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# A note on tensor categories of Lie type $E_9$

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**Abstract**

We consider the problem of decomposing tensor powers of the fundamental level 1 highest weight representation $V$ of the affine Kac–Moody algebra $\mathfrak{g}(E_9)$. We describe an elementary algorithm for determining the decomposition of the submodule of $V^{\otimes n}$ whose irreducible direct summands have highest weights which are maximal with respect to the null-root. This decomposition is based on Littelmann's path algorithm and conforms with the uniform combinatorial behavior recently discovered by H. Wenzl for the series $E_N$, $N \neq 9$.


## 1. Introduction

While a description of the tensor product decompositions for irreducible highest weight modules over affine algebras can be found in the literature (see, e.g., [1]), effective algorithms for computing explicit tensor product multiplicities are scarce. Some partial results in this direction have been obtained by computing characters (see, e.g., [2]) and by employing crystal bases (see, e.g., [5]) or the equivalent technique of Littelmann paths. In this note we look at the particular case of the affine Kac–Moody algebra associated to the Dynkin diagram $E_9$, with any eye towards extending the results of [6].

Let $V$ be the irreducible highest weight representation of the $\mathfrak{g}(E_N)$, $N \geqslant 6$, with highest weight $\Lambda_1$ corresponding to the vertex in the Dynkin diagram furthest from the triple

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Table 1
Notation

| | |
|---|---|
| $\alpha_i$ | $i$th simple root |
| $Q$ | root lattice |
| $\Lambda_i$ | $i$th fundamental weight |
| $P$ | weight lattice |
| $P_+$ | dominant weights |
| $\widehat{P}_+$ | dominant weights (mod $\delta$) |
| $\widehat{P}_+(n)$ | level $n$ dominant weights (mod $\delta$) |
| $n(\lambda)$ | level of $\lambda$ |
| $P(\Lambda_1)$ | weights of $V$ |
| $W \cdot \Lambda_1$ | maximal weights of $V_{\Lambda_1}$ |
| $\Omega$ | set of straight weights |
| $P_+(V^{\otimes n})$ | dominant weights of $V^{\otimes n}$ |
| $[\lambda]_3$ | least residue of $k(\lambda)$ (mod 3) |
| $\pi_\lambda$ | path $t \to t\lambda$ |
| $W$ | (affine) Weyl group |
| $\mathcal{S}(k)$ | level $k$ initial weights |
| $\lambda \to \mu$ | straight weight path |

point. For $N \neq 9$, H. Wenzl [6] has found uniform combinatorial behavior for decomposing a certain submodule $V_{\mathrm{new}}^{\otimes n}$ of $V^{\otimes n}$ using Littelmann paths [4]. These submodules have the property that each irreducible summand of $V_{\mathrm{new}}^{\otimes n}$ appears in $V^{\otimes n}$ for the first time (for $N \leqslant 8$) or last time (for $N \geqslant 10$). The degeneracy of the invariant form was an obstacle to including the affine, $N = 9$ case.

We extend Wenzl's combinatorial description to the case $N = 9$ by finding submodules $\mathcal{M}_n$ analogous to his $V_{\mathrm{new}}^{\otimes n}$. Specifically, we look at the (full multiplicity) direct sum of those submodules of $V^{\otimes n}$ whose highest weights have maximal null-root coefficient. Not surprisingly, these summands appear *only* in $V^{\otimes n}$. The particular utility of considering this submodule is that whereas decomposing the full tensor power $V^{\otimes n}$ into its simple constituents would require an infinite path basis, only a finite sub-basis (consisting of 200 straight paths) is needed to determine the decomposition of $\mathcal{M}_n$. Although this note was inspired by the results of [6], the module $\mathcal{M}_n$ appears so naturally that this case may shed some light on the combinatorial behavior described by Wenzl.

This paper is organized in the following way. In Section 2 we give the data and standard definitions for the Kac–Moody algebra $\mathfrak{g}(E_9)$. Section 3 is dedicated to summarizing the general technique of Littelmann paths, while in Section 4 we apply this technique to the present case and present some new definitions. Table 1 gives a glossary of notation for the reader's convenience. All the lemmas we prove are contained in Section 5, and the main theorem and algorithm they lead to is described and illustrated in Section 6. We briefly mention a possible application and a generalization in Section 7, as well as connections to Wenzl's results.

## 2. Notation and definitions

We begin by fixing a realization of the generalized Cartan matrix of $\mathfrak{g}(E_9)$ sometimes denoted in the literature by $\mathfrak{g}(E_8^{(1)})$. Observe that our realization is different than that of

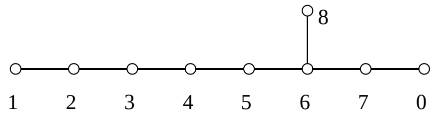

Fig. 1. Dynkin diagram of $E_9$.

Kac [3]. In particular, the vertex Kac labels with a 0 we label with a 1 in our Dynkin diagram (Fig. 1). This is done to conform with the notation of [6].

**Definition 2.1.** Let $\{\varepsilon_0, \delta, \varepsilon_1, \ldots, \varepsilon_8\}$ be an ordered basis for $\mathbb{R}^{10}$, with symmetric bilinear form $\langle\,,\,\rangle$ such that $\langle \varepsilon_i, \varepsilon_j\rangle = \delta_{ij}$ for $1 \leqslant i, j \leqslant 8$ and $\langle \delta, \varepsilon_0\rangle = 1$ with all other pairings 0. The simple roots of $\mathfrak{g}(E_9)$ are defined by

$$\alpha_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1}, & \text{if } 1 \leqslant i \leqslant 7, \\ \varepsilon_7 + \varepsilon_8, & \text{if } i = 8, \\ \frac{1}{2}\left(\delta + \varepsilon_8 - \sum_{i=1}^{7} \varepsilon_i\right), & \text{if } i = 0. \end{cases}$$

The simple roots generate the root lattice $Q := \mathrm{span}_{\mathbb{Z}}\{\alpha_i\}$, and we define coroots

$$\check{\alpha}_i := \frac{2}{\langle \alpha_i, \alpha_i\rangle}\alpha_i.$$

As $\mathfrak{g}(E_9)$ is simply-laced, we abuse notation and identify each coroot with the corresponding root. Since we are only concerned with the combinatorics, we refer the reader to Chapter 6 of the book [3] for the full description of the Kac–Moody algebra $\mathfrak{g}(E_9)$.

**Definition 2.2.** We define the fundamental weights by

$$\Lambda_i = \begin{cases} (i, 0; 1, \ldots, 1, 0, \ldots, 0)\ i \text{ ones}, & \text{if } 1 \leqslant i \leqslant 6, \\ \frac{1}{2}(8, 0; 1, \ldots, 1, -1), & \text{if } i = 7, \\ \frac{1}{2}(6, 0; 1, \ldots, 1), & \text{if } i = 8, \\ (2, 0; 0, \ldots, 0). & \text{if } i = 0. \end{cases}$$

Note that $\langle \alpha_i, \Lambda_j\rangle = \delta_{ij}$, and the fundamental weights of $\mathfrak{g}(E_9)$ are determined up to a multiple of $\delta$ by this relation. The set of dominant weights $P_+$ is the $\mathbb{N}$-span of the fundamental weights plus $\mathbb{C}\delta$, and the $\mathbb{Z}$-span $P$ is called the weight lattice. It will be useful to denote by $\widehat{P}_+$ those dominant weights whose second coordinate is 0.

**Definition 2.3.** We define the Weyl group in the usual way: $W = \langle s_i\colon\ i = 0, \ldots, 8\rangle$ where $s_i(\nu) = \nu - \langle \nu, \alpha_i\rangle\alpha_i$ for $\nu \in \mathbb{R}^{10}$.

The simple reflections $\{s_i\}_{i=1}^8$ generate a finite subgroup $\overline{W}$ acting on the last eight coordinates by permutations and an even number of sign changes. For any $\lambda$ with $\langle\lambda, \alpha_0\rangle > 0$ the vector $s_0(\lambda)$ has a strictly smaller $\delta$-coefficient, and by applying elements of $\overline{W}$ to arrange $\langle\lambda, \alpha_0\rangle > 0$ one can construct an infinite sequence of vectors with strictly decreasing $\delta$-coefficients. Thus one sees that $W$ is an infinite group.

## 3. Littelmann paths

To decompose the tensor powers of $V$ we use the Littelmann path formalism (see [4]). For this section we consider general Kac–Moody algebras $\mathfrak{g}$.

Littelmann considers the space of piecewise linear paths $\pi : [0, 1] \to P_{\mathbb{Q}}$ beginning at 0 and ending at some point in the weight lattice $P$. He defines *root operators* on the space of all such paths $e_i$ and $f_i$ for each simple root $\alpha_i$, which, when applied repeatedly to the straight path $\pi_\lambda$ from 0 to a dominant weight $\lambda$ give a *path basis* $\mathcal{B}_\lambda$ for the corresponding irreducible highest weight module $V_\lambda$. The operators $f_i$ are defined on paths $\pi$ as follows (see [4] for full details): let $h_i(t) = \langle\pi(t), \alpha_i\rangle$, and $m_i = \min(h_i(t))$. If $h_i(1) - m_i \geqslant 1$, split the interval $[0, 1]$ into three pieces: $[0, t_0] \cup [t_0, t_1] \cup [t_1, 1]$ where $t_0$ is the maximal $t$ such that $h_i(t) = m_i$ and $t_1$ is the minimal $t$ such that $h_i(t) = m_i + 1$. Then

$$f_i\pi = \begin{cases} \pi(t) & \text{on } [0, t_0], \\ s_i\big(\pi(t)\big) & \text{on } [t_0, t_1], \\ \pi(t) - \alpha_i & \text{on } [t_1, 1]. \end{cases}$$

If $h_i(1) - m_i < 1$ then $f_i\pi = 0$. The operators $e_i$ are defined similarly. Since all paths begin at the weight 0, we may concatenate paths in the usual way. For any $\lambda \in P$ define the path $\pi_\lambda : [0, 1] \to P_{\mathbb{Q}}$ by $t \to t\lambda$. We denote concatenation by $*$, i.e., $\pi_\lambda * \pi_\mu$ passes through $\lambda$ and terminates at $\lambda + \mu$.

Let $\lambda$ and $\mu$ be dominant weights of a Kac–Moody algebra and $V_\lambda$, $V_\mu$ the corresponding irreducible highest weight modules. We collect together those of Littelmann's results that we will need in:

**Proposition 3.1.**

(a) $\mathcal{B}_\lambda \subset \{f_{1_j} f_{2_j} \cdots f_{s_j} \pi_\lambda\}$, *that is, every path in the basis* $\mathcal{B}_\lambda$ *is obtained from* $\pi_\lambda$ *by applying a finite sequence of the root operators* $f_i$.

(b) *The decomposition rules for the tensor product given as follows*:

$$V_\mu \otimes V_\lambda \cong \bigoplus_\pi V_{\pi(1)},$$

*where* $\pi = \pi_\mu * \pi_i$ *with* $\pi_i \in \mathcal{B}_\lambda$ *and the image of* $\pi$ *contained in the closure of the dominant Weyl chamber.*

(c) *The multiplicity of* $V_\nu$ *in* $V_\lambda^{\otimes n}$ *is equal to the number of paths whose image is contained in the closure of the dominant Weyl chamber that terminate at* $\nu$ *and are obtained by concatenating* $n$ *basis paths.*

## 4. Lie type $E_9$

Now we consider the set $P(\Lambda_1)$ of weights of $V$. Following Kac [3], we call the weights in the Weyl group orbit $W \cdot \Lambda_1$ *maximal* and note that

$$P(\Lambda_1) = \bigcup_{\omega \in W \cdot \Lambda_1} \{\omega - t\delta\colon\ t \in \mathbb{N}\}.$$

Any $V_{\lambda - s\delta}$ that appears in some $V^{\otimes n}$ must be of the form

$$\lambda - s\delta = \sum_{\omega_i \in P(\Lambda_1)} \omega_i$$

with $s \in \frac{1}{2}\mathbb{N}$. It is well known that the maximal weights appear in the multiset $P(\Lambda_1)$ with multiplicity one (for example, see [1]).

The second coordinate (essentially determined by the number of times $s_0$ occurs in a minimal expression) provides a gradation on $W \cdot \Lambda_1$ which motivates the following lemma, the proof of which is a computation.

**Lemma 4.1.** *Every $\omega \in W \cdot \Lambda_1$ is of one of the following* 4 *forms*:

- (I) *Type* I: $(1, 0; \pm\varepsilon_i)$.
- (II) *Type* II: $\frac{1}{2}(2, -1; \pm 1, \ldots, \pm 1)$ *with an even number of minuses among the last eight coordinates.*
- (III) *Type* III: $(1, -1; w(1, 1, 1, 0, \ldots, 0))$ *where $w \in S_8$, the group of permutations on* 8 *symbols.*
- (IV) *Type* IV: *all others, i.e.,* $(1, -j; \nu)$ *where $j \geqslant 1$ and if $j = 1$, $\nu \notin S_8\{(1, 1, 1, 0, \ldots, 0)\}$.*

The weights of types I–III will be particularly useful and we will call them *straight weights* and denote the set of straight weights by $\Omega$. It is a simple but tedious computation to show that for any $\omega \in \Omega$, the straight line path $\pi_\omega$ from 0 to $\omega$ is in the path basis $\mathcal{B}_{\Lambda_1}$. The idea of the computation is to start with the path $\pi_{\Lambda_1}$ and inductively apply only those operators $f_i$ for which the height function $h_i(t) = t$ so that two of the three intervals in the definition of the operator $f_i$ are degenerate, and the image of the paths remain straight lines. Types I and II weights are in fact *all* maximal weights with second coordinate 0 or $-\frac{1}{2}$, while there are maximal weights with second coordinate $-1$ besides those of type III. Observe that since $\Lambda_1$ is the unique level one dominant weight (modulo $\delta$), all paths obtained from concatenation of basis paths whose image lies in the dominant Weyl chamber must pass through $\Lambda_1$.

**Definition 4.2.** The *level* $n(\lambda)$ of a weight $\lambda$ is the $\varepsilon_0$ coordinate. Note that all weights in $P(\Lambda_1)$ are level 1, thus $\lambda$ has level $n$ iff $\lambda - t\delta$ appears in $V^{\otimes n}$ for some $t$ since $\lambda$ will be a sum of weights in $P(\Lambda_1)$. Denote by $\widehat{P}_+(n)$ the set of level $n$ dominant weights modulo $\delta$.

The following definition appears in [6] and is critical in the sequel.

**Definition 4.3.** We define the function $k : Q \to \mathbb{Z}$ in one of the following equivalent ways:

(a) $k(\omega) = -\langle \omega, 2\hat{\alpha}_0 \rangle$, where $\hat{\alpha}_0 = \alpha_0 - \varepsilon_8$.
(b) If $\omega = \sum_{i=0}^{8} M_i \Lambda_i$, then $k(\omega) = M_8 - M_7 - 2M_0$.

We will also need the quantity $[(\lambda)]_3$ defined to be the remainder of $k(\lambda)$ upon division by 3.

We compute these values for the maximal weights and record them in the following lemma.

**Lemma 4.4.** *The values of the function $k$ for the maximal weights $\omega$ of types* I–III *and* IV (*as in Lemma* 4.1) *satisfy*:

(I) *Type* I: $k(\omega) \in \{0, -2\}$.
(II) *Type* II: $k(\omega) \in \{3, 1, -1, -3, -5\}$.
(III) *Type* III: $k(\omega) = 2$.
(IV) *Type* IV: $k(\omega) \leqslant (6j - 6)$ *where* $\omega = (1, -j; \nu)$ *and* $1 \leqslant j \in \frac{1}{2}\mathbb{Z}$.

The dominant weights are only defined up to a multiple of $\delta$, but we are interested in those that appear in $P_+(V^{\otimes n})$ which motivates:

**Definition 4.5.** A level $n$ dominant weight $\lambda - m_\lambda \delta$ is called *initial* if $m_\lambda$ is minimal such that $V_{\lambda - m_\lambda \delta}$ appears in $V^{\otimes n}$.

**Remark 4.6.** It is easy to see that there are finitely many initial weights of a fixed level $n$, since there is a one-to-one correspondence between the finite set $\widehat{P}_+(n)$ and initial weights. The term *initial* comes from the fact that if $\lambda - m_\lambda \delta \in \widehat{P}_+(n)$ so is $\lambda - (m_\lambda + 1)\delta$. Moreover, it is clear that $m_\lambda$ is always a non-negative half-integer, since the coefficient of $\delta$ for any weight $\omega \in P(\Lambda_1)$ is a non-positive half integer.

We will eventually show that the $m_\lambda$ is computed from the value of $k(\lambda)$ via the function:

**Definition 4.7.** Let $\lambda \in \widehat{P}_+$:

$$\Delta(\lambda) = \begin{cases} 0, & \text{if } k(\lambda) \leqslant 0 \text{ and even}, \\ \dfrac{1}{2}, & \text{if } k(\lambda) \leqslant 1 \text{ and odd}, \\ \dfrac{1}{6}\big(k(\lambda) + 2[\lambda]_3\big), & \text{if } k(\lambda) \geqslant 1. \end{cases} \tag{4.1}$$

Observe that when $k(\lambda) = 1$ we have $\frac{1}{6}(k(\lambda) + 2[\lambda]_3) = \frac{1}{2}$ so $\Delta$ is well-defined.

**Definition 4.8.** Define $\mathcal{M}_n$ to be the largest submodule of $V^{\otimes n}$ such that all irreducible direct summands have highest weights of the form $\lambda - m_\lambda \delta$ (i.e., initial weights).

We illustrate this definition with an example.

**Example 4.9.** The highest weight module $V_{\Lambda_8}$ does not appear in $V^{\otimes 3}$ as it is not a sum of 3 type I weights. However, $V_{\Lambda_8-\delta/2}$ does appear in $V^{\otimes 3}$ as

$$\Lambda_8 - \frac{\delta}{2} = \Lambda_0 + \frac{1}{2}(2,-1;1,\dots,1) = (1,0;\varepsilon_1) + (1,0;-\varepsilon_1) + \frac{1}{2}(2,-1;1,\dots,1).$$

Notice also that $V_{\Lambda_8-t\delta/2}$ will also appear in $V^{\otimes 3}$ for any $t \geqslant 1$, but only $V_{\Lambda_8-\delta/2}$ will appear in $\mathcal{M}_3$.

The complete reducibility of $V^{\otimes n}$ (see, e.g., [1]) allows us to write:

$$V^{\otimes n} \cong \mathcal{M}_n \oplus Z_n,$$

where $Z_n$ consists of those simple submodules whose highest weights are not initial.

## 5. Lemmas

In this section we describe the combinatorial rules for decomposing the modules $\mathcal{M}_n$. The first two lemmas show that $m_\lambda = \Delta(\lambda)$, while the two that follow show that one may determine $\mathcal{M}_{n+1}$ from $\mathcal{M}_n$ and the (finitely many) straight weights.

**Lemma 5.1.** *Let $\lambda \in \widehat{P}_+$ so that $\lambda - \Delta(\lambda)\delta$ is a level $n$ dominant weight. Then $V_{\lambda-\Delta(\lambda)\delta}$ appears in $V^{\otimes n}$. Moreover, there is a straight weight path from $0$ to $\lambda$ passing through only weights of the form $\mu - \Delta(\mu)\delta$ with $\mu \in \widehat{P}_+$.*

**Proof.** Since we are not concerned with computing multiplicities, it suffices to construct a piecewise linear straight weight path from 0 to $\lambda - \Delta(\lambda)\delta$ contained entirely within the dominant Weyl chamber. Assume $\lambda = \sum_{i=0}^{8} M_i \Lambda_i$. We will construct the required path in reverse by starting from the weight $\lambda - \Delta(\lambda)\delta$ and removing path segments until we reach the weight 0. By concatenating the paths we remove in reverse order we obtain the desired path. The first set of useful paths are the sub-paths of

$$\pi_1 : 0 \to \Lambda_1 \to \Lambda_2 \to \Lambda_3 \to \Lambda_4 \to \Lambda_5 \to \Lambda_6 \to (\Lambda_7 + \Lambda_8) \to (\Lambda_0 + 2\Lambda_8)$$

constructed by concatenating straight paths $\pi_\omega$ terminating at $\omega = (1,0;\varepsilon_i)$, $i = 1,\dots,8$. We denote by $\pi_1^{(i)}$ the $i$th sub-path of $\pi_1$. In a similar fashion we construct

$$\pi_2 : 0 \to \Lambda_1 \to \Lambda_0 \quad \text{and} \quad \pi_3 : 0 \to \Lambda_1 \to \cdots \to (\Lambda_7 + \Lambda_8) \to 2\Lambda_7$$

again using only paths with type I straight weight segments. The affect of removing these path segments on the value of $k(\lambda)$ is as follows (where the value of a path at 1 is $\lambda$):

(1) $k(\pi * \pi_1^{(i)}(1)) = k(\pi(1))$, i.e., deleting sub-paths of $\pi_1$ has no affect on the value of $k(\lambda)$.
(2) $k(\pi * \pi_2(1)) = k(\pi(1)) - 2$ so deleting the path $\pi_2$ increases the value of $k(\lambda)$ by 2.
(3) $k(\pi * \pi_3(1)) = k(\pi(1)) - 2$ so deleting the path $\pi_3$ increases the value of $k(\lambda)$ by 2.

**Case I.** $k(\lambda) = M_8 - M_7 - 2M_0 \leqslant 0$ is even.

In this case $\Delta(\lambda) = 0$. Since $k(\lambda)$ does not depend on $M_i$, $1 \leqslant i \leqslant 6$, we can reduce the case $M_i = 0$, $1 \leqslant i \leqslant 6$, using sub-paths of $\pi_1$. For $\lambda$ with $M_8 = M_7 = M_0 = 0$ we are done. If not, we observe that $M_7$ and $M_8$ have the same parity. Again using the sub-path of $\pi_1$ terminating at $\Lambda_7 + \Lambda_8$ as many times as is necessary, we may assume either $M_7 = 0$ or $M_8 = 0$.

*Case I.1.* $M_7 = 0$. In this case we have $M_8 \leqslant 2M_0$ and $M_8$ even. So by removing path segments $\pi_1$ as many times is as necessary we can reduce to $M_8 = 0$ with $k(\lambda) = -2M_0$ unchanged. At this point we are left with the case $\lambda = M_0\Lambda_0$, to which we remove the path segments $\pi_2$ as many times as necessary to reduce to 0.

*Case I.2.* $M_8 = 0$. Here we have that $M_7 \geqslant -2M_0$ and $M_7$ is even, so we reduce by $\pi_3$ until $M_7 = 0$ and then reduce by the path $\pi_2$ until $M_0 = 0$ and we are left with the weight 0. Observe that $k(\lambda) = -M_7 - 2M_0$ in this case so while deleting path segments $\pi_2$ or $\pi_3$ result in a raised $k$-value, it will always be non-positive and even, regardless.

**Case II.** $k(\lambda) = M_8 - M_7 - 2M_0 \leqslant 1$ is odd.

Here $M_8$ and $M_7$ have opposite parity; so, as in Case I, we reduce by sub-paths of $\pi_1$ until either $M_7 = 0$ or $M_8 = 0$. Then we reduce by paths as in Case I until we are left with two cases: $\lambda = \Lambda_7 - \delta/2$ and $\lambda = \Lambda_8 - \delta/2$. These are achieved by the paths:

$$0 \to \Lambda_1 \to \Lambda_2 \to \Lambda_3 \to \left(\Lambda_7 - \frac{\delta}{2}\right) \quad \text{and} \quad 0 \to \Lambda_2 \to \left(\Lambda_8 - \frac{\delta}{2}\right)$$

using the straight weights $\frac{1}{2}(2, -1; -1, -1, -1, 1, \dots, 1, -1)$ and $\frac{1}{2}(2, -1; -1, -1, 1, \dots, 1)$, respectively.

**Case III.** $k(\lambda) = M_8 - M_7 - 2M_0 \geqslant 2$.

In this case $M_8 \geqslant 2 + M_7 + 2M_0$, so we can use sub-paths of $\pi_1$ to reduce to $M_7 = 0$ and then $M_0 = 0$ without changing the value of $k(\lambda)$ and we are left with the task of constructing a path terminating at $\lambda = M_8\Lambda_8 - \Delta(M_8\Lambda_8)$, where $k(\lambda) = M_8 \geqslant 2$. The weight $3\Lambda_8 - \delta/2$ is of the form $\mu - \Delta(\mu)\delta$ with $\mu \in \widehat{P}_+$ and the path:

$$\pi_1 * \left[(2\Lambda_8 + \Lambda_0) \to \left(3\Lambda_8 - \frac{\delta}{2}\right)\right]$$

allows us to reduce to $M_8 \leqslant 2$ since:

$$\Delta\big((M_8-3\ell)\Lambda_8\big)=\frac{1}{6}\big(M_8-3\ell+2\big[(M_8-3\ell)\Lambda_8\big]_3\big)=\Delta(M_8\Lambda_8)-\frac{\ell}{2} \qquad (5.1)$$

so that

$$M_8\Lambda_8-\Delta(M_8\Lambda_8)\delta-\ell\left(3\Lambda_8-\frac{\delta}{2}\right)=(M_8-3\ell)\Lambda_8-\Delta\big((M_8-3\ell)\Lambda_8\big)\delta.$$

Now the cases $M_8=0,1$ were covered in Cases I and II, respectively, so we need only construct a path to $2\Lambda_8-\delta$. But this is nothing more than a doubling of the path

$$0\to\Lambda_2\to\left(\Lambda_8-\frac{\delta}{2}\right)$$

constructed above. This completes the proof. □

**Lemma 5.2.** $\Delta(\lambda)=m_\lambda$ *for all* $\lambda\in\widehat{P}_+$.

**Proof.** By Lemma 5.1 it is sufficient to show that $m_\lambda \geqslant \Delta(\lambda)$ since $\lambda-\Delta(\lambda)\delta$ appears in $V^{\otimes n}$ hence $m_\lambda\leqslant\Delta(\lambda)$. Again we consider cases.

**Case I.** $k(\lambda)\leqslant 0$ and even.

Since $m_\lambda\geqslant 0=\Delta(\lambda)$ there is nothing to prove.

**Case II.** $k(\lambda)\leqslant 1$ and odd.

We need only show that $m_\lambda\neq 0$. The only way that this can occur is if $\lambda$ can be expressed as a sum of type I weights. But $k(\omega)=0$ or $-2$ for $\omega$ of type I, so if $\lambda$ were a sum of type I weights $k(\lambda)$ would be even.

**Case III.** $k(\lambda)\geqslant 2$.

In this case we will reduce to the case where $\lambda=M_8\Lambda_8-t\delta$ using sub-paths of $\pi_1$ as in the proof of Lemma 5.1. Suppose

$$\lambda=\sum_{i=0}^{8}M_i\Lambda_i-t\delta$$

with $t$ minimal and $t\in\frac{1}{2}\mathbb{N}$. We compute $k(\lambda)=M_8-M_7-2M_0\geqslant 2$, so that $M_8>M_7+2M_0$. We can reduce to the case where $M_1=M_2=\cdots=M_6=M_7=0$ using sub-paths of $\pi_1$ and observing that

$$\lambda'=M_0\Lambda_0+(M_8-M_7)\Lambda_8-t\delta$$

has $k(\lambda') = k(\lambda)$ and $t$ minimal for $\lambda'$ if and only if $t$ is minimal for $\lambda$. Setting $M_0' = M_0$ and $M_8' = (M_8 - M_7)$ we have $k(\lambda') = M_8' - 2M_0' \geqslant 2$. Reducing by the path $\pi_1$ and setting $M_8'' = M_8' - 2M_0'$, we see that

$$\lambda'' = M_8'' \Lambda_8 - t\delta$$

has $t$ minimal if and only if $t$ is minimal for $\lambda'$. So we are left with showing that $m_\lambda \geqslant \Delta(\lambda)$ for $\lambda = M_8 \Lambda_8$. This will follow by an induction argument once we show it for the cases $M_8 = 1, 2$, and 3.

$M_8 = 1$. This case was already covered in Case II above.

$M_8 = 2$. If $2\Lambda_8$ were a sum of type I weights, we would have $k(2\Lambda_8) \leqslant 0$ so we must have at least one weight of type II, III, or IV. By considering the values of $k$ on these weights, we see that $t = 1$ is minimal.

$M_8 = 3$. Again considering the values of $k$, we see that type I weights are not sufficient and that $t = \frac{1}{2}$ is minimal.

Observing that $\Delta((M_8 - 3\ell)\Lambda_8) + \frac{\ell}{2} = \Delta(M_8\Lambda_8)$ (see Eq. (5.1)) the case $M_8 > 3$ follows by induction and we are done. □

**Remark 5.3.** We may now redefine *initial weight* to be any dominant weight of the form $\lambda - \Delta(\lambda)\delta$, and we denote the set of initial weights of level $n$ by $\mathcal{S}(n) = \{\lambda - \Delta(\lambda)\delta\colon \lambda \in \widehat{P}_+(n)\}$.

**Lemma 5.4.** *If $\lambda - \Delta(\lambda)\delta$ is an initial weight and $\omega \in \Omega$ then $\lambda - \Delta(\lambda)\delta - \omega$ is either an initial weight or not in the dominant Weyl chamber.*

**Proof.** Let $\mu - t\delta = \lambda - \Delta(\lambda)\delta - \omega$ where $\mu \in \widehat{P}_+$. Assume that $\mu$ is in the dominant Weyl chamber. We must demonstrate that $t = \Delta(\mu)$. By Lemma 5.2, we have that $t \geqslant \Delta(\mu)$ as $t \leqslant \Delta(\mu)$ would contradict the minimality of $\Delta(\mu)$. Using Lemma 4.4, we have

$$t = \begin{cases} \Delta(\lambda), & \text{if } \omega \text{ is of type I,} \\ \Delta(\lambda) - \dfrac{1}{2}, & \text{if } \omega \text{ is of type II,} \\ \Delta(\lambda) - 1, & \text{if } \omega \text{ is of type III.} \end{cases} \tag{5.2}$$

It is sufficient to show that $\Delta(\mu) \geqslant t$ for all of these cases. We organize them by considering the value of $k(\lambda)$ as follows:

**Case I.** $k(\lambda) \leqslant 0$ and even.

Here we have $\Delta(\lambda) = 0$. Since $\Delta(\mu) \geqslant 0$ the only possibility is that $\omega$ is of type I, for which it is clear.

**Case II.** $k(\lambda) \leqslant 1$ and odd.

The only possibilities are $\omega$ of type I or II, since $\Delta(\lambda) = \frac{1}{2}$ in this case and $\Delta(\mu) \geqslant 0$. If $\omega$ is of type II, then $t = 0$, hence $\Delta(\mu) \geqslant t$ is obvious. If $\omega$ is of type I, then $t = \frac{1}{2}$ and Lemma 4.4 implies $k(\mu) = k(\lambda) - k(\omega) \leqslant 3$ and odd. If $k(\mu) \leqslant 1$ and odd then $\Delta(\mu) = \frac{1}{2}$ and we are done. Otherwise $k(\mu) = 3$, and we compute $\Delta(\mu) = \frac{1}{2}$ as required.

**Case III.** $k(\lambda) \geqslant 2$.

Here there are 3 cases depending on the type of $\omega$. The computations are somewhat tedious, but straightforward.

*Case III.1.* $\omega$ is of type I. If $k(\omega) = 0$, then $k(\mu) = k(\lambda)$ hence $\Delta(\mu) = \Delta(\lambda)$ and we are done. If $k(\omega) = -2$, then $k(\mu) = k(\lambda) + 2$ and we must check the three 3 cases corresponding to the values of $[\mu]_3$ (depending on $[\lambda]_3$) by evaluating $\Delta(\mu) = \frac{1}{6}(k(\mu) + 2[\mu]_3)$.

*Case III.2.* $\omega$ is of type II. We must show that $\Delta(\mu) \geqslant \Delta(\lambda) - \frac{1}{2}$. This is the most involved case as $k(\omega) \in \{3, 1, -1, -3, -5\}$ and we must check a total of 15 subcases corresponding to the 3 values of $[\lambda]_3$ and 5 values of $k(\omega)$. As an example of what is involved we work out the cases where $[\lambda]_3 = 1$ and $k(\omega) = -1$. Then $k(\mu) = k(\lambda) + 1$, $[\mu]_3 = 2$ and

$$\Delta(\mu) = \frac{1}{6}\big(k(\lambda) + 1 + 2[\mu]_3\big) = \frac{1}{6}\big(k(\lambda) + 1 + 4\big) \geqslant \frac{1}{6}\big(k(\lambda) + 2\big) = \Delta(\lambda) > \Delta(\lambda) - \frac{1}{2}.$$

Notice that in this case $\mu$ is not dominant. The remaining cases are handled similarly.

*Case III.3.* $\omega$ is of type III. We must show that $\Delta(\mu) \geqslant \Delta(\lambda) - 1$. Here $k(\omega) = 2$ so $k(\mu) = k(\lambda) - 2$ and we must again check cases by evaluating $\Delta(\mu)$. □

**Lemma 5.5.** *If $\lambda - \Delta(\lambda)\delta$ is an initial weight and $\lambda - \Delta(\lambda)\delta + \omega$ is also initial, then $\omega$ is a straight weight.*

Before giving a proof, we mention a *caveat*: the requirement that $\lambda - \Delta(\lambda)\delta + \omega$ is initial is not superfluous. For example, $2\Lambda_8 - \delta$ is an initial weight and $(1, 0; -\varepsilon_8)$ is a straight weight, but $2\Lambda_8 - \delta + (1, 0; -\varepsilon_8) = \Lambda_7 + \Lambda_8 - \delta$ is not initial.

**Proof.** It is enough to show that $\lambda - \Delta(\lambda)\delta + \omega$ is not initial if $\omega$ is not straight. The key fact here is from Lemma 4.4: $k(\omega) \leqslant 6j - 6$ for the type IV weight $\omega = (1, -j; \nu)$ where $j \geqslant 1$ is a half-integer. Let $\mu - s\delta = \lambda - \Delta(\lambda)\delta + \omega$ for such a weight $\omega$. Observing that $s = \Delta(\lambda) + j$ we will show that $s \neq \Delta(\mu)$.

**Case I.** $k(\mu) \leqslant 1$.

Since $s \geqslant j \geqslant 1$ and $\Delta(\mu) \leqslant \frac{1}{2}$, it is clear that $s \geqslant \Delta(\mu)$.

**Case II.** $k(\lambda) \leqslant 1$.

Here we have that

$$\Delta(\mu) = \frac{1}{6}\big(k(\lambda) + k(\omega) + 2\big[k(\lambda) + k(\omega)\big]_3\big) \leqslant \frac{1}{6}(1 + 6j - 6 + 4) = \frac{6j-1}{6} < j \leqslant s$$

so once again $\Delta(\mu) \neq s$.

**Case III.** $k(\lambda) \geqslant 1$ and $k(\mu) \geqslant 1$.

Computing as above, we have

$$\begin{aligned}\Delta(\mu) &= \frac{1}{6}\big(k(\lambda) + k(\omega) + 2\big[k(\lambda) + k(\omega)\big]_3\big) \leqslant \frac{1}{6}\big(k(\lambda) + 6j - 6 + 4\big)\\ &\leqslant \frac{1}{6}\big(k(\lambda)\big) + \frac{6j-2}{6} < \Delta(\lambda) + j = s.\end{aligned}$$

So we see that $\Delta(\mu) \neq s$ in all cases and we are done. □

## 6. Them main theorem and an algorithm

The following theorem is a immediate corollary of the lemmas in the previous section.

**Theorem 6.1.** *If $\lambda - \Delta(\lambda)\delta \in \mathcal{S}(n)$, then any straight weight path from $0$ to $\lambda - \Delta(\lambda)\delta$ passes through only initial weights. Thus*

$$\mathcal{M}_n \cong \bigoplus_{\lambda \in \widehat{P}_+(n)} c_\lambda V_{\lambda - \Delta(\lambda)\delta},$$

*where the multiplicities $c_\lambda$ are determined by counting the straight weight paths terminating at $\lambda - \Delta(\lambda)\delta$.*

Applying the results, we have the following simple inductive algorithm for decomposing $\mathcal{M}_n$ as a sum of simple highest weight modules:

Step 1. Initialize with $\mathcal{M}_1 \cong V_{\Lambda_1}$.

Step 2. Having determined the multiplicities $c_\lambda$ so that

$$\mathcal{M}_n \cong \bigoplus_{\lambda \in \widehat{P}_+(n)} c_\lambda V_{\lambda - \Delta(\lambda)\delta}$$

compute the set $A_\lambda = \{\lambda - \Delta(\lambda)\delta + \omega\colon\ \omega \in \Omega\}$ for each $\lambda \in \widehat{P}_+(n)$.

Step 3. Compute the set $\mathcal{S}(n+1)$. The size of $\mathcal{S}(k)$ is computed from the generating function:

$$\prod_{0\leqslant i\leqslant 8}\frac{1}{1-x^{n(\Lambda_i)}} = 1 + x + 3x^2 + 5x^3 + 10x^4 + 15x^5 + 27x^6 + 39x^7 + 63x^8 + O\big[x^9\big]. \tag{6.1}$$

Step 4. For each $\mu - \Delta(\mu)\delta \in \mathcal{S}(n+1)$, let $B_\mu = \{\lambda \in \widehat{P}_+(n)\colon\ \mu - \Delta(\mu)\delta \in A_\lambda\}$. Then

$$c_\mu = \sum_{\lambda\in B_\mu} c_\lambda.$$

**Remark 6.2.** The formula in Step 3 is valid since the level of a dominant weight $\lambda$ is determined by the decomposition $\lambda = \sum_i M_i \Lambda_i$ and the levels $n(\Lambda_i)$ of the fundamental weights $\Lambda_i$ (see Definition 4.2). One identifies a level $n$ dominant weight with a partition of $n$ into parts whose sizes are in the multi-set $\{n(\Lambda_i)\}$, and standard combinatorics lead to Eq. (6.1). For arbitrary $N$ the highest weight module $V_\lambda$ appears in $V^{\otimes n(\lambda)}$ where the formula for $n(\lambda)$ is given in [6, Eq. (3.1)], in case $N \neq 9$. However, his formula breaks into three cases which depend on $k(\lambda)$ in a way that makes the problem of constructing a generating function valid for all $N$ rather complicated combinatorially.

As an application we compute the decompositions of the first few $\mathcal{M}_n$:

$$\begin{aligned}
\mathcal{M}_2 &\cong V_{\Lambda_0} \oplus V_{\Lambda_2} \oplus V_{2\Lambda_1},\\
\mathcal{M}_3 &\cong V_{3\Lambda_1} \oplus 2V_{\Lambda_1+\Lambda_2} \oplus 3V_{\Lambda_0+\Lambda_1} \oplus V_{\Lambda_3} \oplus 2V_{\Lambda_8-\delta/2},\\
\mathcal{M}_4 &\cong V_{4\Lambda_1} \oplus V_{\Lambda_4} \oplus 6V_{\Lambda_0+2\Lambda_1} \oplus 3V_{\Lambda_2+2\Lambda_1} \oplus 6V_{\Lambda_7-\delta/2} \oplus 6V_{\Lambda_0+\Lambda_2}\\
&\quad \oplus 3V_{2\Lambda_0} \oplus 8V_{\Lambda_1+\Lambda_8-\delta/2} \oplus 3V_{\Lambda_1+\Lambda_3} \oplus 2V_{2\Lambda_2}.
\end{aligned}$$

## 7. Connections and further directions

### 7.1. $E_N$ series

Wenzl introduces a generic labeling set, $\Gamma$, for the dominant integral weights of $\mathfrak{g}(E_N)$, $N \neq 9$, consisting of triples $(n, \mu, i)$ where $n \in \mathbb{N}$, $\mu$ a Young diagram with $|\mu| \leqslant n$ and $i \in \{0, 1, 2\}$, subject to some further conditions (see [6, Section 2]). The labeling is realized via a map $\Phi$ assigning an element of $\Gamma$ to each integral dominant weight. The ambiguity in the dominant weights due to the null-root precludes extending $\Phi$ directly to the excluded case; however, the set of integral dominant weights of $\mathfrak{g}(E_9)$ whose image under $\Phi$ is in $\Gamma$ is precisely the set of initial weights! Thus one sees that our submodule $\mathcal{M}_n$ must be the "missing link" replacing $V_{\text{new}}^{\otimes n}$ required to extend Wenzl's main combinatorial result [6, Proposition 3.10] for the $E_N$, $N \geqslant 6$, series to the $N = 9$ case:

**Proposition 7.1.** *Assume $N > n$. Then the branching rules for $V_{\text{new}} \subset V_{\text{new}}^{\otimes 2} \subset \cdots \subset V_{\text{new}}^{\otimes n}$ do not depend on $N$.*

This proposition implies that when $k < 9$ and $N > k$ the combinatorial formula given is Step 3 of the algorithm holds.

### 7.2. Braid representations

For generic $q$, the tensor product rules for the quantum group $U_q\mathfrak{g}(E_N)$ are the same as those of the Kac–Moody algebra $\mathfrak{g}(E_N)$. Wenzl was also able to show that, for $N \neq 9$, the centralizer algebra of the corresponding $U_q\mathfrak{g}(E_N)$-module $V_{\mathrm{new}}^{\otimes n}$ is generated by the image of the braid group $B_n$ (acting by $R$-matrices) and one more operator called the *quasi-Pfaffian*. It should be possible to extend this result to the $N = 9$ case using the quantum group version of the modules $\mathcal{M}_n$ together with the specific knowledge of the decomposition rules.

### 7.3. Other Lie types

It may be possible to use the same approach to derive a similar algorithm for decomposing the tensor powers of low-level highest weight modules for any affine Kac–Moody algebra. By defining the submodules analogous to $\mathcal{M}_n$ one would just need to determine the subset of maximal weights corresponding to the set $\Omega$ of straight weights.

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