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Maximal 2-rainbow domination number of a graph

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Abstract

A *2-rainbow dominating function* (2RDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. A *maximal 2-rainbow dominating function* on a graph G is a 2-rainbow dominating function f such that the set $\{w \in V(G) | f(w) = \emptyset\}$ is not a dominating set of G. The *weight* of a maximal 2RDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *maximal 2-rainbow domination number* of a graph G, denoted by $\gamma_{mr}(G)$, is the minimum weight of a maximal 2RDF of G. In this paper we initiate the study of maximal 2-rainbow domination number in graphs. We first show that the decision problem is NP-complete even when restricted to bipartite or chordal graphs, and then, we present some sharp bounds for $\gamma_{mr}(G)$. In addition, we determine the maximal rainbow domination number of some graphs.

Keywords: Maximal domination; Rainbow domination; Maximal rainbow domination

1. Introduction

For terminology and notation on graph theory not given here, the reader is referred to [1–3]. In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *k-regular* if $d(v) = k$ for each vertex v of G. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *tree* is an acyclic connected graph. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, P_n for a *path* of order n and C_n for a *cycle* of length n.

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A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D is said to be a *maximal dominating set* (MDS) if $V - D$ is not a dominating set of G. The *maximal domination number* $\gamma_m(G)$ is the minimum cardinality of a maximal dominating set of G. The definition of the maximal domination was given by Kulli and Janakiram [4]. For more information on maximal domination we refer the reader to [5,6].

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined in [7,8] as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. A Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. A *maximal Roman dominating function* (MRDF) on a graph G is a Roman dominating function $f = (V_0, V_1, V_2)$ such that V_0 is not a dominating set of G. The *maximal Roman domination number* of a graph G, denoted by $\gamma_{mR}(G)$, equals the minimum weight of an MRDF on G. A $\gamma_{mR}(G)$-*function* is a maximal Roman dominating function of G with weight $\gamma_{mR}(G)$. The maximal Roman domination was introduced by Ahangar et al. in [9] and has been studied in [10].

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The *k-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A $\gamma_{rk}(G)$-*function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [11] and has been studied by several authors [12–20].

A 2-rainbow dominating function $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ can be represented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ (or $(V_0^f, V_1^f, V_2^f, V_{1,2}^f)$ to refer f) of V, where $V_0 = \{v \in V \mid f(v) = \emptyset\}$, $V_1 = \{v \in V \mid f(v) = \{1\}\}$, $V_2 = \{v \in V \mid f(v) = \{2\}\}$, $V_{1,2} = \{v \in V \mid f(v) = \{1, 2\}\}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|$.

A *maximal 2-rainbow dominating function* (M2RDF) on a graph G is a 2-rainbow dominating function $f = (V_0, V_1, V_2, V_{1,2})$ such that V_0 is not a dominating set of G. The *maximal 2-rainbow domination number* of a graph G, denoted by $\gamma_{mr}(G)$, equals the minimum weight of an M2RDF on G. A $\gamma_{mr}(G)$-*function* is a maximal 2-rainbow dominating function of G with weight $\gamma_{mr}(G)$. As $f = (\emptyset, V(G), \emptyset, \emptyset)$ is a maximal 2-rainbow dominating function of G and since every maximal 2-rainbow dominating function is a 2-rainbow dominating function, we have

$$\gamma_{r2}(G) \le \gamma_{mr}(G) \le n. \tag{1}$$

Since $V_1 \cup V_2 \cup V_{1,2}$ is a maximal dominating set when $f = (V_0, V_1, V_2, V_{1,2})$ is an M2RDF, and since assigning $\{1, 2\}$ to the vertices of a maximal dominating set yields an M2RDF, we observe that

$$\gamma_m(G) \le \gamma_{mr}(G) \le 2\gamma_m(G). \tag{2}$$

We note that maximal 2-rainbow domination number differs significantly from 2-rainbow domination number. For example, for $n \ge 2$, $\gamma_{r2}(K_n) = 2$ and $\gamma_{mr}(K_n) = n$.

Our purpose in this paper is to initiate the study of maximal 2-rainbow domination number in graphs. We first show that the decision problem is NP-complete even when restricted to bipartite or chordal graphs, and then we study basic properties and bounds for the maximal 2-rainbow domination number of a graph. In addition, we determine the maximal 2-rainbow domination number of some classes of graphs.

We make use of the following results in this paper.

Proposition A (*[12]*)**.** *For $n \ge 2$, $\gamma_{r2}(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.*

Proposition B (*[12]*)**.** *For $n \ge 3$, $\gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$.*

Proposition C (*[9]*)**.** *Let G be a connected graph of order $n \ge 3$. Then $\gamma_m(G) = n - 1$ if and only if $G = P_4$ or $G = K_n - M$ where M is a nonempty matching.*

Proposition D ([9]). *Let G be a connected graph G of order $n \geq 2$. Then $\gamma_{mR}(G) = n$ if and only if $G = K_2, P_3, P_4, C_3, C_4, C_5$ or $G = K_n - M$, where M is a matching of G.*

Observation 1. *For $n \geq 1$, $\gamma_{mr}(K_n) = \gamma_{mr}(\overline{K_n}) = n$.*

Proof. Obviously, $\gamma_{mr}(\overline{K_n}) = n$. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{mr}(K_n)$-function. As every vertex of K_n dominates all vertices, we must have $V_0 = \emptyset$ and hence $\gamma_{mr}(K_n) = |V_1| + |V_2| + 2|V_{1,2}| \geq |V_1| + |V_2| + |V_{1,2}| = n$. By (1) we have $\gamma_{mr}(K_n) = n$. ■

Observation 2. *For $n \geq 4$ and any non-empty matching M of K_n, $\gamma_{mr}(K_n - M) = n - 1$.*

Proof. Let $G = K_n - M$. It follows from (2) and Proposition C that $\gamma_{mr}(G) \geq n - 1$. Let $uv \in M$ and let $w \in V(G) - \{u, v\}$. Then the function $f = (\{u\}, V(G) - \{u, w\}, \{w\}, \emptyset)$ is obviously a maximal rainbow dominating function of G of weight $n - 1$ and hence $\gamma_{mr}(G) = n - 1$. This completes the proof. ■

2. Complexity of maximal 2-rainbow domination problem

In this section we consider the following decision problem regarding the maximal 2-rainbow domination number of a graph.

MAXIMAL 2-RAINBOW DOMINATION PROBLEM (M2RD-PROBLEM):

INSTANCE: A graph G and a positive integer $k \leq |V(G)|$.

QUESTION: Is $\gamma_{mr}(G) \leq k$?

To prove that the decision problem for maximal 2-rainbow domination is NP-complete, we use a polynomial time reduction from 2-rainbow domination problem.

2-RAINBOW DOMINATION PROBLEM (2RD-PROBLEM):

INSTANCE: A graph G and a positive integer $k \leq |V(G)|$.

QUESTION: Is $\gamma_{r2}(G) \leq k$?

As shown in [12], the 2-rainbow domination problem remains NP-complete even when restricted to bipartite or chordal graphs.

In order to present our results we need to introduce some additional terminology and notation. Given a graph G of order n and a graph H with root vertex v, the rooted product graph $G \circ_v H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and identifying the vertex u_i of G with the vertex v in the ith copy of H for every $1 \leq i \leq n$ [21]. More formally, assuming that $V(G) = \{u_1, \ldots, u_n\}$ and that the root vertex of H is v, we define the rooted product graph $G \circ_v H = (V, E)$, where $V = V(G) \times V(H)$ and

$$E = \bigcup_{i=1}^{n} \{(u_i, b)(u_i, y) : \ by \in E(H)\} \cup \{(u_i, v)(u_j, v) : \ u_i u_j \in E(G)\}.$$

Fig. 1 shows an example of the rooted product of graphs.

Theorem 3. *M2RD-PROBLEM problem is NP-complete, even when restricted to bipartite or chordal graphs.*

Proof. Let G be a graph of order n. M2RD-PROBLEM is a member of NP, since for a given function $f = (V_0, V_1, V_2, V_{1,2})$ of G such that $\omega(f) \leq n$, we can check in polynomial time that f is a 2-rainbow dominating function of G and that V_0 does not dominate G.

Now, we consider a rooted product graph $G \circ_{v_1} H$, where G is a graph of order n with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and H is a graph with root v_1 constructed as follows. We begin with a cycle C_4 with set of vertices $V(C_4) = \{v_1, v_2, v_3, v_4\}$ and set of edges $E(C_4) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. To obtain the graph H, we add three vertices $\{x_1, x_2, x_3\}$, and edges v_3x_1, v_3x_2 and v_3x_3. Notice that $G \circ_{v_1} H$ can be done in polynomial time.

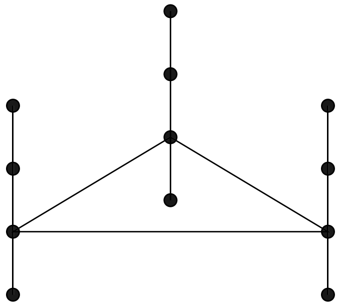

Fig. 1. Rooted product $C_3 \circ_v P_4$, where v has degree two in P_4.

Let g be a $\gamma_{r2}(G)$-function and consider the function $f = (V_0, V_1, V_2, V_{1,2})$ on $G \circ_{v_1} H$ such that:

- $f(u_i, v_1) = g(u_i)$ for $i \in \{1, 2, \ldots, n\}$;
- $f(u_i, v_2) = f(u_i, v_4) = \emptyset$ for $i \in \{1, 2, \ldots, n\}$;
- $f(u_i, v_3) = \{1, 2\}$ for $i \in \{1, 2, \ldots, n\}$;
- $f(u_1, x_1) = \{1\}$ and $f(u_i, x_1) = \emptyset$ for $i \in \{2, 3, \ldots, n\}$;
- $f(u_i, x_2) = f(u_i, x_3) = \emptyset$ for $i \in \{1, 2, \ldots, n\}$.

Clearly f is a maximal 2-rainbow dominating function of $G \circ_{v_1} H$, since (u_1, x_1) is not dominated by V_0. Thus $\gamma_{mr}(G \circ_{v_1} H) \leq 2n + 1 + \gamma_{2r}(G)$.

On the other hand, let f' be a $\gamma_{r2}(G \circ_{v_1} H)$-function. From the structure of $G \circ_{v_1} H$, for any $i \in \{1, \ldots, n\}$ we deduce that either $f'(u_i, v_3) = \{1, 2\}$ or we have three vertices (u_i, x_1), (u_i, x_2), and (u_i, x_3) to which f' does not assign $\emptyset$. Thus,

$$\left|V_1 \cap \bigcup_{i=1}^{n} \{(u_i, v_3), (u_i, x_1), (u_i, x_2), (u_i, x_3)\}\right| + \left|V_2 \cap \bigcup_{i=1}^{n} \{(u_i, v_3), (u_i, x_1), (u_i, x_2), (u_i, x_3)\}\right| + 2\left|V_{1,2} \cap \bigcup_{i=1}^{n} \{(u_i, v_3), (u_i, x_1), (u_i, x_2), (u_i, x_3)\}\right| \geq 2n.$$

Moreover, for all $i \in \{1, \ldots, n\}$ the vertex (u_i, v_1) has to be 2-rainbowly dominated. So, it follows that

$$\left|V_1 \cap \bigcup_{i=1}^{n} \{(u_i, v_1)\}\right| + \left|V_2 \cap \bigcup_{i=1}^{n} \{(u_i, v_1)\}\right| + 2\left|V_{1,2} \cap \bigcup_{i=1}^{n} \{(u_i, v_1)\}\right| \geq \gamma_{r2}(G).$$

Thus $\gamma_{r2}(G \circ_{v_1} H) \geq 2n + \gamma_{r2}(G)$. According to the structure of $G \circ_{v_1} H$, once more, it is straightforward to observe that every 2-rainbow dominated function $h = (V_0', V_1', V_2', V_{1,2}')$ of $G \circ_{v_1} H$, such that $\omega(h) = 2n + \gamma_{r2}(G)$, has the following form.

- $h(u_i, x_1) = h(u_i, x_2) = h(u_i, x_3) = \emptyset$ for $i \in \{1, 2, \ldots, n\}$;
- $h(u_i, v_3) = \{1, 2\}$ for $i \in \{1, 2, \ldots, n\}$;
- $h(u_i, v_2) = h(u_i, v_4) = \emptyset$ for $i \in \{1, 2, \ldots, n\}$;
- $h(u_i, v_1) = g'(u_i)$ for $i \in \{1, 2, \ldots, n\}$, where g' is any $\gamma_{r2}(G)$-function.

Hence, V_0' is a dominating set of $G \circ_{v_1} H$, and, as a consequence, $\gamma_{mr}(G \circ_{v_1} H) > \gamma_{r2}(G \circ_{v_1} H) = 2n + \gamma_{r2}(G)$. So, the equality $\gamma_{mr}(G \circ_{v_1} H) = 2n + \gamma_{r2}(G) + 1$ is obtained.

If G is a bipartite, then $G \circ_{v_1} H$ is a bipartite. If G is a chordal graph, then we construct a graph $G \circ_{v_1} H'$, where $V(G \circ_{v_1} H') = V(G \circ_{v_1} H)$ and $E(G \circ_{v_1} H') = E(G \circ_{v_1} H) \cup \bigcup_{i=1}^{n} (u_i, v_2)(u_i, v_4)$. Clearly $G \circ_{v_1} H'$ is chordal. By an analogous procedure, the equality $\gamma_{mr}(G \circ_{v_1} H) = 2n + \gamma_{r2}(G) + 1$ is derived. Therefore, for $j = 2n + 1 + k$, we infer that $\gamma_{r2}(G) \leq k$ if and only if $\gamma_{mr}(G) \leq j$, which completes the reduction of the M2RD-PROBLEM from the 2RD-PROBLEM. ■

3. Basic properties and bounds

In this section we study properties of maximal 2-rainbow domination and present some sharp bounds.

Proposition 4. *For any nonempty graph G of order $n \geq 4$, $\gamma_{mr}(G) \geq 3$ with equality if and only if $\Delta(G) = n - 1$ and $\delta(G) = 1$ or $\Delta(G) = n - 2$ and $\delta(G) = 0$ or there are two vertices v, w such that $N(v) \cap N(w)$ has a subset of size $n - 3$ which is not a dominating set of G.*

Proof. By (1), $\gamma_{mr}(G) \geq \gamma_{r2}(G) \geq 2$. If $\gamma_{mr}(G) = 2$ and $f = (V_0, V_1, V_2, V_{1,2})$ is a $\gamma_{mr}(G)$-function, then clearly either $|V_{1,2}| = 0$, $|V_1| = |V_2| = 1$ and $|V_0| = n - 2$ or $|V_{1,2}| = 1$, $|V_1| = |V_2| = 0$ and $|V_0| = n - 1$. It is easy to see that in each case, V_0 is a dominating set of G, a contradiction. Hence $\gamma_{mr}(G) \geq 3$.

If $\Delta(G) = n - 2$ and $\delta(G) = 0$, then let v be a vertex of degree $n - 2$ and suppose that u is an isolated vertex. Clearly, the function $f = (V(G) - \{u, v\}, \{u\}, \emptyset, \{v\})$ is an M2RDF of G and hence $\gamma_{mr}(G) = 3$. If $\Delta(G) = n - 1$ and $\delta(G) = 1$ then as above, we have $\gamma_{mr}(G) = 3$. Suppose now that there are two vertices v, w such that $N(v) \cap N(w)$ has a subset D of size $n - 3$ which is not a dominating set of G. If u is not dominated by D, then obviously $f = (D, \{u, v\}, \{w\}, \emptyset)$ is an M2RDF of G and hence $\gamma_{mr}(G) = 3$.

Conversely, let $\gamma_{mr}(G) = 3$. Assume that $f = (V_0, V_1, V_2, V_{1,2})$ is a $\gamma_{mr}(G)$-function. Then, we may assume, without loss of generality, that $|V_1| = |V_{1,2}| = 1$ or $|V_1| = 2$ and $|V_2| = 1$. First let $|V_1| = |V_{1,2}| = 1$. Let $V_1 = \{u\}$ and $V_{1,2} = \{v\}$. Since v must dominate all vertices in V_0, we have $\Delta(G) \geq \deg(v) \geq n - 2$. Since f is an M2RDF of G, u has no neighbor in V_0, otherwise V_0 dominates $V(G)$ which is a contradiction. If $uv \in E(G)$, then $\Delta(G) = n - 1$ and $\delta(G) = 1$, and if $uv \notin E(G)$, then $\Delta(G) = n - 2$ and $\delta(G) = 0$. Now let $|V_1| = 2$ and $|V_2| = 1$. Let $V_1 = \{u, v\}$ and $V_2 = \{w\}$. Clearly, each vertex in V_0 is adjacent to w. Since f is an M2RDF of G, we may assume u has no neighbor in V_0. It follows that each vertex in V_0 is adjacent to v. Thus, V_0 is a subset of $N(v) \cap N(w)$ of size $n - 3$ which does not dominate $V(G)$. This completes the proof. ■

Proposition 5. *For any graph G without isolated vertex,*

$$\gamma_{mr}(G) \leq \gamma_{r2}(G) + \delta(G).$$

Furthermore, this bound is sharp.

Proof. Let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{r2}(G)$-function and let v be a vertex of minimum degree. Then either $v \in V_1 \cup V_2 \cup V_{1,2}$ or $v \in V_0$. If $v \in V_0$, then v has a neighbor in $V_{1,2}$ or v has a neighbor in V_1 and a neighbor in V_2. It is clear that $g = (V_0 - N[v], V_1 \cup (N[v] - (V_2 \cup V_{1,2})), V_2, V_{1,2})$ is a maximal 2-rainbow dominating function on G and hence $\gamma_{mr}(G) \leq \gamma_{r2}(G) + \delta(G)$. If $v \in V_1 \cup V_2 \cup V_{1,2}$, then the function $g = (V_0 - N(v), V_1 \cup (N(v) \cap V_0), V_2, V_{1,2})$ is a maximal 2-rainbow dominating function on G and hence $\gamma_{mr}(G) \leq \gamma_{r2}(G) + \delta(G)$.

To prove the sharpness, let G be the graph obtained from K_n by adding a new vertex and joining it to exactly one vertex of K_n. Then $\gamma_{r2}(G) = 2$ and $\gamma_{mr}(G) = 3$ and the proof is complete. ■

Corollary 6. *For any tree T of order $n \geq 2$, $\gamma_{mr}(T) \leq \gamma_{r2}(T) + 1$.*

Next we present an upper bound for maximal 2-rainbow domination number of a graph in terms of its order and minimum degree.

Proposition 7. *Let G be a connected graph of order n with* $\operatorname{diam}(G) \geq 4$. *Then*

$$\gamma_{mr}(G) \leq n - \delta(G) + 1.$$

Proof. Consider a diametral path $P = x_1 x_2 \ldots x_{\operatorname{diam}(G)+1}$ in G. Then, the function $f = (N(x_2), \emptyset, V(G) - N[x_2], \{x_2\})$ is an M2RDF of G and hence $\gamma_{mr}(G) \leq \omega(f) = |V_1| + |V_2| + 2|V_{1,2}| = n - \deg(x_2) + 1$. Thus $\gamma_{mr}(G) \leq n - \delta(G) + 1$ and the proof is complete. ■

Proposition 8. *For any graph G,*

$$\gamma_{mr}(G) \leq 2\gamma_m(G) - 1.$$

Furthermore, this bound is sharp.

Proof. Let D be a $\gamma_m(G)$-set. Since D is an MDS, there is a vertex $u \in D$ not dominated by $V - D$. Define $f : V(G) \to \mathcal{P}(\{1, 2\})$ by $f(u) = \{1\}$, $f(x) = \{1, 2\}$ for $x \in D - \{u\}$ and $f(x) = \emptyset$ otherwise. It is easy to see that f is an M2RDF of G and hence $\gamma_{mr}(G) \leq 2(|D| - 1) + 1 = 2\gamma_m(G) - 1$.

To prove the sharpness, let G be the graph obtained from the complete K_n by adding a new vertex and joining it to exactly one vertex of K_n. ■

In (1) we observe that $\gamma_{r2}(G) \leq \gamma_{mr}(G) \leq n$. In the rest of this section we characterize all extremal graphs.

Lemma 9. *For a graph G, $\frac{2}{3}\gamma_{mR}(G) \leq \gamma_{mr}(G) \leq \gamma_{mR}(G)$.*

Proof. If $f = (V_0, V_1, V_2)$ is a $\gamma_{mR}(G)$-function, then obviously $(V_0, V_1, \emptyset, V_2)$ is a M2RDF of G and hence $\gamma_{mr}(G) \leq \gamma_{mR}(G)$.

To prove the lower bound, let f be a $\gamma_{mr}(G)$-function and let $X_i = \{v \in V(G) \mid i \in f(v)\}$ for $i = 1, 2$. We may assume that $|X_1| \leq |X_2|$. Then $|X_1| \leq (|X_1| + |X_2|)/2 = \gamma_{mr}(G)/2$. Define $g : V(G) \to \{0, 1, 2\}$ by $g(u) = 0$ if $f(u) = \emptyset$, $g(u) = 1$ when $f(u) = \{2\}$ and $g(u) = 2$ if $1 \in f(u)$. Obviously, g is a maximal Roman dominating function on G with $\omega(g) \leq 2|X_1| + |X_2| \leq \frac{3}{2}\gamma_{mr}(G)$ and the result follows. ■

Theorem 10. *Let G be a connected graph G of order $n \geq 2$. Then $\gamma_{mr}(G) = n$ if and only if $G = K_2, P_3, C_3$ or $G = K_n$.*

Proof. If $G = K_2, P_3, C_3$ or $G = K_n$, then clearly $\gamma_{mr}(G) = n$. Let $\gamma_{mr}(G) = n$. Then $\gamma_{mR}(G) = n$ by Lemma 9. It follows from Proposition D that $G = K_2, P_3, P_4, C_3, C_4, C_5$ or $G = K_n - M$, where M is a matching of G. Since $\gamma_{mr}(G) \leq n - 1$ for $G = P_4, C_4, C_5$ or $G = K_n - M$ where M is a nonempty matching of G, we deduce that $G = K_2, P_3, C_3$ or $G = K_n$ and the proof is complete. ■

Theorem 11. *Let G be a connected graph of order at least 3. Then $\gamma_{mr}(G) = \gamma_{r2}(G)$ if and only if G has a non-cut vertex u such that*

(a) $\gamma_{r2}(G - u) = \gamma_{r2}(G) - 1$,
(b) *$G - u$ has a $\gamma_{r2}(G - u)$-function f such that assigns 1 to all neighbors of u in G.*

Proof. If (a) and (b) hold, then we can extend $\gamma_{r2}(G - u)$-function f to a 2RDF of G by defining $f(u) = 1$. Clearly, f is an M2RDF of G and so $\gamma_{mr}(G) \leq \gamma_{r2}(G - u) + 1 = \gamma_{r2}(G)$. Thus $\gamma_{mr}(G) = \gamma_{r2}(G)$.

Conversely, let $\gamma_{mr}(G) = \gamma_{r2}(G)$. Assume $f = (V_0, V_1, V_2, V_{1,2})$ is a $\gamma_{mr}(G)$-function such that $|V_0|$ is maximum. Let V_A be the set of vertices which are not dominated by V_0. Since V_0 dominates $V_0 \cup V_{1,2}$, we have $V_A \subseteq V_1 \cup V_2$. If $V_A \cap V_2 \neq \emptyset$, then the function $g = (V_0, V_1 \cup (V_A \cap V_2), V_2 \setminus V_A, V_{1,2})$ is a $\gamma_{mr}(G)$-function such that $|V_0|$ is maximum and all vertices not dominated by V_0 belong to V_1. Thus we may assume, without loss of generality, that $V_A \subseteq V_1$. If some vertex $v \in V_A$, has a neighbor in $V_{1,2}$ or has a neighbor in V_1 and a neighbor in V_2, then $(V_0 \cup \{v\}, V_1 - \{v\}, V_2, V_{1,2})$ is a 2RDF of G of weight less than $\omega(f) = \gamma_{r2}(G)$ which is a contradiction. Hence, $N(V_A) \subset V_1 \cup V_2$ and $N(v) \subseteq V_1$ or $N(v) \subseteq V_2$ for each $v \in V_A$.

Claim 1. $G[V_A]$ is a complete graph.

Assume to the contrary that $uv \notin E(G)$ for some $u, v \in V_A$. Since G is connected and $N(u) \subseteq V_1$ or $N(u) \subseteq V_2$, we may assume that u has a neighbor w in V_1. Then $g = (V_0 \cup \{u\}, V_1 - \{u, w\}, V_2, V_{1,2} \cup \{w\})$ is a $\gamma_{mr}(G)$-function which contradicts the choice of f.

Claim 2. $|V_A| = 1$.

Let $|V_A| \geq 2$. If $|V_A| \geq 3$ then for each $u \in V_A$, the function $(V_0 \cup (V_A - \{u\}), V_1 - V_A, V_2, V_{1,2} \cup \{u\})$ is a 2RDF of G of weight less than $\omega(f) = \gamma_{r2}(G)$ which is a contradiction. Suppose $|V_A| = 2$ and $V_A = \{u, v\}$. Since G is connected of order at least 3, we may assume $\deg(u) \geq 2$. Since $N(u) \subseteq V_1$, the function $(V_0 \cup \{u\}, V_1 - \{u, v\}, V_2 \cup \{v\}, V_{1,2})$ is a 2RDF of G of weight less than $\omega(f) = \gamma_{r2}(G)$, a contradiction again.

Let $V_A = \{u\}$. We may assume $N(u) \subseteq V_1$. We claim that u is not a cut vertex. Suppose to the contrary that u is a cut vertex and $G_1, G_2, \ldots, G_k$ are the components of $G - u$. Obviously, $f|_{V(G_i)}$ if a 2RDF of G_i for each i. Define g by $g(u) = \emptyset$, $g(x) = \{1\}$ if $x \in V(G_1) \cap V_2$, $g(x) = \{2\}$ if $x \in V(G_1) \cap V_1$ and $g(x) = f(x)$ otherwise. It is easy to see that g is a 2RDF of G of weight less than $\omega(f) = \gamma_{r2}(G)$, a contradiction.

Thus u is a non-cut vertex. Obviously, the function f, restricted to $G-u$, is a 2RDF of G of weight $\gamma_{r2}(G)-1$ which assigns 1 to all neighbors of u in G. Hence $\gamma_{r2}(G-u) \le \gamma_{r2}(G)-1$. It remains to prove that $\gamma_{r2}(G-u) = \gamma_{r2}(G)-1$. Suppose to the contrary that $\gamma_{r2}(G-u) < \gamma_{r2}(G)-1$ and let h be a $\gamma_{r2}(G-u)$-function. Then we can extend h to a 2RDF of G by defining $h(u) = 1$ implying that $\gamma_{r2}(G) \le \gamma_{r2}(G-u)+1 < \gamma_{r2}(G)$ which is a contradiction. This completes the proof. ■

4. Special values of maximal 2-rainbow domination numbers

In this section we determine the exact value of maximal 2-rainbow domination number of some classes of graphs including paths, cycles and complete multipartite graphs.

Proposition 12. *For $m \ge n \ge 2$, $\gamma_{mr}(K_{m,n}) = n+1$ and $\gamma_{mr}(K_{m,1}) = 3$ for $m \ge 2$.*

Proof. Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_m\}$ be the bipartite sets of $K_{m,n}$. First let $n = 1$. It is easy to see that the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(x_1) = \{1,2\}$, $f(y_1) = \{1\}$ and $f(x) = \emptyset$ otherwise, is an M2RDF of weight 3 and hence $\gamma_{mr}(K_{m,1}) = 3$ by Proposition 4.

If $n = 2$, then clearly the function f defined by $f(x_1) = \{2\}$, $f(x_2) = f(y_1) = \{1\}$ and $f(x) = \emptyset$ otherwise, is an M2RDF of G of weight 3 and it follows from Proposition 4 that $\gamma_{mr}(K_{2,m}) = 3$.

Finally, let $n \ge 3$. First note that the function f defined by $f(x_1) = \{2\}$, $f(x_2) = \cdots = f(x_n) = f(y_1) = \{1\}$ and $f(x) = \emptyset$ otherwise, is an M2RDF of G of weight $n+1$ and hence $\gamma_{mr}(K_{m,n}) \le n+1$. Now let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{mr}(K_{m,n})$-function. If $V_0 \cap X \ne \emptyset$ and $V_0 \cap Y \ne \emptyset$, then clearly V_0 is a dominating set of $K_{m,n}$, a contradiction. Let $V_0 \cap X = \emptyset$. If $V_0 \cap Y = \emptyset$, then $\omega(f) \ge m+n > n+1$ which is a contradiction. Hence $V_0 \cap Y \ne \emptyset$ that implies f assigns 1 and 2 to some vertices in X. If $Y = V_0$, then V_0 is a dominating set, a contradiction. Thus $V_0 \subset Y$ implying that $\gamma_{mr}(K_{m,n}) = \omega(f) \ge |X|+1 = n+1$. Similarly, if $V_0 \cap Y = \emptyset$, then $\gamma_{mr}(K_{m,n}) \ge m+1$. In each case, $\gamma_{mr}(K_{m,n}) \ge n+1$ and hence $\gamma_{mr}(K_{m,n}) = n+1$. This completes the proof. ■

Proposition 13. *Let $G = K_{m_1,m_2,\dots,m_n}$ be the complete n-partite graph with $m_n \ge 2$ and $m_1 \le m_2 \le \cdots \le m_n$. Then $\gamma_{mr}(G) = 1 + \sum_{i=1}^{n-1} m_i$.*

Proof. Suppose $X_1, X_2, \dots, X_n$ are the partite sets of the complete n-partite graph G with $|X_i| = m_i$, and let $X_i = \{x_1^i, x_2^i, \dots, x_{m_i}^i\}$. It is easy to see that the function f defined by $f(x_1^n) = \{1\}$, $f(x_2^n) = \cdots = f(x_{m_n}^n) = \emptyset$ and $f(x) = \{2\}$ otherwise, is an M2RDF of G and so $\gamma_{mr}(G) \le 1 + \sum_{i=1}^{n-1} m_i$.

Now let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{mr}(G)$-function. If $V_0 \cap X_i \ne \emptyset$ and $V_0 \cap X_j \ne \emptyset$ for some $i \ne j$, then V_0 is a dominating set of G which is a contradiction. As in the proof of Proposition 12, one can verify that $\gamma_{mr}(G) \ge 1 + \sum_{i=1}^{n-1} m_i$ and hence $\gamma_{mr}(G) = 1 + \sum_{i=1}^{n-1} m_i$. ■

Proposition 14. *For $n \ge 2$, $\gamma_{mr}(P_n) = \lceil \frac{n+1}{2} \rceil$ if n is even and $\gamma_{mr}(P_n) = \lceil \frac{n+1}{2} \rceil + 1$ if n is odd.*

Proof. First let n is even. Then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(v_n) = \{1\}$, $f(v_{4i+1}) = \{1\}$ for $0 \le i \le \lceil \frac{n}{4} \rceil - 1$, $f(v_{4i+3}) = \{2\}$ for $0 \le i \le \lceil \frac{n-2}{4} \rceil - 1$ and $f(x) = \emptyset$ otherwise, is an M2RDF of P_n of weight $\lceil \frac{n+1}{2} \rceil$ and hence $\gamma_{mr}(P_n) \le \lceil \frac{n+1}{2} \rceil$. Since $\gamma_{mr}(P_n) \ge \gamma_{r2}(P_n)$, we deduce from Proposition A that $\gamma_{mr}(P_n) = \lceil \frac{n+1}{2} \rceil$.

Now let n be odd. Then the functions f and g defined by

$$f(v_{4i+1}) = \{1\} \quad \text{for } 0 \le i \le \left\lceil \frac{n}{4} \right\rceil - 1, \qquad f(v_{4i+3}) = \{2\} \quad \text{for } 0 \le i \le \left\lceil \frac{n-2}{4} \right\rceil - 1, \quad \text{and}$$
$$f(x) = \emptyset \text{ otherwise}$$

and

$$g(v_{4i+1}) = \{2\} \quad \text{for } 0 \le i \le \left\lceil \frac{n}{4} \right\rceil - 1, \qquad g(v_{4i+3}) = \{1\} \quad \text{for } 0 \le i \le \left\lceil \frac{n-2}{4} \right\rceil - 1, \quad \text{and}$$
$$g(x) = \emptyset \text{ otherwise}$$

are the unique $\gamma_{r2}(P_n)$-functions. Obviously, f and g are not M2RDF on P_n. Thus $\gamma_{mr}(P_n) \ge \gamma_{r2}(P_n) + 1$. On the other hand, the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(v_{n-1}) = 1$, $f(v_{4i+1}) = \{1\}$ for $0 \le i \le \lceil \frac{n}{4} \rceil - 1$,

$f(v_{4i+3}) = \{2\}$ for $0 \le i \le \lceil \frac{n-2}{4} \rceil - 1$, and $f(x) = \emptyset$ otherwise, is an M2RDF of weight $\lceil \frac{n+1}{2} \rceil + 1$ and hence $\gamma_{mr}(P_n) \le \lceil \frac{n+1}{2} \rceil + 1$. Thus $\gamma_{mr}(P_n) = \lceil \frac{n+1}{2} \rceil + 1$ for odd n and the proof is complete. ■

Proposition 15. *For $n \ge 3$, $\gamma_{mr}(C_n) = \gamma_{r2}(C_n)$ if $n \equiv 2 \pmod 4$ and $\gamma_{mr}(C_n) = \gamma_{r2}(C_n)+1$ if $n \equiv 0, 1, 3 \pmod 4$.*

Proof. Let $C_n = (v_1, v_2, \dots, v_n)$ be a cycle on n vertices. If $n \equiv 2 \pmod 4$, then the function $f : V(C_n) \to \mathcal{P}(\{1,2\})$ defined by $f(v_n) = 1$, $f(v_{4i+1}) = \{1\}$ for $0 \le i \le \lceil \frac{n}{4} \rceil - 1$, $f(v_{4i+3}) = \{2\}$ for $0 \le i \le \lceil \frac{n-2}{4} \rceil - 1$, and $f(x) = \emptyset$ otherwise, is obviously an M2RDF of C_n of weight $\gamma_{r2}(C_n)$ implying that $\gamma_{mr}(C_n) = \gamma_{r2}(C_n)$.

Now let $n \not\equiv 2 \pmod 4$. It is easy to see that $\gamma_{r2}(C_n - v_i) = \gamma_{r2}(P_{n-1}) = \lceil \frac{n}{2} \rceil = \gamma_{r2}(C_n)$ for each i. It follows from Theorem 11 and (1) that $\gamma_{mr}(C_n) \ge \gamma_{r2}(C_n) + 1$.

If $n \equiv 0 \pmod 4$, then define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(v_n) = \{1\}$, $f(v_{4i+1}) = \{1\}$ for $0 \le i \le \frac{n}{4} - 1$, $f(v_{4i+3}) = \{2\}$ for $0 \le i \le \frac{n}{4} - 1$ and $f(x) = \emptyset$ otherwise. Obviously, f is an M2RDF of G of weight $\gamma_{r2}(C_n) + 1$ which implies that $\gamma_{mr}(C_n) = \gamma_{r2}(C_n) + 1$.

If $n \equiv 1 \pmod 4$, then define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(v_2) = \{1\}$, $f(v_{4i+1}) = \{1\}$ for $0 \le i \le \lceil \frac{n}{4} \rceil - 1$, $f(v_{4i+3}) = \{2\}$ for $0 \le i \le \lfloor \frac{n}{4} \rfloor - 1$ and $f(x) = \emptyset$ otherwise. Clearly, f is an M2RDF of G of weight $\gamma_{r2}(C_n) + 1$ which implies that $\gamma_{mr}(C_n) = \gamma_{r2}(C_n) + 1$.

Let $n \equiv 3 \pmod 4$. Define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(v_2) = \{1\}$, $f(v_{4i+1}) = \{1\}$ for $0 \le i \le \lceil \frac{n}{4} \rceil - 1$, $f(v_{4i+3}) = \{2\}$ for $0 \le i \le \lceil \frac{n}{4} \rceil - 1$ and $f(x) = \emptyset$ otherwise. It is easy to see that f is an M2RDF of G of weight $\gamma_{r2}(C_n) + 1$ and so $\gamma_{mr}(C_n) = \gamma_{r2}(C_n) + 1$. ■

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