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On the global existence and finite time blow-up of shadow systems

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A B S T R A C T

Shadow systems are often used to approximate reaction–diffusion systems when one of the diffusion rates is large. In this paper, we study the global existence and blow-up phenomena for shadow systems. Our results show that even for these fundamental aspects, there are serious discrepancies between the dynamics of the reaction–diffusion systems and that of their corresponding shadow systems.

1. Introduction

Reaction–diffusion systems of the following form have been used extensively in modeling various phenomena in many branches of science

$$\begin{cases} u_t = d_1 \Delta u + f(u,v) & \text{in } \Omega \times (0,T), \\ \tau v_t = d_2 \Delta v + g(u,v) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the usual Laplace operator, Ω is a bounded smooth domain in $\mathbf{R}^n$ with unit outward normal vector ν on its boundary $\partial\Omega$; d_1, d_2 are two positive constants representing the diffusion rates of the two substances u, v respectively, $\tau > 0$ is related to the response rate of v

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versus the change in u, and f and g are two smooth functions generally referred to as the reaction terms.

Roughly speaking, mathematical progress on (1) is still limited to this date despite much effort in the past century.

When one of the diffusion rates, say, d_2 is very large, one attempt in understanding (1) is to let d_2 tend to ∞ and formally reduces (1) to the following "shadow system"

$$\begin{cases} u_t = d_1 \Delta u + f(u, \xi) & \text{in } \Omega \times (0, T), \\ \tau \xi_t = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega g(u, \xi)\, dx & \text{in } (0, T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ \xi(0) = \xi_0, \quad u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2}$$

where $|\Omega|$ is the measure of Ω and $\xi(t)$ is the formal limit of $v(x,t)$ as $d_2 \to \infty$. The idea, due to Keener [9], is to study (2) first, which seems "easier", as its second equation is now an ordinary differential equation (although nonlocal), and then derive the desired properties of (1) at least when d_2 is sufficiently large. Such an approach is often typical in activator–inhibitor models, and has been partially successful in various cases, notably for the steady state solutions of the well-known Gierer–Meinhardt system [5,13].

The stability properties of steady states of (2) have been analyzed by [11]. In fact, the linearized stability of steady states of (2) in a more general form, allowing x-dependence in the reaction terms f and g, has been studied fairly thoroughly in [11]. (See the references in [11] for related papers.) Various other properties, in particular, the existence of compact attractors, have also been studied. See [6] and the references therein.

The main purpose of this paper is to compare the dynamics of shadow systems (2) with their original reaction–diffusion systems (1). Our results indicate, however, that there are serious discrepancies between (1) and (2) even as the fundamental aspects, global existence and finite time blow-up, are concerned. Thus, one must proceed with great care while using the shadow system (2) to understand (1).

To illustrate our results, we again consider the Gierer–Meinhardt system

$$\begin{cases} u_t = d_1 \Delta u - u + \dfrac{u^p}{v^q} & \text{in } \Omega \times (0, T), \\ \tau v_t = d_2 \Delta v - v + \dfrac{u^r}{v^s} & \text{in } \Omega \times (0, T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \geqslant 0, \quad v(x, 0) = v_0(x) > 0 & \text{in } \Omega, \end{cases} \tag{3}$$

where the exponents p, q, r are positive and s is nonnegative, and satisfy

$$0 < \frac{p-1}{r} < \frac{q}{s+1}. \tag{4}$$

The system (3) was proposed in 1972 [5], based on Turing's idea of "diffusion-driven instability" in 1952 [20], to model the regeneration phenomena of *hydra*. This system has attracted a lot of attention since the publications of [16,17] in early 1990s in which spike-layer steady states were established. We refer the readers to the survey papers [13,14,21] for further details.

The dynamics of (3) is very complicated, and is far from being understood at this time. It is interesting to note that even the fundamental question of global existence of (3) remained largely open until recently, and to this date, it is still not completely resolved. First result in this direction was due to Rothe in 1984 [18], but only for a very special case $n = 3$, $p = 2$, $q = 1$, $r = 2$ and $s = 0$. In 1987, a result for a related system was obtained in [12]. The nearly optimal resolution for the global existence issue came in 2006 with an elementary and elegant proof by Jiang [8]. In [8], the global existence of (3) was established for the range $\frac{p-1}{r} < 1$ by an ingenious argument only involving Hölder inequality which was used to control the nonlinear term $\frac{u^\alpha}{v^\beta}$. This only leaves the critical case $\frac{p-1}{r} = 1$ still open, as it has been known already that in case $\frac{p-1}{r} > 1$ even for the corresponding ordinary differential equations, i.e. when u_0 and v_0 are suitable constants in (3), (3) blows up at finite time. (See [15] for a complete description of all solutions to the corresponding kinetic system of (3).)

However, the situation for the global existence of the corresponding shadow system

$$\begin{cases} u_t = d_1 \Delta u - u + \dfrac{u^p}{\xi^q} & \text{in } \Omega \times (0,T), \\ \tau \xi_t = -\xi + \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \frac{u^r}{\xi^s}\, dx & \text{in } (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ \xi(0) = \xi_0 > 0, \quad u(x,0) = u_0(x) \geqslant 0 & \text{in } \Omega, \end{cases} \tag{5}$$

is much less understood. Obviously, the finite time blow-up results obtained in [15] for the corresponding ordinary differential equations still apply to (5). Other than this, not much is known.

In this paper, we establish both global existence and finite time blow-up for (5). Our existence result reads as follows.

Theorem 1. *If $\frac{p-1}{r} < \frac{2}{n+2}$, then every solution of the shadow system* (5) *exists for all time $t > 0$.*

Compared to the global existence result of Jiang for (3) mentioned above, Theorem 1 seems quite modest, and it seems natural to ask what would be the *optimal* condition for the global existence of (5): Is it possible to improve the upper bound for $\frac{p-1}{r}$ from $\frac{2}{n+2}$ to 1 in Theorem 1?

While our result on finite time blow-up, Theorem 2 below, gives a negative answer to the question above, we still do not know the optimal condition to guarantee the global existence. Nonetheless, Theorem 2 below does demonstrate a serious gap between the shadow system (5) and the original Gierer–Meinhardt system (3).

Theorem 2. *Suppose that Ω is the unit ball $B_1(0)$, and that $p = r$, $\tau = s + 1 - q$ and $0 < \frac{p-1}{r} < \frac{q}{s+1} < 1$. If $\frac{p-1}{r} > \frac{2}{n}$, $n \geqslant 3$, then* (5) *always has finite time blow-up solutions for suitable choices of initial values u_0 and ξ_0.*

The range $\frac{2}{n} \geqslant \frac{p-1}{r} \geqslant \frac{2}{n+2}$ remains open. In proving Theorem 2, we first reduce the shadow system (5) to a *single nonlocal equation*. We then use the idea in [7] to construct a sequence of solutions with the blow-up times shrinking to 0.

Theorem 1 is established in Section 2, and Section 3 is devoted to the proof of Theorem 2. Some miscellaneous remarks are included in Section 4.

2. Shadow systems: Global existence

We will prove Theorem 1 in this section. Throughout this section we assume that $\frac{p-1}{r} < \frac{2}{n+2}$, and let $(0,T)$ be the maximal time interval for which the solution $(u(x,t), \xi(t))$ of (5) exists. Suppose that $T < \infty$, we shall derive a contradiction. For simplicity, we will assume $|\Omega| = 1$ and denote d_1 by d throughout this section.

Lemma 2.1. *$\xi(t) \geqslant \xi_0 e^{-t/\tau}$ for all $t > 0$.*

Proof. From the equation for ξ in (5)

$$\tau \xi^s \xi_t = -\xi^{s+1} + \frac{1}{|\Omega|} \int_\Omega u^r \, dx \geqslant -\xi^{s+1}$$

since $u > 0$ in $\Omega \times (0, T)$. Thus

$$\left(e^{\frac{s+1}{\tau} t} \xi^{s+1}\right)_t \geqslant 0$$

and it follows that

$$e^{\frac{s+1}{\tau} t} \xi^{s+1} \geqslant \xi_0^{s+1},$$

and our conclusion holds. □

Our next lemma is inspired by [12]. First, for $0 < t' \leqslant T$ set

$$C_{\ell,a}(t') \equiv \int_0^{t'} e^{-\ell(t'-t)} \left(\int_\Omega \frac{u^r(x,t)}{\xi^{s+1+a}(t)} \, dx \right) dt.$$

Lemma 2.2. *$C_{\ell,a}(T) < \infty$ for all $\ell > 0$, $a > 0$.*

Proof. For fixed $t' < T$, set

$$\zeta(t) = \frac{e^{-\ell(t'-t)}}{\xi^a(t)}, \quad 0 < t < t'.$$

Then

$$\tau \zeta_t(t) = (\tau \ell + a)\zeta - a \frac{e^{-\ell(t'-t)}}{\xi^{s+1+a}(t)} \int_\Omega u^r(x,t) \, dx.$$

Integrating t from 0 to t' we obtain, by Lemma 2.1,

$$\tau \zeta(t') - \tau \zeta(0) = (\tau \ell + a) \int_0^{t'} \zeta(t) \, dt - a C_{\ell,a}(t') \leqslant C - a C_{\ell,a}(t').$$

Thus

$$a C_{\ell,a}(t') \leqslant C + \tau \zeta(0) - \tau \zeta(t) \leqslant C + \tau \zeta(0),$$

and our conclusion follows by letting $t' \to T$. □

To derive a contradiction, by Lemma 2.1 and standard parabolic regularity estimates, *it suffices to prove that for ℓ large there exists a constant $C_\ell(T)$ such that*

$$\left\|u(\cdot,t)\right\|_{L^\ell(\Omega)} \leqslant C_\ell(T) < \infty \tag{6}$$

for all $0 < t < T$.

To prove (6), we set $w = u^{\ell/2}$ and compute

$$\begin{aligned}\frac{d}{dt}\int_\Omega w^2\,dx = \frac{d}{dt}\int_\Omega u^\ell\,dx &= \ell\int_\Omega u^{\ell-1}\left[d\Delta u - u + \frac{u^p}{\xi^q}\right]dx \\ &= -\frac{4d(\ell-1)}{\ell}\int_\Omega |\nabla w|^2\,dx - \ell\int_\Omega w^2\,dx + \ell\int_\Omega \frac{u^{p-1+\ell}}{\xi^q}\,dx. \end{aligned}\tag{7}$$

Write

$$\frac{u^{p-1+\ell}}{\xi^q} = \left(\frac{u^r}{\xi^{s_0}}\right)^{q/s_0} u^{p-1+\ell-\frac{qr}{s_0}},$$

where $s_0 > s+1$ is chosen such that

$$\frac{p-1}{r} < \frac{q}{s_0} < \frac{2}{n+2}.$$

This can be achieved by (4) and our assumption that $\frac{p-1}{r} < \frac{2}{n+2}$. Letting $a = s_0 - (s+1)$ and $\rho = \frac{q}{s_0}$, we have

$$\frac{u^{p-1+\ell}}{\xi^q} = \left(\frac{u^r}{\xi^{s+1+a}}\right)^\rho \left(w^2\right)^\theta,$$

where $\theta = \frac{1}{\ell}(p-1+\ell-\rho r) = 1 - \frac{1}{\ell}[\rho r - (p-1)]$. Note that $\theta < 1$, and $\theta > 0$ if ℓ is large. (In fact, $\theta \uparrow 1$ as $\ell \uparrow \infty$.) Since $\rho < \frac{2}{n+2} < 1$, Hölder inequality implies that

$$\int_\Omega \frac{u^{p-1+\ell}}{\xi^q}\,dx \leqslant \left(\int_\Omega \frac{u^r}{\xi^{s+1+a}}\,dx\right)^\rho \left(\int_\Omega w^{\frac{2\theta}{1-\rho}}\,dx\right)^{1-\rho}.$$

For convenience, we denote

$$g_a \equiv \frac{u^r}{\xi^{s+1+a}}$$

and (7) becomes

$$\frac{d}{dt}\|w\|^2_{L^2(\Omega)} \leqslant -\frac{4d(\ell-1)}{\ell}\|\nabla w\|^2_{L^2(\Omega)} - \ell\|w\|^2_{L^2(\Omega)} + \ell\|g_a\|^\rho_{L^1(\Omega)}\|w\|^{2\theta}_{L^{\frac{2\theta}{1-\rho}}(\Omega)}. \tag{8}$$

By Gagliardo–Nirenberg inequality (see e.g. [3])

$$\|w\|_{L^{\frac{2\theta}{1-\rho}}} \leqslant C\|w\|^\gamma_{W^{1,2}}\|w\|^{1-\gamma}_{L^2},$$

where $\gamma = n(\rho+\theta-1)/(2\theta) \in (0,1)$ for ℓ sufficiently large, we can control the last term in (8) as follows:

$$\ell\|g_a\|_{L^1}^{\rho}\|w\|_{L^{\frac{2\theta}{1-\rho}}}^{2\theta} \leqslant \ell C\|g_a\|_{L^1}^{\rho}\big[\|\nabla w\|_{L^2}^{\gamma}\|w\|_{L^2}^{1-\gamma}+\|w\|_{L^2}\big]^{2\theta}$$

$$\leqslant \big(\epsilon\|\nabla w\|_{L^2}^{2\gamma\theta}\big)^{\frac{1}{\gamma\theta}}(\gamma\theta)+\left[\frac{1}{\epsilon}\ell C\|g_a\|_{L^1}^{\rho}\|w\|_{L^2}^{2\theta(1-\gamma)}\right]^{\frac{1}{1-\gamma\theta}}(1-\gamma\theta)+\ell C\|g_a\|_{L^1}^{\rho}\|w\|_{L^2}^{2\theta}$$

by Young's inequality since $0<\gamma\theta<1$ for ℓ large. Choosing $\epsilon>0$ such that $\epsilon^{\frac{1}{\gamma\theta}}(\gamma\theta)=\frac{4d(\ell-1)}{\ell}$, we have

$$\frac{d}{dt}\|w\|_{L^2}^2 \leqslant -\ell\|w\|_{L^2}^2+C\big[\|g_a\|_{L^1}^{\rho}\|w\|_{L^2}^{2\theta(1-\gamma)}\big]^{\frac{1}{1-\gamma\theta}}+\ell C\|g_a\|_{L^1}^{\rho}\|w\|_{L^2}^{2\theta}. \tag{9}$$

To simplify the notation, we set $\eta\equiv\|w\|_{L^2(\Omega)}^2$, and (9) becomes

$$\frac{d}{dt}\eta \leqslant -\ell\eta+C\|g_a\|_{L^1}^{\frac{\rho}{1-\gamma\theta}}\eta^{\frac{(1-\gamma)\theta}{1-\gamma\theta}}+C\|g_a\|_{L^1}^{\rho}\eta^{\theta}, \tag{10}$$

where C denotes generic constants which also depend on ℓ. Notice that the exponents $\frac{(1-\gamma)\theta}{1-\gamma\theta}<1$, and $\frac{\rho}{1-\gamma\theta}<1$ since $\rho<\frac{2}{n+2}$.

For $0<t<t'<\tilde{t}<T$, integrating (10) gives

$$\int_0^{t'} e^{-\ell(t'-t)}\left(\frac{d}{dt}\eta+\ell\eta\right)dt \leqslant C\sup_{[0,\tilde{t}]}\eta^{\frac{(1-\gamma)\theta}{1-\gamma\theta}}\int_0^{t'} e^{-\ell(t'-t)}\|g_a\|_{L^1}^{\frac{\rho}{1-\gamma\theta}}\,dt+C\sup_{[0,\tilde{t}]}\eta^{\theta}\int_0^{t'} e^{-\ell(t'-t)}\|g_a\|_{L^1}^{\rho}\,dt$$

$$\leqslant Ce^{\ell T}\sup_{[0,\tilde{t}]}\eta^{\frac{(1-\gamma)\theta}{1-\gamma\theta}}\int_0^{T} e^{-\ell(T-t)}\big(\|g_a\|_{L^1}+1\big)\,dt$$

$$+Ce^{\ell T}\sup_{[0,\tilde{t}]}\eta^{\theta}\int_0^{T} e^{-\ell(T-t)}\big(\|g_a\|_{L^1}+1\big)\,dt$$

$$\leqslant C\Big(\sup_{[0,\tilde{t}]}\eta\Big)^{\frac{(1-\gamma)\theta}{1-\gamma\theta}}+C\Big(\sup_{[0,\tilde{t}]}\eta\Big)^{\theta} \tag{11}$$

by Lemma 2.2. The left-hand side of (11) equals

$$\int_0^{t'}\frac{d}{dt}\big[e^{-\ell(t'-t)}\eta\big]\,dt=\eta(t')-e^{-\ell t'}\eta(0).$$

Hence

$$\eta(t')\leqslant \eta(0)+C\Big(\sup_{[0,\tilde{t}]}\eta\Big)^{\frac{(1-\gamma)\theta}{1-\gamma\theta}}+C\Big(\sup_{[0,\tilde{t}]}\eta\Big)^{\theta}. \tag{12}$$

Since (12) holds for every $t'<\tilde{t}$, we have

$$\sup_{[0,\tilde{t}]}\eta\leqslant \eta(0)+C\Big(\sup_{[0,\tilde{t}]}\eta\Big)^{\frac{(1-\gamma)\theta}{1-\gamma\theta}}+C\Big(\sup_{[0,\tilde{t}]}\eta\Big)^{\theta}. \tag{13}$$

This implies that $\sup_{[0,\tilde{t}]} \eta$ is bounded, *independent of* $\tilde{t} < T$, as both the exponents θ and $\frac{(1-\gamma)\theta}{1-\gamma\theta}$ in (13) are less than 1. Therefore, $\sup_{[0,T)} \eta$ is bounded, and our assertion (6) is established. This completes the proof of Theorem 1.

3. Shadow systems: Blow-up

This section is devoted to the study of finite time blow-up behavior of the system (5). In particular, we will prove Theorem 2 here. Throughout this entire section, we will take Ω in (5) to be the unit ball $B_1(0)$.

First, we reduce (5) to a nonlocal single equation by considering special initial values for ξ.

Multiplying the equation for ξ in (5) by ξ^{s-q}, we have

$$\left(\xi^{s-q+1}\right)_t + \xi^{s-q+1} = \frac{1}{|B_1(0)|} \int\limits_{B_1(0)} \frac{u^p}{\xi^q}\, dx, \tag{14}$$

since $p = r$ and $\tau = s + 1 - q$. On the other hand, integrating the equation for u in (5), we obtain

$$\bar{u}_t + \bar{u} = \frac{1}{|B_1(0)|} \int\limits_{B_1(0)} \frac{u^p}{\xi^q}\, dx, \tag{15}$$

where $\bar{u}$ denotes the spatial average of u

$$\bar{u}(t) = \frac{1}{|B_1(0)|} \int\limits_{B_1(0)} u(x,t)\, dx.$$

From (14) and (15) we deduce that

$$\left(\bar{u} - \xi^{s-q+1}\right)_t + \left(\bar{u} - \xi^{s-q+1}\right) = 0,$$

i.e.

$$e^t\left(\bar{u} - \xi^{s-q+1}\right) = \bar{u}(0) - \xi^{s-q+1}(0) \quad \text{for all } t > 0.$$

Thus, if we choose the initial value $\xi_0^{s-q+1} = \bar{u}_0$, it follows that $\xi^{s-q+1}(t) \equiv \bar{u}(t)$ and the shadow system (5) reduces to

$$\begin{cases} u_t = d_1 \Delta u - u + \dfrac{u^p}{\bar{u}^{q'}} & \text{in } B_1(0) \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1(0) \times (0,T), \\ u(x,0) = u_0(x) \geqslant 0 & \text{in } B_1(0), \end{cases} \tag{16}$$

where $q' = q/(s+1-q)$. Theorem 2 now follows from the following result.

Proposition 3.1. *If* $p > \frac{n}{n-2}$ *and* $q' > p - 1$, *then* (16) *always has finite time blow-up solutions.*

The proof is lengthy. To begin with, we set $w = e^t u(x,t)$ and (16) becomes

$$\begin{cases} w_t = \Delta w + K(t) w^p & \text{in } B_1(0) \times (0,T), \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial B_1(0) \times (0,T), \\ w(x,0) = w_0(x) \geqslant 0 & \text{in } B_1(0), \end{cases} \tag{17}$$

where $K(t) = e^{(q'-p+1)t} \bar{w}^{-q'}$, and, for simplicity, we have taken $d_1 = 1$. Obviously, $\bar{w}(t)$ is increasing and

$$\bar{w}(t) \geqslant \bar{w}(0) = \bar{u}_0. \tag{18}$$

To choose the initial value u_0, we first consider

$$\varphi(r) = \begin{cases} r^{-\alpha}, & \delta \leqslant r \leqslant 1, \\ \delta^{-\alpha}(1 + \frac{\alpha}{2}) - \frac{\alpha}{2} \delta^{-(\alpha+2)} r^2, & 0 \leqslant r < \delta, \end{cases}$$

where $\alpha = \frac{2}{p-1}$ and $\delta > 0$ is small. Direct computations show that $\varphi \in C^1([0,1])$ and

$$\frac{1}{|B_1(0)|} \int\limits_{B_1(0)} \varphi^p \, dx = \frac{n}{n - \alpha p} + O\left(\delta^{n-\alpha p}\right), \tag{19}$$

$$\frac{1}{|B_1(0)|} \int\limits_{B_1(0)} \varphi \, dx = \frac{n}{n - \alpha} + O\left(\delta^{n-\alpha}\right). \tag{20}$$

Moreover,

$$\varphi_{rr} + \frac{n-1}{r} \varphi_r = \begin{cases} -\alpha(n-2-\alpha)\varphi^p, & \delta \leqslant r \leqslant 1, \\ -n\alpha\varphi^p(\delta), & 0 \leqslant r < \delta. \end{cases}$$

Thus

$$\varphi_{rr} + \frac{n-1}{r} \varphi_r + n\alpha\varphi^p \geqslant 0$$

holds for $r \in [0,1]$ *in the weak sense*. Now, setting $u_0 = \lambda\varphi$, we compute, by $q' > p-1$,

$$(u_0)_{rr} + \frac{n-1}{r}(u_0)_r + \frac{u_0^p}{(2\bar{u}_0)^{q'}} \geqslant \left[-\frac{n\alpha}{\lambda^{p-1}} + \frac{1}{\lambda^{q'} \{\frac{2n}{n-\alpha} + O(\delta^{n-\alpha})\}^{q'}} \right] u_0^p \geqslant 2u_0^p \tag{21}$$

for all δ small, say, $0 < \delta \leqslant \delta_0$, provided that λ is sufficiently small. For the rest of this section λ will be fixed (so that (21) holds for $0 < \delta \leqslant \delta_0$), and our strategy is to show that the maximal existence time interval for the solution $w(x,t;u_0)$ of (17) will shrink to 0 as δ goes to 0. We remark that the choice of φ and initial values are inspired by the work of B. Hu and H.M. Yin [7].

For $\delta < \delta_0$, let $(0, T_\delta)$, $T_\delta \leqslant \infty$, be the maximal (time) interval for the solution $w(x,t;u_0)$ to exist. We claim that *there exists a constant $C > 0$, independent of* $0 < \delta < \delta_0$, *such that*

$$T_\delta \leqslant C\delta^2. \tag{22}$$

As a preliminary step, we have the following estimate.

Lemma 3.2. *$w(r,t) \leqslant \frac{\bar{w}(t)}{r^n}$, for all $0 < r < 1$ and $0 < t < T_\delta$.*

Proof. Define the operator

$$L[\psi] \equiv \psi_t - \psi_{rr} + \frac{n-1}{r}\psi_r - pK(t)w^{p-1}\psi, \tag{23}$$

where $\psi = r^{n-1}w_r$. Straightforward calculations show that

$$\begin{cases} L\psi = 0 & \text{for } 0 < r < 1,\ 0 < t < T_\delta, \\ \psi = 0 & \text{for } r = 0, 1,\ \text{and } 0 < t < T_\delta. \end{cases}$$

Since $\psi(r,0) < 0$ for $0 < r < 1$, it follows from the maximum principle that $\psi < 0$ in $0 < r < 1$, $0 < t < T_\delta$. (The singularity in the term $\frac{n-1}{r}\psi_r$ does not cause any complication since $\psi = 0$ at $r = 0$.) In particular, we have $w_r \leqslant 0$ in $0 < r < 1$, $0 < t < T_\delta$. Hence

$$w(r,t)r^n = w(r,t)\int_0^r nz^{n-1}\,dz \leqslant \frac{1}{|B_1(0)|}\int_0^r w(z,t)n\omega_n z^{n-1}\,dz \leqslant \bar{w}(t),$$

where ω_n is the volume of the unit ball, and our conclusion follows. □

Next, observe that in $\frac{1}{2} < r < 1$, $0 < t < T_\delta$, $pK(t)w^{p-1}$ is uniformly bounded (for all $0 < \delta < \delta_0$) by (18), (20) and Lemma 3.2. Comparing ψ with the solution of

$$\begin{cases} \rho_t - \rho_{rr} + \frac{n-1}{r}\rho_r = 0 & \text{in } \frac{1}{2} < r < 1,\ 0 < t < T_\delta, \\ \rho = 0 & \text{at } r = \frac{1}{2}, 1,\ \text{and } 0 < t < T_\delta, \\ \rho(r,0) = \psi(r,0) & \text{in } \frac{1}{2} < r < 1, \end{cases}$$

we see that $\psi \leqslant \rho$ in $\frac{1}{2} < r < 1$ and $0 < t < T_\delta$. In particular,

$$w_r\left(\frac{3}{4}, t\right) \leqslant \left(\frac{4}{3}\right)^{n-1}\rho\left(\frac{3}{4}, t\right) \leqslant -C_0, \quad 0 < t < T_\delta, \tag{24}$$

where the constant C_0 is *independent of* $0 < \delta < \delta_0$.

The key ingredient in our proof is the following

Lemma 3.3. *There exists $0 < t_0 \leqslant 1$, independent of $0 < \delta < \delta_0$, such that for $0 < t < \min\{t_0, T_\delta\}$*

$$2C_1\bar{w}^\gamma \geqslant \frac{1}{|B_1(0)|}\int_{B_1(0)} w^p\,dx \geqslant \frac{1}{2}C_2\bar{w}^\gamma, \tag{25}$$

where

$$C_1 = \sup_{0<\delta<\delta_0} \frac{1}{\bar{u}_0^\gamma |B_1(0)|} \int_{B_1(0)} u_0^p \, dx < \infty,$$

$$C_2 = \inf_{0<\delta<\delta_0} \frac{1}{\bar{u}_0^\gamma |B_1(0)|} \int_{B_1(0)} u_0^p \, dx > 0,$$

and

$$\gamma = \frac{p(q'+1)}{k-1} \tag{26}$$

with $1 < k < p$ *chosen such that* $n > \frac{2p}{k-1}$.

Note that both C_1, C_2 are of the order

$$\lambda^{p-\gamma}\left(\frac{n}{n-\alpha p}\right)\left[\frac{1}{n/(n-\alpha)}\right]^\gamma + o(1) \tag{27}$$

for $\delta > 0$ small, by (19) and (20).

Proof of Lemma 3.3. To simplify our notation, set $\ell = q' + 1$. We define the following auxiliary function

$$\eta = r^{n-1} w_r + \epsilon r^n \frac{w^k}{\bar{w}^\ell},$$

where $\epsilon > 0$ will be chosen later. Direct computations yield that

$$\begin{aligned} L\eta &= L\left[\epsilon r^n \frac{w^k}{\bar{w}^\ell}\right] \\ &\leqslant -\epsilon \ell r^n K(t) \frac{w^k}{\bar{w}^{\ell+1}} \frac{1}{|B_1(0)|} \int_{B_1(0)} w^p \, dx - 2\epsilon k r^{n-1} \frac{w^{k-1}}{\bar{w}^\ell} w_r - \epsilon(p-k) K(t) r^n \frac{w^{p-1+k}}{\bar{w}^\ell} \\ &= -2\epsilon k \frac{w^{k-1}}{\bar{w}^\ell} \eta + \epsilon r^n \frac{w^k}{\bar{w}^{2\ell}}\left[2\epsilon k w^{k-1} - \ell \frac{e^{(q'-p+1)t}}{|B_1(0)|} \int_{B_1(0)} w^p - (p-k) e^{(q'-p+1)t} w^{p-1} \bar{w}\right]. \end{aligned} \tag{28}$$

We now begin to prove (25). For each $0 < \delta < \delta_0$, let $t_0(\delta)$ be the maximal time interval for which (25) holds. Clearly $T_\delta \geqslant t_0(\delta) > 0$, for each $0 < \delta < \delta_0$. If $t_0(\delta) \geqslant 1$, we take $t_0 = 1$ and the statement of Lemma 3.3 automatically holds. We now proceed with the case $t_0(\delta) \leqslant 1$.

In $0 < t < t_0(\delta)$, (25) implies that

$$L\eta \leqslant -2\epsilon k \frac{w^{k-1}}{\bar{w}^\ell} \eta + \epsilon r^n \frac{w^k}{\bar{w}^{2\ell}}\left[2\epsilon k w^{k-1} - \frac{C_2}{2} \ell \bar{w}^\gamma - (p-k) w^{p-1} \bar{w}\right].$$

By Young's inequality,

$$w^{k-1} \leqslant \frac{k-1}{p-1}\left(w^{k-1}\right)^{\frac{p-1}{k-1}} + \frac{p-k}{p-1} 1^{\frac{p-1}{p-k}} = \frac{k-1}{p-1} w^{p-1} + \frac{p-k}{p-1},$$

we see that in view of (18), there is an $\epsilon > 0$, *independent of* $0 < \delta < \delta_0$, such that

$$L\eta \leqslant -2\epsilon k \frac{w^{k-1}}{\bar{w}^{\ell}} \eta$$

for $0 < r < \frac{3}{4}$ and $0 < t < t_0(\delta)$. Observe that $\eta(0,t) = 0$. At $r = \frac{3}{4}$, from (18), (24) and Lemma 3.2 it follows that

$$\eta\left(\frac{3}{4}, t\right) \leqslant -C_0\left(\frac{3}{4}\right)^{n-1} + \epsilon\left(\frac{3}{4}\right)^{n-nk} \bar{w}(0)^{k-\ell} < 0$$

(since $k < p < \ell$) provided that ϵ is sufficiently small, *but still independent of* $0 < \delta < \delta_0$. Finally, for the initial value $t = 0$ and $0 < r < \delta$, we have

$$\begin{aligned}\eta(r,0) &= r^{n-1}\left[\lambda\varphi_r + \epsilon r \lambda^{k-\ell} \frac{\varphi^k}{\bar{\varphi}^{\ell}}\right] \\ &\leqslant r^{n-1}\left[-\frac{\lambda\alpha r}{\delta^{\alpha+2}} + \epsilon r \lambda^{k-\ell} \frac{(1+\frac{\alpha}{2})^k}{\delta^{\alpha k}} \frac{1}{\{\frac{n}{n-\alpha} + O(\delta^{n-\alpha})\}^{\ell}}\right] < 0\end{aligned}$$

(since λ is fixed and $\alpha + 2 = \alpha p > \alpha k$) if ϵ is sufficiently small (*but still independent of* $0 < \delta < \delta_0$). For $t = 0$, $\delta < r \leqslant \frac{3}{4}$, obviously

$$\eta(r,0) = r^{n-1}\left[-\frac{\lambda\alpha}{r^{\alpha+1}} + \epsilon r \lambda^{k-\ell} \frac{1}{r^{\alpha k}\{\frac{n}{n-\alpha} + O(\delta^{n-\alpha})\}^{\ell}}\right] < 0$$

(since $\alpha + 1 > \alpha k - 1$) if ϵ is sufficiently small (*but independent of* $0 < \delta < \delta_0$). Summing up, we have

$$\begin{cases} L\eta \leqslant -2\epsilon k \dfrac{w^{k-1}}{\bar{w}^{\ell}} \eta & \text{in } 0 < r < \dfrac{3}{4},\ 0 < t < t_0(\delta), \\ \eta \leqslant 0 & \text{at } r = 0, \dfrac{3}{4} \text{ and } 0 < t < t_0(\delta), \\ \eta \leqslant 0 & \text{in } 0 < r < \dfrac{3}{4} \text{ and } t = 0. \end{cases}$$

Again, the maximum principle implies that $\eta \leqslant 0$ in $0 \leqslant r \leqslant \frac{3}{4}$, $0 \leqslant t < t_0(\delta)$; i.e.

$$w_r \leqslant -\epsilon r \frac{w^k}{\bar{w}^{\ell}},$$

and we derive that, for $0 \leqslant r \leqslant \frac{3}{4}$, $0 \leqslant t < t_0(\delta)$, an "improved" estimate

$$w \leqslant \left[\frac{2\bar{w}^{\ell}}{\epsilon(k-1)}\right]^{\frac{1}{k-1}} r^{-\frac{2}{k-1}}. \tag{29}$$

(The idea of using auxiliary functions to obtain desired estimates in blow-up problems goes back to [4].) Now, for any $0 < R < \frac{3}{4}$, we have

$$\frac{1}{|B_1(0)|} \int\limits_{B_R(0)} w^p \, dx \leqslant n \left[\frac{2}{\epsilon(k-1)}\right]^{\frac{p}{k-1}} \frac{R^{n-\frac{2p}{k-1}}}{n - \frac{2p}{k-1}} \bar{w}^{\gamma}$$

for $0 \leqslant t < t_0(\delta)$. Thus we can choose R so small that

$$\frac{1}{|B_1(0)|} \int\limits_{B_R(0)} w^p \, dx \leqslant \frac{C_2}{8} \bar{w}^\gamma \quad \text{for } 0 \leqslant t < t_0(\delta). \tag{30}$$

On the other hand, from (17) and (25) we have

$$\bar{w}_t = e^{(q'-p+1)t} \bar{w}^{-q'} \frac{1}{|B_1(0)|} \int\limits_{B_1(0)} w^p \, dx \leqslant C_3 \bar{w}^{\gamma - q'},$$

where $C_3 = 2C_1 e^{q'-p+1}$, since $t \leqslant \min\{t_0(\delta), 1\}$. It follows that

$$\bar{w}(t) \leqslant \left[\bar{u}_0^{1+q'-\gamma} - C_3(\gamma - q' - 1)t\right]^{-\frac{1}{\gamma - q' - 1}}$$

since $\gamma - q' > 1$. It is easy to see that

$$\left[\bar{u}_0^{1+q'-\gamma} - C_3(\gamma - q' - 1)t\right]^{-\frac{1}{\gamma - q' - 1}} \leqslant 2\bar{u}_0$$

if

$$t \leqslant \min\left\{ \frac{1 - 2^{1+q'-\gamma}}{C_3(\gamma - q' - 1)} \bar{u}_0^{1+q'-\gamma}, 1 \right\}.$$

Therefore, we have

$$\bar{w}(t) \leqslant 2\bar{u}_0 \leqslant 2M \equiv 2 \sup_{0<\delta<\delta_0} \bar{u}_0 \tag{31}$$

if $t < \min\{t_0(\delta), t_1\}$, where

$$t_1 = \min\left\{ \frac{1 - 2^{1+q'-\gamma}}{C_3(\gamma - q' - 1)} M^{1+q'-\gamma}, 1 \right\}, \tag{32}$$

which is independent of $0 < \delta < \delta_0$.

Next, observe that the function $w/\bar{w}^\theta$, $\theta = \frac{\gamma}{p}$ $(= \frac{q'+1}{k-1} > 1)$, satisfies

$$\left(\frac{w}{\bar{w}^\theta}\right)_t = \Delta\left(\frac{w}{\bar{w}^\theta}\right) + e^{(q'+1-p)t}\left(\frac{w^p}{\bar{w}^{\theta+q'}} - \theta \frac{w}{\bar{w}} \frac{1}{|B_1(0)|} \int\limits_{B_1(0)} \frac{w^p}{\bar{w}^{\theta+q'}} \, dx\right),$$

and, in the domain $[B_1(0) \backslash B_R(0)] \times (0, \min\{t_0(\delta), t_1\})$, the terms

$$\frac{w}{\bar{w}^\theta}, \quad \frac{w^p}{\bar{w}^{\theta+q'}}, \quad \frac{w}{\bar{w}}, \quad \text{and} \quad \int\limits_{B_1(0)} \frac{w^p}{\bar{w}^{\theta+q'}} \, dx$$

are all uniformly bounded in view of the estimates (18), Lemma 3.2, (25), (31), and that $\gamma > \theta + q' > p$. Hence the standard parabolic regularity estimates (the DeGiorgi–Nash–Moser estimates, see e.g. [10]) imply that there is $t_2 > 0$, *independent of* $0 < \delta < \delta_0$, such that

$$\left|\frac{1}{|B_1(0)|}\int\limits_{B_1(0)\setminus B_R(0)}\frac{w^p}{\bar{w}^\gamma}\,dx-\frac{1}{|B_1(0)|}\int\limits_{B_1(0)\setminus B_R(0)}\frac{w_0^p}{\bar{w}_0^\gamma}\,dx\right|<\frac{C_2}{8}\tag{33}$$

for $0\leqslant t<\min\{t_0(\delta),t_1,t_2\}$.

Suppose that there exists $\tilde{\delta}\in(0,\delta_0)$ such that

$$t_0(\tilde{\delta})<\min\{t_1,t_2,T_{\tilde{\delta}}\}.$$

Then, from (30) and (33) we obtain that, for $0<t<t_0(\tilde{\delta})$,

$$\begin{aligned}
&\left|\frac{1}{|B_1(0)|}\int\limits_{B_1(0)}\frac{w^p}{\bar{w}^\gamma}\,dx-\frac{1}{|B_1(0)|}\int\limits_{B_1(0)}\frac{u_0^p}{\bar{u}_0^\gamma}\,dx\right|\\
&\quad\leqslant\left|\frac{1}{|B_1(0)|}\int\limits_{B_R(0)}\frac{w^p}{\bar{w}^\gamma}\,dx-\frac{1}{|B_1(0)|}\int\limits_{B_R(0)}\frac{u_0^p}{\bar{u}_0^\gamma}\,dx\right|\\
&\qquad+\left|\frac{1}{|B_1(0)|}\int\limits_{B_1(0)\setminus B_R(0)}\frac{w^p}{\bar{w}^\gamma}\,dx-\frac{1}{|B_1(0)|}\int\limits_{B_1(0)\setminus B_R(0)}\frac{u_0^p}{\bar{u}_0^\gamma}\,dx\right|\\
&\quad\leqslant\frac{C_2}{8}+\frac{C_2}{8}+\frac{C_2}{8}=\frac{3}{8}C_2,
\end{aligned}$$

i.e.

$$\frac{11}{8}C_1\geqslant\frac{1}{|B_1(0)|}\int\limits_{B_1(0)}\frac{w^p}{\bar{w}^\gamma}\,dx\geqslant\frac{5}{8}C_2.$$

Then, since $t_0(\tilde{\delta})<T_{\tilde{\delta}}$, we can extend $t_0(\tilde{\delta})$ further. This contradicts the maximality assumption of $t_0(\tilde{\delta})$, and therefore (25) holds for $0<t<\min\{t_0,T_\delta\}$, where $t_0=\min\{t_1,t_2\}$. □

We now continue the proof of Proposition 3.1. From (31) we have, for $0<t<\min\{t_0,T_\delta\}$, $0<r<1$,

$$w_t=\Delta w+e^{(q'-p+1)t}\frac{w^p}{\bar{w}^{q'}}\geqslant\Delta w+C_4w^p,$$

where $C_4=(2M)^{-q'}$. It then follows from the comparison principle that $w(x,t)\geqslant w_*(x,t)$ for $0<r<1$ and $0<t<\min\{t_0,T_\delta\}$, where w_* is the solution of

$$\begin{cases}
w_{*t}=\Delta w_*+C_4w_*^p & \text{in } B_1(0)\times\bigl(0,\min\{t_0,T_\delta\}\bigr),\\
\dfrac{\partial w_*}{\partial\nu}=0 & \text{on } \partial B_1(0)\times\bigl(0,\min\{t_0,T_\delta\}\bigr),\\
w_*(x,0)=w_0(x) & \text{in } B_1(0).
\end{cases}$$

Setting $\zeta=w_{*t}-w_*^p$, we have $\zeta>0$ at $t=0$ by (21), provided that δ_0 is sufficiently small. Moreover,

$$\zeta_t=\Delta\zeta+p(p-1)w_*^{p-2}|\nabla w_*|^2+C_4pw_*^{p-1}\zeta\geqslant\Delta\zeta+C_4pw_*^{p-1}\zeta$$

with $\frac{\partial\zeta}{\partial\nu} = 0$ on $\partial B_1(0) \times (0, \min\{T_\delta, t_0\})$. Again, the maximum principle implies that $\zeta > 0$ in $B_1(0) \times (0, \min\{T_\delta, t_0\})$; i.e.

$$w_{*t} \geqslant w_*^p$$

in $B_1(0) \times (0, \min\{t_0, T_\delta\})$. Straightforward integration gives

$$w_*(r,t) \geqslant \left[\frac{1}{w_0^{p-1}(r)} - (p-1)t\right]^{-\frac{1}{p-1}}.$$

In particular,

$$w_*(0,t) \geqslant \left\{\frac{1}{w_0^{p-1}(0)} - (p-1)t\right\}^{-\frac{1}{p-1}} = \left\{\frac{\delta^{\alpha(p-1)}}{[\lambda(1+\frac{\alpha}{2})]^{p-1}} - (p-1)t\right\}^{-\frac{1}{p-1}}$$

which clearly becomes ∞ at

$$t = \frac{1}{p-1}\left[\lambda\left(1+\frac{\alpha}{2}\right)\right]^{1-p}\delta^2.$$

Therefore $T_\delta \leqslant C\delta^2$ for δ small and our proof of (22) is complete.

Remark. From Lemma 3.2 and (31), it follows that the solution blows up only at the origin at $t = T_\delta$.

4. Miscellaneous remarks

Semilinear parabolic equations with nonlinearities involving nonlocal terms also naturally arise in other applications in science, see e.g. [1,2,19]. Local existence and uniqueness can be handled in a more or less standard fashion, see e.g. [19]. We also refer the readers to [19] and the references therein for a brief survey.

The nonlocal equation in Section 3

$$\begin{cases} u_t = d\Delta u - u + \dfrac{u^p}{\bar{u}^q} & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \Omega, \end{cases} \tag{34}$$

seems of independent interest. With different ranges of p and q, (34) exhibits various phenomena. When $q < p-1$, there are obviously finite time blow-up solutions even for $u_0 \equiv$ constant (which reduces (34) to a simple ordinary differential equation). Our results, Theorems 1 and 2, imply that in the case $q > p-1$, we have

(i) if $p < \frac{n+2}{n}$, then all solutions of (34) exist for all time $t > 0$;
(ii) if $p > \frac{n}{n-2}$, then there are finite time blow-up solutions.

The range $\frac{n+2}{n} \leqslant p \leqslant \frac{n}{n-2}$ remains open. We will return to the problem (34) in a future paper.

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