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A generalized recurrence for Bell polynomials: An alternate approach to Spivey and Gould–Quaintance formulas

Hacène Belbachir, Miloud Mihoubi

USTHB, Faculty of Mathematics, P.B. 32 El Alia, 16111, Algiers, Algeria

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ABSTRACT

Letting $B_n(x)$ the n-th Bell polynomial, it is well known that B_n admit specific integer coordinates in the two following bases $\left\{x^i\right\}_{i=0,\ldots,n}$ and $\{xB_i(x)\}_{i=0,\ldots,n-1}$ according, respectively, to Stirling numbers and binomial coefficients. Our aim is to prove that, for $r+s=n$, the sequence $\left\{x^jB_k(x)\right\}_{\substack{j=0,\ldots,r\\k=0,\ldots,s}}$ is a family of bases of the $\mathbb{Q}$-vectorial space formed by polynomials of $\mathbb{Q}[X]$ for which B_n admits a Binomial Recurrence Coefficient.

1. Introduction and main result

In [1,2] the first author and Bencherif established that Chebyshev polynomials of first and second kind – and more generally bivariate polynomials associated with recurrence sequences of order two, including Jacobsthal polynomials, Vieta polynomials, Morgan–Voyce polynomials and others – admit remarkable integer coordinates in a specific basis. What about Bell polynomials?

The Bell polynomials $\{B_n(x)\}_{n\geq 0}$ are defined by their generating function

$$\sum_{n\geq 0} B_n(x)\frac{t^n}{n!} = \exp\left(x\left(\mathrm{e}^t-1\right)\right).$$

They also satisfy

$$B_n(x) = \exp(-x)\sum_{i=0}^{\infty}\frac{i^n}{i!}x^i.$$

It is well known that B_n admits integer coordinates in the two following bases $\left\{x^i\right\}_{i=0,\ldots,n}$ and $\{xB_i(x)\}_{i=0,\ldots,n-1}$ as

E-mail addresses: hbelbachir@usthb.dz, hacenebelbachir@gmail.com (H. Belbachir), mmihoubi@usthb.dz, miloudmihoubi@hotmail.com (M. Mihoubi).

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$$B_n(x) = \sum_{i=0}^{n} S(n,i)\, x^i \quad \text{and} \quad B_n(x) = x \sum_{i=0}^{n-1} \binom{n-1}{i} B_i(x)$$

according, respectively, to the Stirling numbers of the second kind and the binomial coefficients.

Our aim is to prove that the sequence $\left\{x^j B_k(x)\right\}_{\substack{j=0,\ldots,r \\ k=0,\ldots,s}}$, with $s+r=n$, is a (anti-diagonal) family of bases of the $\mathbb{Q}$-vectorial space formed by polynomials of $\mathbb{Q}[X]$ for which B_n admits a Binomial Recurrence Coefficient. When $s=0$, we obtain the first expression, and when $r=1$, we obtain the second one.

Theorem 1. *Decomposition of B_n into $\left\{x^j B_k(x)\right\}_{j,k}$*

$$B_{s+r}(x) = \sum_{k=0}^{s} \sum_{j=0}^{r} j^{s-k} S(r,j) \binom{s}{k} x^j B_k(x). \tag{1}$$

As an immediate consequence (for $x=1$), we deduce the Spivey's generalized recurrence for Bell numbers — see [3]. He gives a very nice and attractive combinatorial proof.

Corollary 2. *Bell numbers satisfy the following recurrence relationship*

$$B_{s+r} = \sum_{k=0}^{s} \sum_{j=0}^{r} j^{s-k} S(r,j) \binom{s}{k} B_k.$$

After our paper was submitted we learned that H.W. Gould and Jocelyn Quaintance: Implications of Spivey's Bell Number Formula, J. Integer Sequences, 11 (2008), Article 08.3.7., obtained the same result as ours, independently from us. However, the proofs are different. Gould and Quaintance's proof is a generating function one. Our proof follows a different approach. It consists of establishing that Bell polynomials admit specific integer coordinates in a family of bases. Also, the motivations for the two papers are different and complementary.

2. Proof of the main result

From [4, p. 157], we have

$$\frac{\partial^r F(y)}{\partial x_1 \cdots \partial x_r} = \sum_{j=1}^{r} S(r,j)\, y^{j-1} F^{(j)}(y) \quad \text{with } y = x_1 \cdots x_r \ \ (r \geq 1).$$

Setting $F(y) := \exp(y) B_s(y) = \sum_{i=0}^{\infty} \frac{i^s}{i!} y^i$, with s being a nonnegative integer, we obtain

$$\begin{aligned} y \frac{\partial^r F(y)}{\partial x_1 \cdots \partial x_r} &= y \frac{\partial^r}{\partial x_1 \cdots \partial x_r} \left(\sum_{i=0}^{\infty} \frac{i^s}{i!} (x_1 \cdots x_r)^i \right) \\ &= \sum_{i=0}^{\infty} \frac{i^s}{i!} i^r (x_1 \cdots x_r)^i \\ &= \exp(y) B_{s+r}(y), \end{aligned}$$

also, we have

$$\begin{aligned} y \frac{\partial^r F(y)}{\partial x_1 \cdots \partial x_r} &= \sum_{j=0}^{r} S(r,j)\, y^j F^{(j)}(y) \\ &= \sum_{j=0}^{r} S(r,j)\, y^j \left(\sum_{i=0}^{\infty} \frac{(i+j)^s}{i!} y^i \right) \end{aligned}$$

$$= \sum_{k=0}^{s} \sum_{j=0}^{r} \binom{s}{k} S(r,j) j^{s-k} y^j \left(\sum_{i=0}^{\infty} \frac{i^k}{i!} y^i \right)$$
$$= \exp(y) \sum_{k=0}^{s} \sum_{j=0}^{r} \binom{s}{k} S(r,j) j^{s-k} y^j B_k(y).$$

Then, from the two expressions of $y \frac{\partial^r F(y)}{\partial x_1 \cdots \partial x_r}$ we conclude to

$$B_{s+r}(y) = \sum_{k=0}^{s} \sum_{j=0}^{r} \binom{s}{k} S(r,j) j^{s-k} y^j B_k(y), \quad s \geq 0 \text{ and } r \geq 1. \tag{2}$$

We can verify that the expression given by (1) is true for $r = 0$ and, for $r \geq 1$, we take $x_1 := x$ and $x_2 = \cdots = x_r = 1$ in relation (2).

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