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Note

A note on the spectra of certain skew-symmetric $\{1,0,-1\}$-matrices

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Abstract

We characterize skew-symmetric $\{1,0,-1\}$-matrices with a certain combinatorial property. In particular, we exhibit several equivalent descriptions of this property. These results allow characterizations of unimodular orientations of the complete graph, of rank 2 chirotopes, and of a class of multipartite oriented graphs. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and prerequisites

Throughout this paper, let $A=(a_{xy})$ be a skew-symmetric $(n\times n)$-matrix (i.e., $A^{\mathrm{T}}=-A$) with entries in $\{1,0,-1\}$. We will be interested in particular in such matrices with the additional property

$$a_{wx}a_{yz}+a_{wy}a_{zx}+a_{wz}a_{xy}=a_{wx}a_{wy}a_{wz}a_{xy}a_{xz}a_{yz} \tag{1}$$

for all distinct $w,x,y,z\in E=\{1,\dots,n\}$. At first glance, this property may seem somewhat artificial. However, it will soon become clearer as an equation for a Pfaffian. The property has three equivalent formulations which will be summarized in Theorem 6, which is the main result of this paper.

The *Pfaffian* of A is defined recursively for words $x_1\dots x_{2m}$ of even length over the alphabet E by

$$A[\varepsilon]=1,$$

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where ε is the null word,

$$A[x_1x_2]=a_{x_1x_2}$$

and

$$A[x_1\dots x_{2m}]=\sum_{j=2}^{2m} A[x_1x_j]A[x_{j+1}\dots x_{2m}x_2\dots x_{j-1}] \tag{2}$$

for $m>0$.

Ref. [7] contains further information on Pfaffians and their applications, including a historical survey. Here we need just two basic properties:

$$A[\alpha]=0 \tag{3}$$

if α contains repeated letters and

$$A[x_{\sigma(1)}\dots x_{\sigma(2m)}]=(\operatorname{sign}\sigma)A[x_1\dots x_{2m}] \tag{4}$$

for every permutation of letters σ.

We will also need the square of the Pfaffian applied to sets of indices. If $S=\{x_1,\dots,x_n\}$, we mean by $A[S]^2$ the value $A[x_1\dots x_n]^2$. Because of (4), this is well-defined.

A well-known observation from determinant theory (see for example [8]) asserts that the determinant of a skew-symmetric matrix of even order equals the square of its Pfaffian, i.e.

$$\det A=A[E]^2. \tag{5}$$

In this paper, we call a skew-symmetric matrix *dense* if all but the diagonal entries are non-zero. We slightly rephrase Proposition 1 of [9]:

Lemma 1. *A dense skew-symmetric* $\{1,0,-1\}$*-matrix is regular if and only if* n *is even.*

Hence, for odd n, $\det A=0$ and for even n, $\det A\geqslant 1$, because the determinant is integral and a square. If we additionally define the Pfaffian of odd-length words to be zero, then (5) also holds in this case.

We repeat here the following remarkable result in [7] for matrices (in fact, it holds for arbitrary skew-symmetric maps):

Theorem 2. *If A is a skew-symmetric matrix, then for its Pfaffian the identity*

$$A[x_1\dots x_m]=\prod_{1\leqslant i<j\leqslant m} A[x_ix_j] \tag{6}$$

holds for all even m and all words $x_1\dots x_m$ over E if and only if it holds for all words $x_1x_2x_3x_4$ of length 4 over E.

Note that property (1) is just the case $m=4$ of this equation. In view of (3) this implies that, on the one hand, (1) is trivially satisfied if any of the w,x,y,z coincide and, on the other hand, if (1) holds for some w, x, y, z, then it also holds for all permutations thereof, because both sides are alternating. We will tacitly use this observation to simplify the subsequent proofs.

2. Equivalent formulations of (1)

Following oriented matroid terminology, we call an element $x \in E$ a loop if $a_{xy} = 0$ for all y. Two elements $x, y \in E$ are said to be parallel ($x \parallel y$) if neither is a loop and for some $q \in \{1, -1\}$ we have $a_{xz} = q a_{yz}$ for all z. Clearly, parallelity is an equivalence relation on the set of non-loops of A.

Lemma 3. *If A satisfies (1), x and y are no loops of A and $a_{xy} = 0$, then x and y are parallel. Moreover, for each set of indices $F \subseteq E$ not containing loops or parallel elements, $A|_F$ is dense.*

We omit the straightforward proof.

A is called a *principal unimodular* (*PU*) *matrix if the principal minors* $\det A|_F$ *are in* $\{1, 0, -1\}$ *for all* $F \subseteq E$, *where*

$$A|_F = (a_{xy})_{x,y \in F}.$$

PU matrices have received some attention in the literature, see for example [3,4,6]. It is clear by Theorem 2 that if A satisfies (1) then $A[\alpha] \in \{1, 0, -1\}$ for all words α. Hence we have

Corollary 4. *If A satisfies (1), then it is PU. More precisely, all principal minors are equal to* 0 *or* 1.

The converse of this corollary is false. For example, the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

is PU, but (1) fails to hold. However, the converse can be established using an additional condition:

Theorem 5. *For $n = 4$, assume that* (*a*) *for non-loops x and y, $a_{xy} = 0$ implies that $x \parallel y$, and* (*b*) $\det A \in \{0, 1\}$. *Then* (1) *holds for all permutations of* $1, 2, 3, 4$.

Proof. If there are loops or distinct parallel elements, then (1) is trivially satisfied. Otherwise the first assumption ensures that A is dense. Lemma 1 together with the second assumption then implies that $A[E]^2 = \det A = 1$. We abbreviate as follows:

$$e = a_{12}a_{34},$$

$$f = a_{13}a_{24},$$

$$g = a_{14}a_{23}.$$

Because A is dense, we have $e^2 = f^2 = g^2 = 1$. Moreover,

$$\begin{aligned} 1 = A[1234]^2 &= (e - f + g)^2 \\ &= e^2 + f^2 + g^2 - 2ef + 2eg - 2fg \end{aligned}$$

and hence

$$ef - eg + fg = 1.$$

Multiplication by efg yields

$$e - f + g = efg$$

which is (1) with our abbreviations applied. □

For $1 \leqslant i \leqslant k$, the ith elementary symmetric function of $b_1, \ldots, b_k$ is defined as

$$S_i(b_1, \ldots, b_k) = \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant k} b_{j_1} \ldots b_{j_i},$$

$$S_0(b_1, \ldots, b_k) = 1.$$

Let $B_1, \ldots, B_k$ denote the k distinct equivalence classes of parallelity (on the set of non-loops of A) and $b_i = |B_i|$. We define the polynomial

$$q_A(\lambda) = \lambda^{n-k}(-1)^n((\lambda + b_1)\cdots(\lambda + b_k) + (\lambda - b_1)\cdots(\lambda - b_k))/2.$$

If we write $q_A(\lambda) = q_k\lambda^{n-k} + q_{k-1}\lambda^{n-(k-1)} + \cdots + q_0\lambda^n$, we can compute for $0 \leqslant i \leqslant k$

$$(-1)^n q_i = \begin{cases} S_i(b_1, \ldots, b_k) & \text{if } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Our main result is the following

Theorem 6. *The following conditions are equivalent for a skew-symmetric* $\{1, 0, -1\}$-*matrix* A:

(i) *A satisfies* (1),
(ii) *the characteristic polynomial is*

$$\det(A - \lambda I) = q_A(\lambda), \tag{8}$$

(iii) *the eigenvalues are* $\sqrt{-1}$ *times the unique solutions of*

$$\sum_{j=1}^{k} \operatorname{arccot} \frac{x}{b_j} = \frac{2l-1}{2}\pi \tag{9}$$

for $l = 1, \ldots, k$, *plus zero if* A *has loops or distinct parallel elements.*

Proof. A general matrix-theoretic result (e.g. [5, p. 66]) asserts that for the characteristic polynomial of a matrix A,

$$p(\lambda) = \det(A - \lambda I) = p_n + p_{n-1}\lambda + \cdots + p_0\lambda^n,$$

we have

$$(-1)^{n-i} p_i = \sum_{F \subseteq E,\ |F| = i} \det A|_F \tag{10}$$

for $0 \leqslant i \leqslant n$. Note that for odd i all principal minors of A are zero and hence $p_i = 0$. Let E' denote the set of non-loops of A. If F contains a loop or parallel elements, then $\det A|_F = 0$. Thus, using (5), we get

$$(-1)^{n-i} p_i = \sum_{\substack{F \subseteq E',\ |F| = i \\ |F \cap B_j| \leqslant 1 \text{ for } 1 \leqslant j \leqslant k}} A[F]^2. \tag{11}$$

Note that this sum extends over exactly $S_i(b_1, \ldots, b_k)$ terms.

Assume now that A satisfies (1). Then, by Knuth's result (6) it follows that

$$A[F]^2 = \prod_{1 \leqslant l < m \leqslant i} A[x_l x_m]^2$$

if $F = \{x_1, \ldots, x_i\}$. Because of Lemma 3, $A|_F$ is dense and hence the right-hand side equal to 1. Therefore, $(-1)^n p_i = (-1)^{n-i} p_i = S_i(b_1, \ldots, b_k)$ for all even i, establishing (ii) $\Rightarrow$ (i).

For (i) $\Rightarrow$ (ii), we will apply Lemma 5 to all (4×4)-submatrices of A. First, we verify that $a_{xy} = 0$ implies that x and y are parallel. Because of (7) on the one hand, and (11) on the other hand, we have

$$S_2(b_1, \ldots, b_k) = \sum_{\substack{x, y \in E' \\ x \nparallel y}} a_{xy}^2.$$

Clearly the right-hand side has $S_2(b_1, \ldots, b_k)$ summands, each of which is at most one. Because the sum is $S_2(b_1, \ldots, b_k)$, each a_{xy}^2, where $x \nparallel y$, is 1. Therefore $a_{xy} = 0$ is indeed only possible if x and y are parallel.

Now, let $F \subseteq E'$ be an arbitrary 4-subset of the non-loops. If F contains parallel elements, then $\det A|_F = 0$. Otherwise, the above reasoning implies that $A|_F$ is dense and hence $\det A|_F \geqslant 1$ by Lemma 1. However, by (8) and (7) for $i=4$ we have

$$S_4(b_1,\ldots,b_k)=\sum_{\substack{F\subseteq E',\ |F|=4\\ |F\cap B_j|\leqslant 1\ \text{for}\ 1\leqslant j\leqslant k}} A[F]^2,$$

which has $S_4(b_1,\ldots,b_k)$ summands each of which must hence be in fact equal to 1.

Altogether we have $\det A|_F \in \{0,1\}$ and can now apply Lemma 5 to $A|_F$, yielding the result.

We omit the straightforward proof of (ii) $\Leftrightarrow$ (iii), which only amounts to finding the (purely imaginary) roots of q_A. (The characteristic polynomial determines the eigenvalues and vice versa.) □

In case of a dense matrix, the situation is even simpler, since we have $k=n$ and $b_1=\cdots=b_n=1$. Hence the characteristic polynomial is

$$\det(A-\lambda I)=(-1)^n((\lambda+1)^n+(\lambda-1)^n)/2 \tag{12}$$

and (9) assumes the simple form

$$\operatorname{arccot} x=\frac{2l-1}{2n}\,\pi.$$

We summarize in the following.

Corollary 7. *The following conditions are equivalent for a dense skew-symmetric* $\{1,0,-1\}$*-matrix*:

(i) *A satisfies* (1).
(ii) *A is PU.*
(iii) *All* 4×4 *principal minors of A are equal to* 1.
(iv) *The characteristic polynomial of A satisfies* (12).
(v) *The eigenvalues of A are*

$$i\cot\frac{2l-1}{2n}\,\pi,\quad l=1,\ldots,n.$$

Proof. (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) is just Theorem 6 specialized for dense matrices. (i) $\Rightarrow$ (ii) is Corollary 4. To show that (ii) implies (iv), note that if A is PU, then the principal minors of even order are equal to 1 because of Lemma 1. Together with (10), (iv) follows. (iii) is a special case of (ii) and (iii) $\Rightarrow$ (i) is Theorem 5. □

Concluding this section, we note that Corollary 7 can be used to show that the inverse of an *even-order* matrix satisfying (1) also satisfies the same condition. The proof of this fact can be found in [10].

3. Applications in graph and oriented matroid theory

Any skew-symmetric $\{1,0,-1\}$-matrix may be considered as the adjacency matrix of an orientation of a simple graph. Bouchet [3,4] studied the question which graphs can be equipped with a *unimodular orientation*, i.e. an orientation such that the adjacency matrix is PU. As a first application of our results, we note that Corollary 7 characterizes the unimodular orientations of the complete graph.

Chirotopes are one form of appearance of *Oriented Matriods* which were first extensively treated in the book [1]. Here we will only consider chirotopes of rank 2. Although being a fairly trivial structure, rank 2 chirotopes can also be, to a certain extent, regarded as fundamental, since the definition of chirotopes of arbitrary rank can be reduced to matroids and rank 2 chirotopes, cf. Theorem 3.6.2 in [1].

Theorem 2.12 of [2] implies that a non-zero skew-symmetric $\{1,0,-1\}$-matrix is a chirotope if and only if it satisfies (1). Thus all our equivalence results immediately translate into new characterizations of rank 2 chirotopes. In particular, Corollary 7 yields interesting characterizations of uniform chirotopes (which correspond to dense matrices).

On the other hand, all the results of this paper can also be derived by using the fact that all rank 2 chirotopes are *realizable*, i.e., there are vectors $v_1,\dots,v_n \in \mathbb{R}^2$, such that

$$a_{xy} = \operatorname{sign}\det(v_x, v_y).$$

Using a realization, it is easy to see that each rank 2 chirotope is equivalent, by renumbering and reorientation (meaning multiplication of a row and column of the same index by -1), to a matrix of the form

$$\begin{pmatrix} O_1 & & 1 \\ & \ddots & \\ -1 & & O_k \end{pmatrix}$$

with $(b_i \times b_i)$-blocks of zeros on the diagonal and entirely 1's above and —1's below. The results about principal minors and the spectrum are clearly invariant under renumbering and reorientation, so it suffices to prove them for matrices of this block form.

If one defines switching equivalence of directed graphs by switching directions of arcs on cuts and if, furthermore, one calls a k-partite graph *homogenously acyclic* if it is acyclic and all arcs between any two classes of the partition are directed the same way (then its adjacency matrix has the above form), then Theorem 6 can also be read as follows:

Theorem 8. *A complete k-partite oriented graph G is switching equivalent to the homogeneously acyclic complete k-partite digraph with class sizes $b_1,\dots,b_k$ if and only if G's characteristic polynomial has form* (8).

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