrangian. The approach presented here ensures the completeness of the operator basis at the considered order, it clarifies the systematics of the effective Lagrangian, guarantees a consistent and unambiguous power counting, and it shows how the generalization of the effective field theory to higher orders in ξ has to be performed. We point out that terms of order ξ^2 , which are usually not included in the SILH Lagrangian, are parametrically larger than terms of order $\xi/16\pi^2$ that are retained, as long as $\xi \gtrsim 1/16\pi^2$. Conceptual issues such as custodial symmetry and its breaking are also discussed. For illustration, the minimal composite Higgs model based on the coset $SO(5)/SO(4)$ is considered at next-to-leading order in the chiral expansion. It is shown how the effective Lagrangian for this model is contained as a special case in the electroweak chiral Lagrangian based on $SU(2)_L \otimes SU(2)_R/SU(2)_V$.

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1. Introduction

After the discovery of the Higgs-like boson at the LHC [1–5] the understanding of its precise role in electroweak symmetry breaking has become the prime topic in particle physics. A general approach, independent of any specific high-energy model, is provided by the effective field-theory (EFT) method. The motivation for using EFT is reinforced by the absence (so far) of evidence for new particles below the TeV energy scale. To be fully general and to account for the possibility of Higgs compositeness, the electroweak chiral Lagrangian [6] including a light Higgs [7–10] should be employed.

A widely used effective description of a light (pseudo-Goldstone) composite Higgs particle is the Lagrangian of the strongly-interacting light Higgs (SILH) [11,12]. By definition, this low-energy Lagrangian is constructed under the additional assumption that the electroweak scale v is parametrically smaller than the corresponding scale of the strong dynamics f , and the Lagrangian is restricted to terms of at most first order in $\xi = v^2/f^2$. For clarity we will therefore define the term *SILH Lagrangian* as the effective theory of a light composite Higgs particle through linear order in ξ . Without this restriction, we will use the expression *electroweak chiral Lagrangian* instead.

As will be explained in detail below, the usual derivation of the SILH Lagrangian appears unsatisfactory. An important point is that the power-counting rules postulated in [11] are based on naive dimensional analysis (NDA) [13,14], which are not valid in general [8] in their usual formulation. As a consequence, the counting rules of [11] are not fully consistent and lead to ambiguities in the estimate of the Lagrangian coefficients. A systematic derivation of the SILH Lagrangian can be given starting from the electroweak chiral Lagrangian with a light Higgs. This derivation is the main subject of this paper. In addition to providing a complete SILH-type Lagrangian, we elaborate on further conceptual issues of relevance to phenomenology. Although we agree with many results incorporated in the traditional SILH Lagrangian, we find some notable differences, which we discuss.

Some of the differences between the traditional formulation and the present approach may be seen as reflecting two different perspectives on effective field theory, which we might refer to as the ‘top-down’ and the ‘bottom-up’ point of view [15]. On the one hand, in the top-down applications, effective field theory can be used to construct a low-energy approximation of a given theory, or a certain class of theories, at high energies. A typical example is the derivation of low-energy effective Lagrangians for the weak interactions of the Standard Model (SM), or one of its extensions. In this case the high-energy theory is known and the EFT is used as a systematic tool to simplify the theory in the energy regime of interest. On the other hand, following the bottom-up approach, a low-energy EFT can be constructed from the relevant light degrees of freedom, based on the appropriate symmetries and a consistent power counting, without specifying any details of the high-energy completion. An example is the chiral perturbation theory of pions, where the theory at high energies, QCD, is known in principle, but the nonperturbative hadronic dynamics makes it intractable in practice. Another example is the renormalizable Standard Model itself, which can be extended by operators of higher dimension to account in full generality for the unknown physics in the UV. Even though a top-down construction may capture most of the essential features in the low-energy Lagrangian, it is clear that only the bottom-up framework will guarantee a fully general EFT.

We emphasize that our starting point, the electroweak chiral Lagrangian, is formulated as a bottom-up EFT in this sense, whereas the traditional SILH Lagrangian might be rather considered as following a top-down approach with a class of composite Higgs models in mind. In this work

we will show how the SILH Lagrangian can be consistently derived from a model-independent bottom-up perspective.

The remainder of this paper is organized as follows. In Section 2 we revisit the original SILH Lagrangian as given in [11] and discuss a number of issues related to its power counting. Section 3 outlines the systematics of the electroweak chiral Lagrangian in the limit of small ξ and clarifies the connection between a dimensional expansion (in powers of ξ) and the chiral expansion (in the number of loops). In Section 4 we derive the SILH Lagrangian, identified as the $\mathcal{O}(\xi)$ expansion of the electroweak chiral Lagrangian. Comments on custodial symmetry and its breaking through spurions are given in Section 5. As a concrete illustration, in Section 6 we discuss the basis of bosonic NLO operators for the $SO(5)/SO(4)$ model. Conclusions are given in Section 7 while technical details are collected in Appendix A.

2. Comments on the SILH Lagrangian

The construction of the electroweak chiral Lagrangian as an EFT requires a power-counting prescription in order to be well defined. As discussed in [8], the electroweak chiral Lagrangian mixes weakly-coupled and strongly-coupled interactions, which in isolation have a very different power counting. The strategy followed in [8–10] was to define a power counting such that NLO counterterms account for all the (superficial) divergences coming from the one-loop diagrams built from the leading-order Lagrangian.¹ Such a power counting is thus based on the infrared structure of the theory and in this sense it is the most general one. In particular, it allows us to identify the natural size of the coefficients associated with each order in the EFT expansion.

Let us briefly summarize the basic assumptions and properties of this framework.

- The Goldstone bosons of electroweak symmetry breaking and the light Higgs are treated, in general, as part of a new strong dynamics, to which they are coupled with a strength of $\mathcal{O}(4\pi)$. The scale of the new dynamics is given by the Goldstone-boson decay constant f .
- The transverse gauge bosons and the fermions of the Standard Model are weakly coupled among themselves and to the strong sector, that is with couplings of $\mathcal{O}(1)$.
- The general effective theory for the light fields mentioned above (the fields of the SM) is an electroweak chiral Lagrangian. This theory is nonrenormalizable and is valid below a cut-off² $\Lambda = 4\pi f$. The terms in the Lagrangian are organized as a loop expansion, which is equivalent to a counting of terms according to their chiral dimension [8,17,18]. The assignment of chiral dimensions to fields and couplings is 0 for Higgs, Goldstone and gauge fields, 1/2 for fermions, and 1 for derivatives and weak couplings (gauge or Yukawa).
- In full generality, the electroweak scale v and the scale f can be taken to be of the same order, $\xi \equiv v^2/f^2 = \mathcal{O}(1)$. An expansion in ξ can be performed for $\xi \ll 1$. In this case a counting by canonical dimension is recovered. The new dynamics decouples in the limit $\xi \rightarrow 0$.

¹ Contrary to the chiral Lagrangian in the strong sector, the loop expansion for the electroweak chiral Lagrangian cannot be cast as a derivative expansion. Quite generally, a derivative expansion is valid only when the field content of the theory is restricted to Goldstone fields, but fails when other fields are present. The Yukawa interactions, for instance, clearly cannot be accounted for in terms of a pure derivative expansion.

² If new states with mass of order $f \gg v$ are present, the chiral Lagrangian with SM fields will only be valid up to this scale f . The value $4\pi f$ for the cut-off Λ of the strong dynamics should be viewed as an order-of-magnitude estimate. Details of the strong dynamics and additional pseudo-Goldstones at scale f may lead to an actual cut-off $\Lambda_U < 4\pi f$, as has been discussed for Little-Higgs theories in [16].

Knowledge or partial knowledge of non-infrared physics, *i.e.*, extra symmetries or additional particle content around or beyond the cutoff $\Lambda = 4\pi f$, refine the power-counting estimate and allow for additional information on the size of the operator coefficients. However, one should keep in mind that incorporating UV information goes beyond the EFT power counting and introduces some degree of model dependence.

The traditional SILH Lagrangian [11] is constructed by assuming a set of (infrared) power-counting rules based on NDA [13,14] supplemented with information on the UV completion. Specifically, the UV completion is assumed to contain a heavy vector with $m_V \lesssim 4\pi f = \Lambda$, implemented as a gauge field of some hidden local symmetry (HLS) [19,20]. The NDA rules for operator building in [11] are only defined relative to the leading-order (SM) Lagrangian, rather than in an absolute sense, and read: (i) extra powers of the Higgs doublet H receive a suppression by $1/f$; (ii) SM gauge fields and derivatives receive a $1/m_V$ suppression.

With these assumptions, the NLO Lagrangian is written as [11]:

$$\begin{aligned}
\mathcal{L}_{\text{SILH}} = & \frac{c_H}{2f^2} \partial^\mu (H^\dagger H) \partial_\mu (H^\dagger H) - \frac{c_6 \lambda}{f^2} (H^\dagger H)^3 + \left(\frac{c_Y y f}{f^2} H^\dagger H \bar{f}_L H f_R + \text{h.c.} \right) \\
& + \frac{c_T}{2f^2} (H^\dagger \overleftrightarrow{D}^\mu H) (H^\dagger \overleftrightarrow{D}_\mu H) \\
& + i g \frac{c_W}{2m_V^2} (H^\dagger \overleftrightarrow{D}_\mu^a H) (D^\nu W_{\mu\nu})^a + i g' \frac{c_B}{2m_V^2} (H^\dagger \overleftrightarrow{D}_\mu H) (\partial^\nu B_{\mu\nu}) \\
& + i g \frac{c_{HW}}{(4\pi f)^2} D^\mu H^\dagger W_{\mu\nu} D^\nu H + i g' \frac{c_{HB}}{(4\pi f)^2} D^\mu H^\dagger D^\nu H B_{\mu\nu} \\
& + g'^2 \frac{c_\gamma}{(4\pi f)^2} \frac{g^2}{g_V^2} H^\dagger H B_{\mu\nu} B^{\mu\nu} + g_s^2 \frac{c_g}{(4\pi f)^2} \frac{y_t^2}{g_V^2} H^\dagger H G_{\mu\nu}^a G^{\mu\nu a} \\
& - g^2 \frac{c_{2W}}{2(g_V m_V)^2} D_\mu W^{\mu\nu a} D^\rho W_{\rho\nu}^a - g'^2 \frac{c_{2B}}{2(g_V m_V)^2} \partial_\mu B^{\mu\nu} \partial^\rho B_{\rho\nu} \\
& - g_s^2 \frac{c_{2G}}{2(g_V m_V)^2} D_\mu G^{\mu\nu A} D^\rho G_{\rho\nu}^A \\
& + g^3 \frac{c_{3W}}{(4\pi m_V)^2} \langle W^{\mu\nu} W_{\nu\rho} W_\mu^\rho \rangle + g_s^3 \frac{c_{3G}}{(4\pi m_V)^2} \langle G^{\mu\nu} G_{\nu\rho} G_\mu^\rho \rangle \quad (1)
\end{aligned}$$

where $\langle \dots \rangle$ denotes the trace.

The first two lines collect the operators that are sensitive to the breaking scale f , whereas the remaining lines gather operators generated either by tree-level resonance exchange or at one loop.

If one is aiming at a general description of strongly-coupled EWSB scenarios, the previous setting is unsatisfactory for a number of reasons. The first one is the absence of fermionic operators, which were recently included in [12]. In this paper we will focus instead on issues that mostly affect the foundations and systematics of the SILH construction.

The need for a more systematic power counting can be seen from the fact that the NDA rules given in [11] lead to ambiguities. As a simple example, consider the NLO operator $H^\dagger H B_{\mu\nu} B^{\mu\nu}$. This operator could be built by either (i) applying the first rule to the gauge kinetic term; (ii) applying the second rule to the Higgs potential; or (iii) applying the second rule to the Higgs kinetic term. The size of the corresponding coefficient would be, respectively, $\mathcal{O}(1/f^2)$, $\mathcal{O}(f^2/m_V^4)$ and $\mathcal{O}(1/m_V^2)$. The right counting is the latter, which follows without ambiguities from the rules given in [8,10]. Another example is given by the operator $(H^\dagger \overleftrightarrow{D}^\mu H)^2$, which corresponds to

the T parameter. It can be built with rule (i) applied to the Higgs kinetic term or with rule (ii) applied to the Higgs quartic interaction. The size of the coefficients would then be of $\mathcal{O}(1/f^2)$ or $\mathcal{O}(1/m_V^2)$, respectively. In this case, additional dynamical assumptions are needed to decide between the different possibilities. A more detailed discussion of this operator is given at the end of Section 4.

If Eq. (1) is to describe a consistent EFT, then the scales $\Lambda = 4\pi f \approx m_V \approx g_V f$ may all be identified for the purpose of power counting: numerical differences in the size of the coefficients are expressed in any case through differences in the $\mathcal{O}(1)$ parameters c_i . This follows from rather general principles: unless $m_V \sim \mathcal{O}(\Lambda)$, the naturalness of the EFT would be upset. This is actually one of the conditions to have a natural and predictive strongly-coupled EFT (like chiral perturbation theory).

Based on the previous point, it is apparent that the distinction between tree-level ($1/m_V^2$) vs. loop-suppressed operators ($1/16\pi^2 f^2$) turns out to be of little numerical relevance. However, such a classification is also parametrically misleading: which operators can be generated at tree level and which at one loop mostly depends on the UV completion one adopts. This point has already been discussed in detail in [21].

The pattern displayed in (1) is specific for an UV scenario with vector mesons implemented à la HLS [19,20]. However, there is no compelling reason why such a pattern should be expected. The best counterexample is provided by QCD itself at low energies (see [21] for similar considerations). Within the chiral expansion, the NLO operators that involve gauge fields can be written as [22]

$$\begin{aligned} \mathcal{L}_\chi^{(4)} = & L_9^L \langle F_L^{\mu\nu} D_\mu U D_\nu U^\dagger \rangle + L_9^R \langle F_R^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle \\ & + L_{10} \langle F_L^{\mu\nu} U F_{R\mu\nu} U^\dagger \rangle + H_1^L \langle F_L^{\mu\nu} F_{L\mu\nu} \rangle + H_1^R \langle F_R^{\mu\nu} F_{R\mu\nu} \rangle \end{aligned} \quad (2)$$

where $F_{\mu\nu}^{L,R}$ are generic (non-Abelian) external sources. Since QCD has a global $SU(2)_L \otimes SU(2)_R$ symmetry broken down to $SU(2)_V$, $L_9^L = L_9^R \equiv L_9$ and $H_1^L = H_1^R \equiv H_1$. However, for comparison purposes, it is useful to formally distinguish the left- and right-handed parts. By inspection one can then see that \mathcal{O}_{HW} , \mathcal{O}_{HB} and \mathcal{O}_γ are in correspondence with \mathcal{O}_9^L , \mathcal{O}_9^R and \mathcal{O}_1^R , respectively. \mathcal{O}_W and \mathcal{O}_B , in turn, can be rewritten as linear combinations of the previous operators and, additionally, $H^\dagger H W_{\mu\nu}^a W^{\mu\nu a}$ and $H^\dagger W_{\mu\nu} H B^{\mu\nu}$, which correspond to \mathcal{O}_1^L and \mathcal{O}_{10} .³ Schematically,

$$\mathcal{O}_{HW,HB} \sim \mathcal{O}_9^{L,R}; \quad \mathcal{O}_\gamma \sim \mathcal{O}_1^R; \quad \mathcal{O}_{W,B} \rightarrow \mathcal{O}_{10}, \mathcal{O}_1^L \quad (3)$$

In QCD, all the previous operators are experimentally of the same order $\sim f_\pi^2/\Lambda^2$, they all can be generated by tree-level resonance exchange, and they all are $\mathcal{O}(N_c)$, with no combinations of them being suppressed, i.e., $\mathcal{O}(1)$ [23]. Therefore, QCD does not follow the UV pattern assumed in [11].

A point of phenomenological relevance is the spurion suppression associated with $H^\dagger H B_{\mu\nu} B^{\mu\nu}$ and $H^\dagger H G_{\mu\nu}^A G^{\mu\nu A}$, $(g/g_V)^2$ and $(y_t/g_V)^2$, respectively. This issue is closely related to shift symmetry and its breaking, and will be discussed in more detail in Section 6. The main conclusion is that such a suppression is not present in general.

To summarize, the operators collected in (1) have a simple counting in terms of the breaking scale f and the cutoff scale $4\pi f$ and fall into four main classes:

³ The \mathcal{O}_i are defined as the operators multiplying the coefficients c_i in (1) and L_i, H_i in (2), respectively.

- The first line is suppressed only by $1/f^2$, which in the electroweak chiral Lagrangian corresponds to LO operators.
- The second line (T -parameter) is superficially of order $1/f^2$. If custodial symmetry is weakly broken, as it is usually assumed, the actual coefficient comes with an extra suppression by $1/16\pi^2$.
- The last line, as it stands, would correspond to NNLO operators, since effectively they are two-loop suppressed, $\mathcal{O}(1/(16\pi^2 f)^2)$.
- The remaining operators carry a $1/(4\pi f)^2$ suppression. In the electroweak chiral Lagrangian they appear as NLO operators, and as such they are generated by LO loop diagrams as well as tree-level resonance exchange, in analogy with what happens in QCD. The additional $(g/g_V)^2$ and $(y_i/g_V)^2$ factor suppression in $H^\dagger H B_{\mu\nu} B^{\mu\nu}$ and $H^\dagger H G_{\mu\nu} G^{\mu\nu}$ is not present in a model-independent way.

In the following sections we will substantiate these statements by deriving the SILH Lagrangian as a limiting case of the more general electroweak chiral Lagrangian.

3. The electroweak chiral Lagrangian at small ξ

We will next outline the systematics of the effective theory for standard-model particles with strong dynamics in the Higgs sector. The basic assumptions for the fields and their couplings have been summarized at the beginning of Section 2.

The framework is very general and can be applied to different scenarios. When the appropriate limits are taken, it covers technicolor-like theories, composite-Higgs models, or models with weakly-coupled UV completions. To be specific, we will focus on theories with a pseudo-Goldstone Higgs. In this case we can typically distinguish three relevant energy scales: The electroweak scale v , the scale f of the symmetry breaking that leads to the Goldstone bosons, and the scale $\Lambda = 4\pi f$, where the low-energy description of this dynamics is cut off. The three scales imply two possible expansion parameters, $\xi = v^2/f^2$ and the loop factor $1/(16\pi^2) = f^2/\Lambda^2$.

The resulting picture is sketched in Fig. 1, where we plot the powers of ξ on the vertical axis and the loop order on the horizontal. The dots indicate, schematically, (classes of) operators in the effective Lagrangian or, alternatively, terms in a physical amplitude.

Without expanding in ξ , the effective theory takes the form of a loop expansion as in the usual chiral Lagrangians [24]. This amounts to proceeding from left to right in Fig. 1, order by order in the loop expansion, resumming at each order all terms along the vertical axis.

Alternatively, the expansion may be organized in powers of ξ , proceeding from bottom to top of Fig. 1 and including, in principle, at each power of ξ terms of arbitrary order in the loop expansion. This scheme corresponds to the conventional expansion of the effective theory in terms of the canonical dimension d of operators, where the power of ξ is given by $(d - 4)/2$. Since the dimensional expansion requires only a hierarchy between v and the new-physics scale f , $\xi \ll 1$, it is not restricted to the pseudo-Goldstone Higgs scenarios we are focusing on here.

We emphasize that these observations clarify the relation between an effective theory organized by canonical dimension and the electroweak chiral Lagrangian organized as a loop expansion: The former is constructed row by row, the latter column by column from the terms in Fig. 1.

We now return specifically to the pseudo-Goldstone Higgs scenario with a hierarchy between v and f . The Higgs sector is assumed to be governed by strong dynamics. Its effective description at scale f is then organized in terms of a loop expansion. The electroweak effective Lagrangian

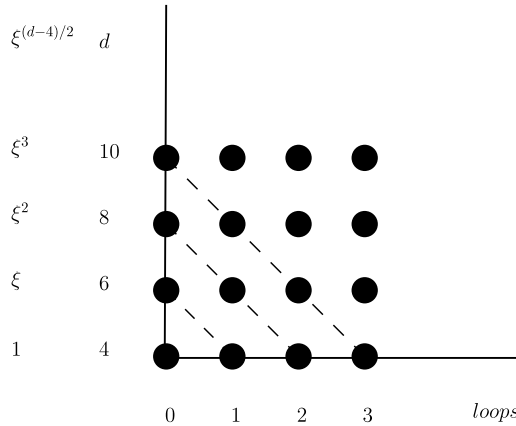


Fig. 1. Systematics of the effective theory with strong dynamics in the Higgs sector. The dots indicate operators in the effective Lagrangian (or terms in a physical amplitude). In general, they may be organized both in powers of $\xi = v^2/f^2$ (vertical axis) and according to their order L in the loop expansion (horizontal axis). The latter is equivalent to the chiral dimension $2L + 2$.

at scale v is further obtained by integrating out the physics at f , which amounts to a dimensional expansion in powers of ξ . Therefore, if ξ is small enough for this expansion to be meaningful,⁴ the effective theory at v can be considered as a *double expansion* in the number of loops and in powers of ξ . Put differently, the expansion is governed simultaneously by chiral and canonical dimensions. Nominally taking ξ and f^2/Λ^2 to be of the same order, the effective theory for a pseudo-Goldstone Higgs sector then becomes an expansion organized as indicated by the dashed lines in Fig. 1.

We note that the conventional SILH Lagrangian [11,12] has been defined as a dimensional expansion up to first order in ξ , with a further scaling of the coefficients of dimension-6 operators either as $1/f^2$ or $1/\Lambda^2$. As discussed in Section 2, the latter scaling essentially reproduces the weighting implied by the loop expansion. However, at second order in the double expansion only terms of order $\xi/16\pi^2$ are retained. The terms of order ξ^2 are not included in SILH, which is not justified as long as ξ is at least of order $1/16\pi^2$. In fact, one typically has $\xi \gg 1/16\pi^2$, in which case ξ^2 is actually more important than $\xi/16\pi^2$. This holds in spite of the fact that ξ^2 terms correspond to operators of canonical dimension 8. Such a behaviour may seem unexpected from the point of view of an effective theory organized primarily in terms of canonical dimensions. It can be understood, however, as a natural consequence of the power counting based on chiral dimensions underlying the EFT of a strongly-coupled sector.

We remark that the terms of order ξ^2 might be included in the conventional SILH Lagrangian by adding the appropriate operators of dimension 8 (and leading chiral dimension). Alternatively, we may simply choose to work throughout with the electroweak chiral Lagrangian, which automatically includes a resummation to all orders in ξ .

⁴ If v is not much smaller than f , the ξ -expansion cannot be performed. The resulting EFT is a chiral Lagrangian at scale f . In this case, if new particles with mass of order f should exist, they would have to be included as additional fields in the EFT. This is of course possible, but it would go beyond our initial assumption of a standard-model particle content.

Finally, we emphasize again the very general nature of the complete electroweak chiral Lagrangian with a light Higgs as presented in [10]. There the Higgs boson is simply described as an electroweak singlet, coupled to the nonlinearly realized electroweak Goldstone bosons and the remaining SM fields. While this covers scenarios where the Higgs is a pseudo-Goldstone boson from an extended symmetry, it is not restricted to them. The Higgs particle might e.g. be a dilaton, or just an ad-hoc singlet, even though this appears unattractive theoretically. In any case, the chiral Lagrangian framework will allow for experimental tests with the minimum amount of theoretical bias.

4. SILH from the electroweak chiral Lagrangian

While EFTs of weakly-coupled dynamics are dimensional expansions in powers of $1/\Lambda$, EFTs of strongly-coupled dynamics are intrinsically loop expansions. As a result, they are expansions in $f^2/\Lambda^2 = 1/(16\pi^2)$, which is a reflection of the nondecoupling nature of the interactions. Scenarios that incorporate the vacuum misalignment mechanism [25,26] allow us to describe the transition from the nondecoupling to the decoupling regime through the parameter $\xi = v^2/f^2$, such that at small ξ one recovers a linear (dimensional) expansion. This means that the electroweak effective Lagrangian, which is generally defined as

$$\mathcal{L}_{\chi EW} = \mathcal{L}_{LO}^{(\xi)} + \mathcal{L}_{NLO}^{(\xi)} + \mathcal{O}\left(\frac{f^4}{\Lambda^4}\right) \quad (4)$$

with $\mathcal{L}_{NLO}^{(\xi)} = \mathcal{O}(f^2/\Lambda^2)$, should satisfy

$$\lim_{\xi \rightarrow 0} \mathcal{L}_{\chi EW} = \mathcal{L}_{(0)} + \xi \mathcal{L}_{(1)} + \mathcal{O}(\xi^2) \equiv \mathcal{L}_{SM} + \xi \bar{\mathcal{L}}_{SILH} + \mathcal{O}(\xi^2) \quad (5)$$

It follows that $\mathcal{L}_{SM} = \mathcal{L}_{LO}^{(\xi=0)}$, while $\mathcal{L}_{SILH} \equiv \xi \bar{\mathcal{L}}_{SILH}$, with

$$\bar{\mathcal{L}}_{SILH} = \left[\frac{d}{d\xi} (\mathcal{L}_{LO}^{(\xi)} + \mathcal{L}_{NLO}^{(\xi)}) \right]_{\xi \rightarrow 0} \quad (6)$$

This nontrivial overlap between LO and NLO operators of the linear and nonlinear bases has already been discussed in [10]. Dimension-six operators coming from \mathcal{L}_{LO} are suppressed by $1/f^2$, whereas dimension-six operators stemming from \mathcal{L}_{NLO} have a $1/\Lambda^2$ suppression. The contribution of \mathcal{L}_{LO} to every order in the ξ -expansion can be easily understood by noticing that powers of $H^\dagger H = (v+h)^2/2$ increase canonical dimensions but leave chiral dimensions unaffected.

The generic dipole $\bar{\psi} \sigma_{\mu\nu} X^{\mu\nu} \psi$ and triple-field-strength $X_{\mu\nu} X^{\nu\lambda} X_\lambda^\mu$ operators are not required as counterterms of the chiral Lagrangian at NLO. Concerning their importance in the EFT we remark that the counting of chiral dimensions is less straightforward than the counting of canonical dimensions since the number of weak couplings (carrying a chiral dimension of 1) is not always obvious from the field content of a given operator. We will next discuss some consequences of this in more detail, considering first the chiral Lagrangian at scale f , where the physics that has been integrated out resides at scale Λ .

The triple-gauge operators X^3 have three derivatives. The gauge fields are weakly coupled to the heavy sector, which implies the presence of (at least) three gauge couplings. This is true irrespective of whether the heavy sector itself is governed by weakly or strongly coupled dynamics.

It follows that $g^3 X^3$ has chiral dimension 6 and therefore enters only at NNLO, that is with a double suppression $\sim 1/(16\pi^2 \Lambda^2)$. We remark that this argument generalizes the corresponding result of [27], obtained for the case of a weakly-coupled UV completion. An explicit example for the $1/(16\pi^2 \Lambda^2)$ scaling of the coefficient of X^3 in the context of a strongly-coupled heavy sector is given by the model discussed in [28]. We emphasize that here the scaling of the coefficient does not automatically follow from the canonical dimension of the operator, which only implies a factor of $1/\Lambda^2$. On the other hand, the presence of the additional loop factor $1/(16\pi^2)$ is consistently accounted for through the counting of chiral dimensions. The situation may change when new states at the scale $f \gg v$ are integrated out to yield the EFT at the scale v . In this case, coefficients of order $\xi/16\pi^2$ could arise for the X^3 operators. Similar comments apply to the dipole operators $m_\psi \bar{\psi}_L \sigma_{\mu\nu} \psi_R g X^{\mu\nu}$.

In addition, some of the four Fermi operators are not needed as one-loop counterterms. However, they can be generated via tree-level exchange of a heavy resonance and are therefore kept at NLO.

Practically, as explained in [10], the list of operators up to linear order in ξ is obtained by taking the full list of dimension-six operators in the linear basis [29,30] and performing the polar decomposition of the doublet

$$\phi = \frac{v+h}{\sqrt{2}} U \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

where $U = \exp(2i\varphi^a T^a/v)$ denotes the Goldstone-boson matrix.

The resulting operators are matched onto the leading and next-to-leading operators of the chiral Lagrangian. An important subtlety is worth mentioning, which affects the whole matching procedure. Since the linear basis is normally expressed in the unbroken phase, while the chiral Lagrangian is written in the broken phase, in the former case there are NLO contributions that renormalize the LO parameters. The modified operators can be brought back to their canonical form by subsequent redefinitions of the fields and couplings. Here we will omit details of such redefinitions and present the final results.

To leading order in chiral dimensions, and to first order in ξ , the SM effective Lagrangian can be written in nonlinear notation as [10]

$$\begin{aligned} \mathcal{L}_2 = & -\frac{1}{2} \langle G_{\mu\nu} G^{\mu\nu} \rangle - \frac{1}{2} \langle W_{\mu\nu} W^{\mu\nu} \rangle - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \bar{q} i \not{D} q + \bar{l} i \not{D} l + \bar{u} i \not{D} u + \bar{d} i \not{D} d + \bar{e} i \not{D} e \\ & + \frac{v^2}{4} \langle L_\mu L^\mu \rangle (1 + F_U(h)) + \frac{1}{2} \partial_\mu h \partial^\mu h - V(h) \\ & - v \left[\bar{q} \left(Y_u + \sum_{n=1}^3 Y_u^{(n)} \left(\frac{h}{v} \right)^n \right) U_{P_+ r} + \bar{q} \left(Y_d + \sum_{n=1}^3 Y_d^{(n)} \left(\frac{h}{v} \right)^n \right) U_{P_- r} \right. \\ & \left. + \bar{l} \left(Y_e + \sum_{n=1}^3 Y_e^{(n)} \left(\frac{h}{v} \right)^n \right) U_{P_- \eta} + \text{h.c.} \right] \tag{8} \end{aligned}$$

with $L_\mu = iUD_\mu U^\dagger$, $P_\pm = 1/2 \pm T_3$, and

$$F_U = (2 - a_2) \frac{h}{v} + (1 - 2a_2) \left(\frac{h}{v} \right)^2 - \frac{4}{3} a_2 \left(\frac{h}{v} \right)^3 - \frac{1}{3} a_2 \left(\frac{h}{v} \right)^4 \tag{9}$$

$$V = \frac{m_h^2}{2} h^2 + \frac{m_h^2 v^2}{2} \left[\left(1 + \frac{4}{3} a_1 - \frac{3}{2} a_2 \right) \left(\frac{h}{v} \right)^3 + \left(\frac{1}{4} + 2a_1 - \frac{25}{12} a_2 \right) \left(\frac{h}{v} \right)^4 + (a_1 - a_2) \left(\frac{h}{v} \right)^5 + \frac{a_1 - a_2}{6} \left(\frac{h}{v} \right)^6 \right] \tag{10}$$

$$Y_f^{(1)} = \left(1 - \frac{a_2}{2} \right) Y_f + 2\bar{Y}_f, \quad Y_f^{(2)} = 3Y_f^{(3)} = -\frac{a_2}{2} Y_f + 3\bar{Y}_f, \quad f = u, d, e \tag{11}$$

For generality, we have included generic flavor matrices \bar{Y}_f arising at NLO. In scenarios with minimal flavor violation [31], $\bar{Y}_f \propto Y_f$.

Here a_1, a_2 and the flavor matrices \bar{Y}_d, \dots correspond to the coefficients of the dimension-6 operators $(\phi^\dagger \phi)^3, \partial(\phi^\dagger \phi)\partial(\phi^\dagger \phi)$ and $\bar{q}\phi d\phi^\dagger \phi, \dots$, respectively. These coefficients are all of order ξ . When they are put to zero, \mathcal{L}_2 reduces to the renormalizable SM.

At chiral dimension 4 (NLO) and to order ξ one finds the Lagrangian

$$\begin{aligned} \mathcal{L}_4 = & -\beta_1 v^2 \langle L_\mu \tau_L \rangle^2 \left(1 + \frac{h}{v} \right)^4 - \frac{c_{Xh1}}{4} B_{\mu\nu} B^{\mu\nu} \left[1 - \left(1 + \frac{h}{v} \right)^2 \right] \\ & - \frac{c_{Xh2}}{2} \langle W_{\mu\nu} W^{\mu\nu} \rangle \left[1 - \left(1 + \frac{h}{v} \right)^2 \right] - \frac{c_{Xh3}}{2} \langle G_{\mu\nu} G^{\mu\nu} \rangle \left[1 - \left(1 + \frac{h}{v} \right)^2 \right] \\ & + c_{XU1} g g' \langle W_{\mu\nu} \tau_L \rangle B^{\mu\nu} \left(1 + \frac{h}{v} \right)^2 + c_{\psi V7} (\bar{l} \gamma^\mu l) \langle L_\mu \tau_L \rangle \left(1 + \frac{h}{v} \right)^2 \\ & + c_{\psi V1} (\bar{q} \gamma^\mu q) \langle L_\mu \tau_L \rangle \left(1 + \frac{h}{v} \right)^2 + c_{\psi V10} (\bar{e} \gamma^\mu e) \langle L_\mu \tau_L \rangle \left(1 + \frac{h}{v} \right)^2 \\ & + c_{\psi V4} (\bar{u} \gamma^\mu u) \langle L_\mu \tau_L \rangle \left(1 + \frac{h}{v} \right)^2 + c_{\psi V5} (\bar{d} \gamma^\mu d) \langle L_\mu \tau_L \rangle \left(1 + \frac{h}{v} \right)^2 \\ & + c_{\psi V6} (\bar{u} \gamma^\mu d) \langle P_{21} U^\dagger L_\mu U \rangle \left(1 + \frac{h}{v} \right)^2 + \text{h.c.} \\ & + c_{\psi Vq} \mathcal{O}_q \left(1 + \frac{h}{v} \right)^2 + c_{\psi Vl} \mathcal{O}_l \left(1 + \frac{h}{v} \right)^2 + \mathcal{L}_{\psi^4} + \mathcal{L}_{\psi^2 X} + \mathcal{L}_{X^3} \end{aligned} \tag{12}$$

where \mathcal{L}_{ψ^4} refers to all baryon-number conserving four-fermion operators, $\mathcal{L}_{\psi^2 X}$ to the dipole operators $\bar{\psi} \sigma_{\mu\nu} X^{\mu\nu} \psi$, and \mathcal{L}_{X^3} to the triple-gauge operators $X_{\mu\nu} X^{\nu\lambda} X_\lambda^\mu$. They can be found in [29]. All coefficients c_i and β_1 scale as $\mathcal{O}(\xi/16\pi^2)$. We used the shorthand notation

$$\begin{aligned} \mathcal{O}_q = & 2(\bar{q} \tau_L \gamma^\mu q) \langle L_\mu \tau_L \rangle + (\bar{q} U P_{12} U^\dagger \gamma^\mu q) \langle P_{21} U^\dagger L_\mu U \rangle \\ & + (\bar{q} U P_{21} U^\dagger \gamma^\mu q) \langle P_{12} U^\dagger L_\mu U \rangle \\ \mathcal{O}_l = & 2(\bar{l} \tau_L \gamma^\mu l) \langle L_\mu \tau_L \rangle + (\bar{l} U P_{12} U^\dagger \gamma^\mu l) \langle P_{21} U^\dagger L_\mu U \rangle \\ & + (\bar{l} U P_{21} U^\dagger \gamma^\mu l) \langle P_{12} U^\dagger L_\mu U \rangle \end{aligned} \tag{13}$$

where $\tau_L = UT_3U^\dagger, P_{12} = T_1 + iT_2, P_{21} = P_{12}^\dagger$.

We note that the result of Eq. (12) relies on the assumption that custodial symmetry and CP are only broken by weak perturbations. Their breaking is thus generated by the gauge and Yukawa couplings. Spurions must then come with an associated weak coupling, which carries chiral

dimension. It then follows that the T parameter is loop-suppressed and CP violating operators can only show up at NNLO.

In this sense, Eq. (12) is by construction the most general next-to-leading-order correction to the chiral electroweak Lagrangian close to the decoupling limit, to first order in ξ . It is therefore a well-defined approach to a systematic derivation of the SILH Lagrangian for generic light Higgs scenarios. Notice that picking the leading dependence in ξ from the chiral electroweak operators brings in a series of correlations between the different coefficients, all of them arising from the doublet structure of the Higgs field that emerges in the decoupling limit. As already noted, Eq. (12) is written in the broken phase and therefore the impact of NLO effects in the LO parameters has been taken care of, which results in some operators not being proportional to $(v+h)^2$. Apart from this notational aspect, a comparison with the original SILH Lagrangian [11] and its recent extension [12] shows that: (i) since the construction of the operators is a purely infrared issue, all model-dependence of the original formulation is inessential and can be removed. As a result, the structure of the Lagrangian gets simplified and the role of the relevant scales in the problem becomes more transparent; (ii) custodial symmetry breaking through the T -parameter comes with an overall coefficient $1/\Lambda^2$, in agreement with the discussion in [11]; (iii) in general there is no extra suppression of the $B_{\mu\nu}B^{\mu\nu}H^\dagger H$ and $G_{\mu\nu}G^{\mu\nu}H^\dagger H$ operators (see the further discussion in Section 6).

We emphasize that the effective Lagrangian contains also terms of higher order in ξ . By definition, those go beyond the SILH approximation. However, some of them, related to the Higgs sector, come without loop suppression and would typically be more important than $\xi/16\pi^2$ terms, as long as $\xi \gg 1/16\pi^2$. In practice, to work them out explicitly, dimension-8 operators would have to be considered. We note that working with the full electroweak chiral Lagrangian automatically includes all orders in ξ .

In the following section we will give a more detailed account of how custodial symmetry breaking is implemented in the EFT. This can be done without relying on the UV dynamics and will therefore lead to a number of model-independent conclusions.

5. Custodial symmetry and its breaking

In this section we consider general properties of custodial symmetry and its violation in the electroweak effective Lagrangian. The concept of custodial symmetry is well known. We review it here to provide the proper context for our subsequent general discussion of its violation by spurions in effective field theory.

We assume that the electroweak sector exhibits the spontaneous breaking of a global symmetry according to the pattern

$$SU(2)_L \otimes SU(2)_R \rightarrow SU(2)_V \quad (14)$$

The associated Goldstone fields φ^a , $a = 1, 2, 3$, parametrize the coset of the symmetry breaking in (14), expressed through the $SU(2)$ matrix field $U = \exp(2i\varphi^a T^a/v)$, where T^a are the generators of $SU(2)$. Under $SU(2)_L \otimes SU(2)_R$ the field U transforms as $U \rightarrow g_L U g_R^\dagger$, with $g_{L,R} \in SU(2)_{L,R}$. The vacuum $U = \mathbf{1}$ breaks this symmetry but remains invariant under $SU(2)_V$, defined by $SU(2)$ transformations that obey $g_L = g_R \equiv g_V$. The residual, global invariance under $SU(2)_V$ is commonly referred to as the *custodial symmetry* [32]. It is useful to distinguish two somewhat different meanings of this term. In the narrow sense, custodial symmetry refers only to the spontaneously broken dynamics itself, that is to the scalar sector (Higgs fields) or the corresponding new strong interactions. In the general sense, custodial symmetry refers to

all interactions, strong (scalar) dynamics and weak perturbations (e.g. from gauge or Yukawa couplings).

When (part of) the symmetry $SU(2)_L \otimes SU(2)_R$ is gauged, some of the gauge fields become massive via the Higgs mechanism. It is instructive to consider the following possibilities of gauging a subgroup of $SU(2)_L \otimes SU(2)_R$ and the resulting spectrum of gauge bosons: a) $SU(2)_L$ (3 massive, degenerate gauge bosons); b) $SU(2)_V$ (3 massless gauge bosons); c) $SU(2)_L \otimes SU(2)_R$ (3 massive, degenerate and 3 massless gauge bosons); d) $SU(2)_L \otimes U(1)_Y$ (Standard Model, 1 massless, 3 massive gauge bosons with $M_W \neq M_Z$). By the assumption of (14), all cases have a custodial symmetry in the narrow sense. In the general sense of the term, custodial symmetry is violated in the Standard Model ($M_W \neq M_Z$), while cases a), b) and c) remain custodially symmetric, despite the weak gauging.

The distinction between custodial symmetry in the general or the narrow sense is of course a matter of definition. However, it clarifies apparently different uses of the term in the existing literature. For instance, among the electroweak oblique corrections, the T parameter, but not the S parameter, is referred to as a measure of custodial symmetry breaking in [33]. On the other hand, also the S parameter is viewed as a violation of custodial symmetry in [34]. The apparent inconsistency is resolved when the former usage of custodial symmetry is understood in the narrow sense, the latter in the general sense of this term.

In the following, unless stated otherwise, we will adopt the meaning of custodial symmetry in the general sense as defined above. Hence, both $M_W \neq M_Z$ and the S parameter violate custodial symmetry, at leading and next-to-leading order, respectively.

In general, the pattern of explicit breaking of custodial symmetry can be described by spurions ω . We will prove that, in the context of (14), the only spurion of custodial symmetry breaking in the effective Lagrangian is given by T_R^3 , the third generator of $SU(2)_R$. As an illustration of this general theorem we point out how it is realized in the electroweak chiral Lagrangian at leading and next-to-leading order, and also in the usual Standard Model with a linearly realized Higgs sector through operators of dimension 6.

For the case at hand, the spurions are *a priori* general 2×2 matrices⁵ with formal transformation properties under $SU(2)_L \otimes SU(2)_R$, such that invariants under this symmetry can be built, in general involving also U and further fields. The transformation properties reflect the physical origin of a given spurion. Keeping ω fixed at its true (constant) value breaks the global symmetry in the appropriate way.

Any spurion must, in general, have one of the three possible transformation rules under the global group:

$$\omega \rightarrow g_L \omega g_R^\dagger \quad (15)$$

$$\omega \rightarrow g_L \omega g_L^\dagger \quad (16)$$

$$\omega \rightarrow g_R \omega g_R^\dagger \quad (17)$$

An invariant $\omega \rightarrow \omega$ is trivial and does not lead to custodial symmetry breaking. The hermitian conjugate version of (15) is understood.

A general 2×2 matrix ω can be written as a linear combination of the unit matrix $\mathbf{1}$ and T^a , with complex coefficients. An essential restriction arises when part of the global symmetry is

⁵ This is because all terms in the Lagrangian are built from fermion bilinears, U and h fields, gauge field strengths and covariant derivatives, all of which come with an even number of $SU(2)_{L,R}$ indices. Invariants can thus only be formed by contracting with matrices rather than with $SU(2)$ doublets as spurions.

gauged, since only spurions consistent with gauge invariance are allowed. In the case of electroweak theory, the entire $SU(2)_L$ and the weak hypercharge subgroup $U(1)_Y$ of $SU(2)_R$ are gauged. A spurion transforming as (15) would break local $SU(2)_L$ and is therefore forbidden. Scenario (16) likewise breaks $SU(2)_L$ unless $\omega \sim \mathbf{1}$, which is the trivial case. Similarly, (17) breaks $U(1)_Y$ unless $\omega \sim \mathbf{1}$ or $\omega \sim T_R^3$. This leaves T_R^3 as the only nontrivial spurion and proves our assertion.

The allowed spurions are different when a different part of the global group is gauged. An example is the chiral perturbation theory of pions, where the spontaneous breaking of the global symmetry also follows (14), gauged under the electromagnetic $U(1)$. The allowed spurions are then $\omega \sim \mathbf{1}$ and $\omega \sim T^3$, each transforming formally under (15), (16) or (17). This amounts to the quark mass term transforming as (15), and the electric charge operator transforming as (16) or (17).

The fact that T_R^3 is the only spurion of custodial breaking under the electroweak gauging of (14), can be illustrated with concrete examples. Consider first the usual (minimal) Standard Model. The SM Lagrangian can be viewed as the low-energy effective theory of any general UV completion that might exist. There are two sources of custodial symmetry breaking: weak hypercharge gauge interactions, and the difference in up- and down-fermion Yukawa couplings. Both are indeed governed by T_R^3 . In order to see that this is not just an accidental feature of the lowest-order Lagrangian, one may inspect the full set of dimension-6 operators as classified in [29]. These can be written in terms of the Goldstone matrix U and the Higgs singlet h , rather than in terms of the Higgs doublet ϕ . This representation has been discussed e.g. in [9,10]. In this way it can be demonstrated explicitly that, again, the only spurion of custodial breaking that appears is T_R^3 . The same observation holds for the electroweak chiral Lagrangian at leading and next-to-leading order described in [10]. Some of the operators in [9,10] are written in terms of the matrices $P_{12} = T_1 + iT_2$ and $P_{21} = T_1 - iT_2$. To make the reduction to the spurion T_3 manifest one may use the identity

$$(P_{12})_{ij}(P_{21})_{kl} = -\frac{1}{4}\delta_{ij}\delta_{kl} + \frac{1}{2}\delta_{il}\delta_{kj} - (T_3)_{ij}(T_3)_{kl} + \frac{1}{2}(T_3)_{il}\delta_{kj} - \frac{1}{2}\delta_{il}(T_3)_{kj} \quad (18)$$

The discussion of this section demonstrates in particular that the presence of T_R^3 as the only spurion of custodial-symmetry breaking in the general electroweak chiral Lagrangian is a fully general, model-independent property of the effective field theory formulation, in contrast to the claims in [12].

6. $SO(5)/SO(4)$ model at NLO in the chiral expansion

We now would like to show how some of the features that we discussed arise in the context of a specific model, the minimal composite Higgs model of [35,36]. This model assumes spontaneous symmetry breaking of $SO(5)$ down to $SO(4)$ at a scale f , which generates four Goldstone bosons. They span the coset space and can be parametrized as [37] (see Appendix A for details)

$$\Sigma(h_{\hat{a}}) = \begin{pmatrix} \frac{s}{2}\langle U\lambda_{\hat{a}}^\dagger \rangle \\ c \end{pmatrix}, \quad \lambda_{\hat{a}} = (i\vec{\sigma}, 1_2), \quad s = \sin \frac{|h|}{f}, \quad c = \cos \frac{|h|}{f} \quad (19)$$

Above we used the fact that $SO(4)$ is isomorphic to $SU(2)_L \otimes SU(2)_R$ to express the $SO(4)$ vector $h_{\hat{a}}$ in terms of the $SU(2)_L \otimes SU(2)_R$ bifundamental field U and the $h_{\hat{a}}$ modulus $|h|$. The custodial-preserving $SU(2)_L \otimes SU(2)_R$ is further broken (explicitly) by the couplings to gauge

bosons and fermions. The spurion for this breaking is t_3^R and is accompanied by powers of g' and/or Yukawa couplings y_f . For simplicity, in the following we will set fermions aside and focus on the CP-even bosonic sector.

The leading-order Lagrangian (chiral dimension $\chi = 2$) takes the form

$$\mathcal{L} = \frac{f^2}{2} \Sigma^T \Sigma^\mu - V = \frac{1}{2} \partial_\mu |h| \partial^\mu |h| + \frac{f^2}{4} \langle L_\mu L^\mu \rangle s^2 - V \tag{20}$$

where $\Sigma_\mu \equiv D_\mu \Sigma$. The gauge kinetic terms are understood. The leading-order potential is

$$V = \alpha \Sigma^T n - 4\beta \Sigma^T t_3^R t_3^R \Sigma = \alpha c - \beta s^2 \tag{21}$$

where $n = (0, 0, 0, 0, 1)^T$ and t_3^R are the $SO(5)$ -breaking spurions that are consistent with SM gauge invariance. The vector n conserves custodial symmetry, the matrix t_3^R violates it. Both are related through $nn^T = 1 - 4t_3^R t_3^R$.

The coefficients have $\chi = 2$ since they are loop-suppressed and they scale as $\alpha, \beta \sim f^4$. A realization of such a potential in a specific model has been discussed e.g. in [37].

We note that the two terms in (21) are given by the two independent expressions that can be built at leading order from the spurions of $SO(5)$ breaking, n and t_3^R . For $\beta > 0$ and $|\alpha| \leq 2\beta$, the potential in (21) exhibits spontaneous symmetry breaking, generating the vacuum expectation value $\langle |h| \rangle$ via

$$\xi = \frac{v^2}{f^2} = \sin^2 \frac{\langle |h| \rangle}{f} = 1 - \left(\frac{\alpha}{2\beta} \right)^2, \quad v < \langle |h| \rangle < \frac{\pi}{2} v \tag{22}$$

where $\langle |h| \rangle$ ranges from the decoupling to the nondecoupling limit. The resulting mass of the physical scalar boson $h \equiv |h| - \langle |h| \rangle$ is

$$m_h^2 = \frac{2\beta\xi}{f^2} = \mathcal{O}(v^2) \tag{23}$$

To construct the operators at NLO ($\chi = 4$), it is necessary to employ the general method of Callan, Coleman, Wess and Zumino [38,39]. For the case of the $SO(5)/SO(4)$ coset this has been performed in great detail in [40] (see [41] for a recent discussion of this and other cosets). Here we restrict ourselves to quoting the main results, adding some comments and discussing the matching to the electroweak chiral Lagrangian at scale v .

One defines d_μ and E_μ through [40]

$$-i\mathcal{U}^\dagger D_\mu \mathcal{U} = d_\mu^\hat{a} t^{\hat{a}} + E_\mu^a t^a \equiv d_\mu + E_\mu \tag{24}$$

Here $\mathcal{U} = \exp(\sqrt{2}i t^{\hat{a}} h_{\hat{a}}/f)$ and $t^{\hat{a}}$ (t^a) are the broken (unbroken) generators of $SO(5) \rightarrow SO(4)$. $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative with $A_\mu = A_\mu^{\hat{a}} t^{\hat{a}} + A_\mu^a t^a$ in the most general case. (Here the coupling has been absorbed in A_μ .) In practice, we will be mostly interested in gauging the standard-model group, in which case $A_\mu^{\hat{a}} = 0$. The following building blocks are useful [40]:

$$\partial_\mu E_\nu - \partial_\nu E_\mu + i[E_\mu, E_\nu] \equiv E_{\mu\nu} \equiv E_{\mu\nu}^L + E_{\mu\nu}^R \tag{25}$$

$$f_{\mu\nu} = \mathcal{U}^\dagger F_{\mu\nu} \mathcal{U} \equiv f_{\mu\nu}^- + f_{\mu\nu}^L + f_{\mu\nu}^R \tag{26}$$

Here $f_{\mu\nu}^- \equiv f_{\mu\nu}^- \hat{a} t^{\hat{a}}$, $f_{\mu\nu}^L \equiv f_{\mu\nu}^{L,a} t_a^L$, $f_{\mu\nu}^R \equiv f_{\mu\nu}^{R,a} t_a^R$, and similarly for $E_{\mu\nu}^{L,R}$, where the six unbroken generators t^a are decomposed into the generators $t_a^{L,R}$ of $SU(2)_{L,R}$. $F_{\mu\nu}$ is the field strength of A_μ .

The kinetic term in (20) can then be written as

$$\mathcal{L} = \frac{f^2}{2} \Sigma_\mu^T \Sigma^\mu \equiv \frac{f^2}{4} \langle d_\mu d^\mu \rangle \quad (27)$$

The NLO operators can be constructed from the building blocks above. The CP even operators read [40]

$$\begin{aligned} O_1 &= \langle d_\mu d^\mu \rangle^2 \\ O_2 &= \langle d_\mu d_\nu \rangle \langle d^\mu d^\nu \rangle \\ O_3 &= \langle E_{\mu\nu}^L E^{L,\mu\nu} \rangle - \langle E_{\mu\nu}^R E^{R,\mu\nu} \rangle \\ O_4^+ &= \langle (f_{\mu\nu}^L + f_{\mu\nu}^R) i [d^\mu, d^\nu] \rangle \\ O_5^+ &= \langle (f_{\mu\nu}^-)^2 \rangle \\ O_4^- &= \langle (f_{\mu\nu}^L - f_{\mu\nu}^R) i [d^\mu, d^\nu] \rangle \\ O_5^- &= \langle (f_{\mu\nu}^L)^2 - (f_{\mu\nu}^R)^2 \rangle \end{aligned} \quad (28)$$

We remark that the operator O_3 in this list is redundant:

$$O_3 = O_5^- - 2O_4^- \quad (29)$$

This was also noted recently in [41]. The remaining operators can be expressed in terms of the 2×2 Goldstone field U and the Higgs singlet $|h|$ of the chiral Lagrangian based on the coset $SU(2)_L \otimes SU(2)_R / SU(2)_V$. We find

$$\begin{aligned} O_1 &= \left(\frac{2}{f^2} \partial_\mu |h| \partial^\mu |h| + s^2 \langle L_\mu L^\mu \rangle \right)^2 \\ O_2 &= \left(\frac{2}{f^2} \partial_\mu |h| \partial_\nu |h| + s^2 \langle L_\mu L_\nu \rangle \right)^2 \\ O_4^+ &= -s^2 \langle g D_\mu W^{\mu\nu} L_\nu - g' \partial_\mu B^{\mu\nu} \tau_L L_\nu + \frac{g^2}{2} (W_{\mu\nu})^2 + \frac{g'^2}{2} (B_{\mu\nu} T_3)^2 - g' g B_{\mu\nu} W^{\mu\nu} \tau_L \rangle \\ O_5^+ &= s^2 \langle g^2 (W_{\mu\nu})^2 + g'^2 (B_{\mu\nu} T_3)^2 - 2g' g B_{\mu\nu} W^{\mu\nu} \tau_L \rangle \\ O_4^- &= i \frac{c}{2} (s^2 + 2) \langle g W_{\mu\nu} [L^\mu, L^\nu] - g' B_{\mu\nu} \tau_L [L^\mu, L^\nu] \rangle \\ &\quad + 2c \langle g D_\mu W^{\mu\nu} L_\nu + g' \partial_\mu B^{\mu\nu} \tau_L L_\nu + \frac{g^2}{2} (W_{\mu\nu})^2 - \frac{g'^2}{2} (B_{\mu\nu} T_3)^2 \rangle \\ O_5^- &= 2c \langle g^2 (W_{\mu\nu})^2 - g'^2 (B_{\mu\nu} T_3)^2 \rangle \end{aligned} \quad (30)$$

Here the gauging has been restricted to the standard-model group and the couplings have been factored out of the gauge fields. The terms with $D_\mu W^{\mu\nu}$ and $\partial_\mu B^{\mu\nu}$ in O_4^\pm are reducible upon using the equations of motion. Obviously, $O_{4,5}^+ \rightarrow 0$ in the limit $|h| \rightarrow 0$. Note that in the same limit O_4^- also vanishes upon integrating by parts, while O_5^- just renormalizes the gauge kinetic terms.

The operators on the r.h.s. of (30) match the electroweak chiral Lagrangian in the basis of [10] after eliminating redundant terms and expanding around the Higgs vacuum expectation value. Indeed, $O_{1,2}$ correspond to \mathcal{O}_{Di} in [10] with $i = 1, 2, 7, 8, 11$, whereas $O_{4,5}^\pm$ contain $\mathcal{O}_{Xh1,2}$ and $\mathcal{O}_{XU1,7,8}$.

The $SO(5)/SO(4)$ example illustrates how its effective-theory formulation can be expressed in terms of the general chiral Lagrangian of [10]. Expanding the former to first order in ξ provides an explicit realization of the SILH Lagrangian derived in Section 4. It also exhibits the presence of T_R^3 as the only spurion of custodial symmetry breaking, in agreement with the theorem of Section 5.

The representation of the operators in (30) makes it explicit that the Higgs couples to a pair of field-strength factors only in the combinations

$$\begin{aligned} & \langle g^2(W_{\mu\nu})^2 + g'^2(B_{\mu\nu}T_3)^2 - 2g'gB_{\mu\nu}W^{\mu\nu}\tau_L \rangle \\ & \langle g^2(W_{\mu\nu})^2 - g'^2(B_{\mu\nu}T_3)^2 \rangle \end{aligned} \quad (31)$$

These do not contain the photon–photon component $F_{\mu\nu}F^{\mu\nu}$ and hence there is no $h \rightarrow \gamma\gamma$ operator at this order. This has been emphasized in [11] and explained as the consequence of a residual shift symmetry that commutes with the electric charge Q , similar to the absence of $(\pi^0)^2 F_{\mu\nu}F^{\mu\nu}$ at NLO in chiral perturbation theory [42,43]. This feature is valid for the nonlinear (bosonic) Lagrangian defined at scale f and represented at NLO through the terms in (30). However, the electroweak effective Lagrangian is defined at the scale v . In the limit of small $\xi = v^2/f^2$ the physics at scale f is then integrated out, which may induce the local operator $hF_{\mu\nu}F^{\mu\nu}$ with a coefficient of order $\xi/16\pi^2$ as in (12), that is in general without extra suppression. The same holds for the coupling to gluons, $hG_{\mu\nu}G^{\mu\nu}$. An example is provided by fermion representations in minimal composite Higgs models, which induce local $h \rightarrow \gamma\gamma$ and $h \rightarrow gg$ operators with coefficients of size $\xi/16\pi^2$ [44,45]. This is due to an explicit soft breaking of $SO(5)$ in the fermionic sector at scale f .

7. Conclusions

Strongly-coupled scenarios are viable candidates to explain the mechanism of electroweak symmetry breaking. In their minimal version, one assumes a $SU(2)_L \otimes SU(2)_R \rightarrow SU(2)_V$ breaking pattern at a scale v , with a light Higgs (presumably, but not necessarily, a pseudo-Goldstone boson) and new physics starting around the TeV scale. Under these assumptions, the most efficient description of the physics at present-day colliders is provided by the electroweak chiral Lagrangian. In this paper we have explicitly shown that the so-called SILH Lagrangian can be recovered as a special limit of the latter. To the best of our knowledge, this viewpoint offers the first rigorous derivation of SILH and helps to clarify some of its aspects, especially those related with power counting and the breaking of custodial and shift symmetries.

Our main conclusions can be summarized in the following points:

- We emphasize that the small- ξ limit of the electroweak chiral Lagrangian relies on a double expansion in both, powers of ξ and the number of loops. Phenomenologically, terms of order ξ^2 might be larger than the $\xi/16\pi^2$ terms included in the conventional SILH Lagrangian. The electroweak chiral Lagrangian represents the resummation to all orders in ξ .
- The SILH Lagrangian can be understood as the electroweak chiral Lagrangian to first order in $\xi = v^2/f^2$. This allows a systematic construction of the effective-theory operators with a well-defined power counting and without relying on particular UV completions. The resulting set of dimension-6 operators comes from both the LO and NLO chiral Lagrangian and they are suppressed, respectively, by $1/f^2$ and $1/\Lambda^2$.

- In scenarios where the Higgs is a pseudo-Goldstone boson, the $h \rightarrow \gamma\gamma$ and $h \rightarrow gg$ amplitudes at scale v receive local contributions of order $\xi/16\pi^2$. These can arise from integrating out new states at scale f , which may exist in realistic models.
- We prove that, given the $SU(2)_L \otimes SU(2)_R \rightarrow SU(2)_V$ breaking pattern, custodial symmetry breaking is described by a single spurion, namely T_R^3 . If custodial symmetry is assumed to be preserved by the strong sector, and only broken explicitly by the weak sector (gauge and Yukawa couplings), the T -parameter appears as a NLO effect (of chiral dimension 4) and comes with a suppression of $1/\Lambda^2 \ll 1/f^2$.
- As a concrete illustration of the previous points, we have considered the NLO operators of the CP-even bosonic sector of the $SO(5)/SO(4)$ model and matched them to the electroweak chiral Lagrangian.

To summarize, the electroweak chiral Lagrangian, formulated with the vacuum misalignment parameter ξ , gives not only a description of strict nondecoupling scenarios ($\xi \sim 1$), but it is also valid for softly nondecoupling constructions ($\xi \ll 1$), like the SILH Lagrangian. The electroweak chiral Lagrangian thus provides the well-defined starting point for the construction of generic EFT descriptions of electroweak physics with a strong sector. Importantly, in the small- ξ limit, the electroweak chiral Lagrangian implies a pattern for the coefficients of dimension-6 operators characteristic of a strongly-interacting Higgs sector, which can be tested against experiment.

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Appendix A. $SO(5)/SO(4)$ Goldstone field

In this appendix we collect some technical details used in the discussion of Section 6.

When $SO(5)$ is spontaneously broken to $SO(4)$, the Goldstone multiplet can be parametrized by (see e.g. [37])

$$\Sigma(h_{\hat{a}}) = \mathcal{U} \Sigma_0, \quad \Sigma_0 = \begin{pmatrix} 0_4 \\ 1 \end{pmatrix} \quad (\text{A.1})$$

where

$$\mathcal{U} = \exp(\sqrt{2} i t^{\hat{a}} h_{\hat{a}}/f) \quad (\text{A.2})$$

with $h_{\hat{a}}$ ($\hat{a} = 1, \dots, 4$) a $SO(4)$ vector and $t^{\hat{a}}$ the broken generators that span the 4-parameter coset. Using the realization ($i, j = 1, \dots, 5$)

$$t_{ij}^{\hat{a}} = -\frac{i}{\sqrt{2}} \left(\delta_i^{\hat{a}} \delta_j^5 - \delta_j^{\hat{a}} \delta_i^5 \right) \quad (\text{A.3})$$

direct substitution of the generators above yields ($s = \sin |h|/f$, $c = \cos |h|/f$)

$$\Sigma(h_{\hat{a}}) = \begin{pmatrix} \hat{h}_{\hat{a}} s \\ c \end{pmatrix}, \quad \hat{h}_{\hat{a}} = \frac{h_{\hat{a}}}{|h|}, \quad |h| = \sqrt{h_{\hat{a}} h_{\hat{a}}} \quad (\text{A.4})$$

Since $SO(4)$ is isomorphic to $SU(2)_L \otimes SU(2)_R$, one can relate the $SO(4)$ vector to a complex $SU(2)_L \otimes SU(2)_R$ bidoublet H and its polar decomposition into $|h|$ and the $SU(2)$ matrix U ,

$$H = (\tilde{\phi}, \phi) = \begin{pmatrix} h_4 + ih_3 & h_2 + ih_1 \\ -(h_2 - ih_1) & h_4 - ih_3 \end{pmatrix} = h_{\hat{a}} \lambda_{\hat{a}} \equiv |h|U, \quad \lambda_{\hat{a}} = (i\vec{\sigma}, 1_2) \quad (\text{A.5})$$

which implies

$$\hat{h}_{\hat{a}} = \frac{1}{2} \langle U \lambda_{\hat{a}}^\dagger \rangle \quad (\text{A.6})$$

The doublet ϕ corresponds to the SM Higgs. The present definitions ensure that ϕ transforms as a $SU(2)_L$ doublet with weak hypercharge $Y = \frac{1}{2}$, if the $SO(4)$ generators are realized as

$$\begin{aligned} t_1^L &= \frac{i}{2} \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}; & t_2^L &= \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}; & t_3^L &= \frac{i}{2} \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \\ t_1^R &= \frac{i}{2} \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}; & t_2^R &= \frac{i}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; & t_3^R &= \frac{i}{2} \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \end{aligned} \quad (\text{A.7})$$

which satisfy

$$t_a^{L,R} t_b^{L,R} = \frac{1}{4} \delta_{ab} + \frac{i}{2} \varepsilon_{abc} t_c^{L,R}; \quad [t_a^L, t_b^R] = 0 \quad (\text{A.8})$$

The full set of $SO(5)$ generators is then given by (A.3) and the obvious extension of (A.7) to 5×5 matrices [37].

The operators of the $SO(5)/SO(4)$ chiral Lagrangian can be constructed from the building blocks quoted in Section 6, taking into account their chiral dimension:

$$[d_\mu]_c = 1, \quad [E_{\mu\nu}^{L,R}]_c = [f_{\mu\nu}^{-,L,R}]_c = 2 \quad (\text{A.9})$$

The operators in terms of \mathcal{U} can be expressed through $|h|$ and U using (A.6),

$$\mathcal{U} = \left(\frac{1 - (1 - c) \hat{h} \hat{h}^T}{-s \hat{h}^T} \middle| \frac{s \hat{h}}{c} \right) \quad (\text{A.10})$$

and the relations

$$\lambda_{ij}^{\hat{a}} \lambda_{kl}^{\hat{a}\dagger} = 2 \delta_{il} \delta_{kj} \quad (\text{A.11})$$

$$t_{a,\hat{a}\hat{b}}^L \langle U \lambda_{\hat{b}}^\dagger \rangle = \langle T^a U \lambda_{\hat{a}}^\dagger \rangle, \quad t_{3,\hat{a}\hat{b}}^R \langle U \lambda_{\hat{b}}^\dagger \rangle = -\langle U T_3 \lambda_{\hat{a}}^\dagger \rangle \quad (\text{A.12})$$

The resulting operators with chiral dimension 4 are collected in Section 6.

References

- [1] G. Aad, et al., ATLAS Collaboration, Phys. Lett. B 716 (2012) 1, arXiv:1207.7214 [hep-ex].
- [2] S. Chatrchyan, et al., CMS Collaboration, Phys. Lett. B 716 (2012) 30, arXiv:1207.7235 [hep-ex].
- [3] ATLAS Collaboration, ATLAS-CONF-2013-034.
- [4] CMS Collaboration, CMS-PAS-HIG-13-005.
- [5] G. Aad, et al., ATLAS Collaboration, arXiv:1307.1432 [hep-ex].
- [6] T. Appelquist, C.W. Bernard, Phys. Rev. D 22 (1980) 200; A.C. Longhitano, Phys. Rev. D 22 (1980) 1166; T. Appelquist, M.J. Bowick, E. Cohler, A.I. Hauser, Phys. Rev. D 31 (1985) 1676.

- [7] F. Feruglio, *Int. J. Mod. Phys. A* 8 (1993) 4937, arXiv:hep-ph/9301281;
J. Bagger, et al., *Phys. Rev. D* 49 (1994) 1246, arXiv:hep-ph/9306256;
V. Koulovassilopoulos, R.S. Chivukula, *Phys. Rev. D* 50 (1994) 3218, arXiv:hep-ph/9312317;
C.P. Burgess, J. Matias, M. Pospelov, *Int. J. Mod. Phys. A* 17 (2002) 1841, arXiv:hep-ph/9912459;
L.M. Wang, Q. Wang, *Chin. Phys. Lett.* 25 (2008) 1984, arXiv:hep-ph/0605104;
B. Grinstein, M. Trott, *Phys. Rev. D* 76 (2007) 073002, arXiv:0704.1505 [hep-ph];
R. Alonso, M.B. Gavela, L. Merlo, S. Rigolin, J. Yepes, *Phys. Lett. B* 722 (2013) 330, arXiv:1212.3305 [hep-ph];
R. Alonso, M.B. Gavela, L. Merlo, S. Rigolin, J. Yepes, *Phys. Lett. B* 726 (2013) 926 (Erratum).
- [8] G. Buchalla, O. Catà, C. Krause, *Phys. Lett. B* 731 (2014) 80, arXiv:1312.5624 [hep-ph].
- [9] G. Buchalla, O. Catà, *J. High Energy Phys.* 1207 (2012) 101, arXiv:1203.6510 [hep-ph].
- [10] G. Buchalla, O. Catà, C. Krause, *Nucl. Phys. B* 880 (2014) 552, arXiv:1307.5017 [hep-ph].
- [11] G.F. Giudice, C. Grojean, A. Pomarol, R. Rattazzi, *J. High Energy Phys.* 0706 (2007) 045, arXiv:hep-ph/0703164.
- [12] R. Contino, M. Ghezzi, C. Grojean, M. Mühlleitner, M. Spira, *J. High Energy Phys.* 1307 (2013) 035, arXiv:1303.3876 [hep-ph].
- [13] A. Manohar, H. Georgi, *Nucl. Phys. B* 234 (1984) 189.
- [14] H. Georgi, *Phys. Lett. B* 298 (1993) 187, arXiv:hep-ph/9207278.
- [15] H. Georgi, *Annu. Rev. Nucl. Part. Sci.* 43 (1993) 209.
- [16] S. Chang, H.J. He, *Phys. Lett. B* 586 (2004) 95, arXiv:hep-ph/0311177.
- [17] R. Urech, *Nucl. Phys. B* 433 (1995) 234, arXiv:hep-ph/9405341.
- [18] A. Nyffeler, A. Schenk, *Phys. Rev. D* 62 (2000) 113006, arXiv:hep-ph/9907294.
- [19] M. Bando, T. Kugo, S. Uehara, K. Yamawaki, T. Yanagida, *Phys. Rev. Lett.* 54 (1985) 1215.
- [20] M. Bando, T. Kugo, K. Yamawaki, *Phys. Rep.* 164 (1988) 217.
- [21] E.E. Jenkins, A.V. Manohar, M. Trott, *J. High Energy Phys.* 1309 (2013) 063, arXiv:1305.0017 [hep-ph].
- [22] J. Gasser, H. Leutwyler, *Ann. Phys.* 158 (1984) 142.
- [23] G. Ecker, J. Gasser, A. Pich, E. de Rafael, *Nucl. Phys. B* 321 (1989) 311.
- [24] S. Weinberg, *Physica A* 96 (1979) 327.
- [25] D.B. Kaplan, H. Georgi, *Phys. Lett. B* 136 (1984) 183.
- [26] M.J. Dugan, H. Georgi, D.B. Kaplan, *Nucl. Phys. B* 254 (1985) 299.
- [27] C. Arzt, M.B. Einhorn, J. Wudka, *Nucl. Phys. B* 433 (1995) 41, arXiv:hep-ph/9405214.
- [28] A.V. Manohar, *Phys. Lett. B* 726 (2013) 347, arXiv:1305.3927 [hep-ph].
- [29] B. Grzadkowski, M. Iskrzynski, M. Misiak, J. Rosiek, *J. High Energy Phys.* 1010 (2010) 085, arXiv:1008.4884 [hep-ph].
- [30] W. Buchmüller, D. Wyler, *Nucl. Phys. B* 268 (1986) 621.
- [31] G. D’Ambrosio, G.F. Giudice, G. Isidori, A. Strumia, *Nucl. Phys. B* 645 (2002) 155, arXiv:hep-ph/0207036.
- [32] P. Sikivie, L. Susskind, M.B. Voloshin, V.I. Zakharov, *Nucl. Phys. B* 173 (1980) 189.
- [33] R. Barbieri, A. Pomarol, R. Rattazzi, A. Strumia, *Nucl. Phys. B* 703 (2004) 127, arXiv:hep-ph/0405040.
- [34] B. Grinstein, M.B. Wise, *Phys. Lett. B* 265 (1991) 326.
- [35] K. Agashe, R. Contino, A. Pomarol, *Nucl. Phys. B* 719 (2005) 165, arXiv:hep-ph/0412089.
- [36] R. Contino, L. Da Rold, A. Pomarol, *Phys. Rev. D* 75 (2007) 055014, arXiv:hep-ph/0612048.
- [37] R. Contino, arXiv:1005.4269 [hep-ph].
- [38] S.R. Coleman, J. Wess, B. Zumino, *Phys. Rev.* 177 (1969) 2239.
- [39] C.G. Callan Jr., S.R. Coleman, J. Wess, B. Zumino, *Phys. Rev.* 177 (1969) 2247.
- [40] R. Contino, D. Marzocca, D. Pappadopulo, R. Rattazzi, *J. High Energy Phys.* 1110 (2011) 081, arXiv:1109.1570 [hep-ph].
- [41] R. Alonso, I. Brivio, B. Gavela, L. Merlo, S. Rigolin, arXiv:1409.1589 [hep-ph].
- [42] M.V. Terentev, *Sov. J. Nucl. Phys.* 16 (1973) 87, *Yad. Fiz.* 16 (1972) 162.
- [43] J.F. Donoghue, B.R. Holstein, Y.C. Lin, *Phys. Rev. D* 37 (1988) 2423.
- [44] A. Falkowski, *Phys. Rev. D* 77 (2008) 055018, arXiv:0711.0828 [hep-ph].
- [45] M. Carena, L. Da Rold, E. Pontón, *J. High Energy Phys.* 1406 (2014) 159, arXiv:1402.2987 [hep-ph].