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Convergence for pseudo monotone semiflows on product ordered topological spaces ☆

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Abstract

In this paper, we consider a class of pseudo monotone semiflows, which only enjoy some weak monotonicity properties and are defined on product-ordered topological spaces. Under certain conditions, several convergence principles are established for each precompact orbit of such a class of semiflows to tend to an equilibrium, which improve and extend some corresponding results already known. Some applications to delay differential equations are presented.

MSC: 34C12; 34K25; 37C65

Keywords: Pseudo monotone semiflow; Convergence; Precompact orbit; ω-limit set

1. Introduction

In recent years the study of the convergence of precompact orbits as an important subject of the theory of monotone dynamical systems has received amazing achievements. Hirsch [11] established that most orbits of a strongly monotone semiflows on a strongly ordered space tend to the set of equilibria, which extends earlier work of

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Hirsch [9,10] for ordinary differential equations to infinite-dimensional semiflows, and applied this result to parabolic partial differential equations. Those results in [11] were later improved by Matano [15,16], Poláčik [17], and Smith and Thieme [22,23].

The generic convergence principles in the aforementioned work imply that precompact orbits of monotone dynamical systems have a strong tendency to converge to an equilibrium, which therefore inspires many researchers to try to find sufficient conditions for every precompact orbit of monotone dynamical systems convergent to an equilibrium. For instance, Takáč [24] introduced the subhomogeneous hypotheses to establish the global convergence for strongly monotone discrete-time semiflows. Later, the authors in [12,13,26] studied the global convergence for monotone and subhomogeneous systems from different points of view. Some other well-known conditions such as the orbital stability, the first integral, etc. were also utilized by many investigators to prove the global convergence in continuous- and discrete-time monotone dynamical systems (see, e.g., [1,5,8,14,18,20,25,28]). For related work, we refer to the monograph by Zhao [29]. When significantly enriching the theory of monotone dynamical systems, the convergence principles in the above-mentioned literature fail to apply to many differential equations without enjoying a comparison principle. However, it is possible that some differential equations still possess some slightly weaker monotonicity properties and in this case, we might even combine monotonicity arguments with dynamical systems ideas to obtain convergence to equilibrium for precompact orbits. We know that very little has been accomplished in this direction. For instance, Haddock et al. [7] recently introduced a class of eventually strongly pseudo monotone semiflows defined on a function subspace $X \subseteq C(M, R^1)$ which has a topology making its inclusion into $C(M, R^1)$ continuous, where M is a compact topological space and R^1 denotes the set of all real numbers, and proved that each precompact orbit tends to a constant function whenever each constant function is an equilibrium point for such semiflows.

Even though the convergence principle in [7] has been successfully applied to neutral functional differential equations and semilinear parabolic partial differential equations with Neumamn boundary condition, its requirements on the phase space, the set of equilibria and even the monotonicity properties are still too restrictive and therefore, its limitations seem natural. In fact, the convergence principle in [7] cannot be applied to some important examples like the following scalar delay differential equation:

$$x'(t) = -F(x(t)) + G(x(t-r)), \tag{1.1}$$

where r is a positive constant, $F, G \in C(R^1)$, F is nondecreasing, and either $G(x) \geqslant F(x)$ for all $x \in R^1$ or $G(x) \leqslant F(x)$ for all $x \in R^1$. Indeed, (i) if $G \not\equiv F$, then the set of equilibria of (1.1) cannot contain all the constant functions on the space $C([-r, 0], R^1)$; (ii) if $G \equiv F$, then the semiflow generated by (1.1) does not enjoy the monotonicity properties considered by Haddock et al. [7]. It should be pointed out that the convergence principle in [27] cannot be applied to (1.1) either for the similar reasons. Variants of system (1.1) have been used as models for various phenomena such as some population growth, the spread of epidemics, the dynamics of capital stocks, etc. (see, for example, [3,4,6] and the references cited therein).

Motivated by the above discussion and example, we will consider a class of essentially semi-strongly sup-pseudo (sub-pseudo) monotone semiflows (see Section 2 for more details on this definition) defined on product-ordered topological spaces. Under certain conditions, by combining monotonicity arguments and the basic properties of the ω-limit set of precompact orbits (i.e., nonempty, compact, invariant and connected), we obtain several convergence principles, that is, each precompact orbit of such a class of semilows tends to an equilibrium, which extend and improve earlier work of Haddock et al. [7].

The paper is organized as follows. In Section 2, we define several class of pseudo monotone semiflows and establish several convergence principles. In Section 3, some applications of the results obtained in previous section to certain systems of delay differential equations are given.

2. Convergence principles

In this section, we prove several convergence principles. For simplicity here, we begin by introducing some notations and definitions.

Let X_i be a topological space endowed with a closed partial order relation R_i, where $i = 1, 2$, and (X_i, R_i) is also called an ordered topological space. The ordered topological space (X, R) defined by $X = X_1 \times X_2$ and $R = \{(x_1, x_2, y_1, y_2) \in X \times X : (x_i, y_i) \in R_i, i = 1, 2\}$ is called the product ordered topological space of the ordered topological spaces (X_1, R_1) and (X_2, R_2). For any $x_i, y_i \in X_i$, $A_i \subseteq X_i$, the following notations will be used: $x_i \leqslant_i y_i$ iff $(x_i, y_i) \in R_i$, $x_i <_i y_i$ iff $x_i \leqslant_i y_i$ and $x_i \neq y_i$, $x_i \ll_i y_i$ iff $(x_i, y_i) \in \operatorname{Int} R_i$, $x_i \leqslant_i A_i$ iff $x_i \leqslant_i y_i$ for any $y_i \in A_i$, $x <_i A_i$ iff $x_i <_i y_i$ for any $y_i \in A_i$, $x_i \ll_i A_i$ iff $x_i \ll_i y_i$ for any $y_i \in A_i$, where $i = 1, 2$, and $\operatorname{Int} R_i$ denotes the interior of R_i in $X_i \times X_i$. For any $x, y \in X$ and $A \subseteq X$, we write $x \leqslant y$ $(x \ll y)$ iff $x_i \leqslant_i y_i$ $(x_i \ll_i y_i)$ for $i = 1, 2$. Notations such as $x \leqslant y$, $x \ll A$ and so forth, can be defined similarly. In what follows, we shall write "$\leqslant$", "$<$" and "$\ll$" for "$\leqslant_i$", "$<_i$", and "$\ll_i$", respectively, when no confusion results, where $i = 1, 2$.

Let R_+^1 denote the set of all nonnegative real numbers, $\Phi : X \times R_+^1 \to X$ be a semiflow on X, that is, Φ is continuous and $\Phi_t(x) \equiv \Phi(x, t)$ which satisfies:

(i) $\Phi_0(x) = x$ for all $x \in X$;
(ii) $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$ for all $x \in X$ and $t, s \in R_+^1$.

We write $O(x) = \{\Phi_t(x) : t \in R_+^1\}$ for the positive semi-orbit through the point x. The ω-limit set of $O(x)$ is defined by $\omega(x) = \bigcap_{t \in R_+^1} \overline{O(\Phi_t(x))}$. Let $E = \{e \in X : \Phi_t(e) = e,\ t \in R_+^1\}$ be the set of equilibria of Φ.

We now make the following key definitions:

Definition 2.1. Assume that $\sum \subseteq X$ and Φ is a semiflow on X. The semiflow Φ is said to be sup-pseudo monotone with respect to $\sum$ if for any $e \in \sum$, there exists $T = T_e \geqslant 0$ such that for any $x \in X$ with $x \geqslant e$, we have $\Phi_t(x) \geqslant e$ for all $t \geqslant T$. Points of such a $\sum$ are called sup-pseudo equilibria. The semiflow Φ is said to be sub-pseudo monotone with respect to $\sum$ if for any $e \in \sum$, there exists $T = T_e \geqslant 0$ such that for any $x \in X$ with $x \leqslant e$, we have $\Phi_t(x) \leqslant e$ for all $t \geqslant T$. Points of such a $\sum$ are called sub-pseudo equilibria. The semiflow Φ is said to be pseudo monotone with respect to $\sum$ if Φ is both sup-pseudo and sub-pseudo monotone with respect to $\sum$. Points of such a $\sum$ are called pseudo equilibria.

Remark 2.1. Note that if $\sum = E$ and the semiflow Φ is monotone in the sense of Hirsch [11], then Φ is pseudo monotone with respect to $\sum$.

Definition 2.2. Assume that $\sum \subseteq X$ and Φ is a semiflow on X. The semiflow Φ is said to be essentially semi-strongly sup-pseudo monotone with respect to $\sum$ if Φ is sup-pseudo monotone with respect to $\sum$, and for any $e \in \sum$ there exists $T = T_e > 0$ such that for any $x \in X$ with $x \geqslant e$, one of the following holds:

(i) $\Phi_T(x) = e$;
(ii) $\Phi_T(x) \gg e$;
(iii) $(\Phi_T(x))_1 \gg e_1$ and $(\Phi_T(x))_2 = e_2$;
(iv) $(\Phi_T(x))_1 = e_1$ and $(\Phi_T(x))_2 \gg e_2$.

Definition 2.3. Assume that $\sum \subseteq X$ and Φ is a semiflow on X. The semiflow Φ is said to be essentially semi-strongly sub-pseudo monotone with respect to $\sum$ if Φ is sub-pseudo monotone with respect to $\sum$, and for any $e \in \sum$ there exists $T = T_e > 0$ such that for any $x \in X$ with $x \leqslant e$, one of the following holds:

(i) $\Phi_T(x) = e$;
(ii) $\Phi_T(x) \ll e$;
(iii) $(\Phi_T(x))_1 \ll e_1$ and $(\Phi_T(x))_2 = e_2$;
(iv) $(\Phi_T(x))_1 = e_1$ and $(\Phi_T(x))_2 \ll e_2$.

A semiflow Φ is said to be essentially semi-strongly pseudo monotone with respect to $\sum$ if Φ is both essentially semi-strongly sup-pseudo and essentially semi-strongly sub-pseudo monotone with respect to $\sum$.

We will always assume that the map $I_i : R^1 \to X_i$ is continuous and satisfies that $I_i(\alpha_i) \ll I_i(\alpha_i')$ for all $\alpha_i' > \alpha_i$ and that for any $x_i \in X_i$, there exist $\alpha_i, \alpha_i' \in R^1$ such that $I_i(\alpha_i) \leqslant x_i \leqslant I_i(\alpha_i')$, where $i = 1, 2$. Let $F : R^1 \to R^1$ be continuous and nondecreasing. Also, let

$$\begin{aligned} D_F &= \left\{(\alpha, \beta) \in R^2 : F(\alpha) = F(\beta)\right\}, \\ \widehat{D_F} &= \left\{(I_1(\alpha), I_2(\beta)) \in X : (\alpha, \beta) \in D_F\right\}, \\ s_F(\alpha) &= \sup\{\beta \in R^1 : F(\beta) = F(\alpha)\}, \\ i_F(\alpha) &= \inf\{\beta \in R^1 : F(\beta) = F(\alpha)\}. \end{aligned}$$

Remark 2.2. We cannot rule out the possibility that $s_F(\alpha) = +\infty$ and $i_F(\alpha) = -\infty$. In fact, if F is a constant function, then $s_F(\alpha) = +\infty$ and $i_F(\alpha) = -\infty$.

It is further assumed that $\widehat{D_F} \subseteq \sum \subseteq X$.

For the sake of simplicity, we introduce the following assumptions:

(H_1) Let the semiflow Φ be essentially semi-strongly sup-pseudo monotone with respect to $\sum$, the set Ω be the ω-limit set of some precompact positive semi-orbit of Φ, and $(\alpha_1, \alpha_2) \in D_F$ with $(I_1(\alpha_1), I_2(\alpha)) < \Omega$. If there exists $i \in \{1, 2\}$ such that $I_i(\alpha_i) \ll q_i$ for all $q \in \Omega$ and $\alpha_i = s_F(\alpha_i)$, then there exists $q \in \Omega$ such that $(I_1(\alpha_1), I_2(\alpha_2)) \ll q$.

(H_2) Let the semiflow Φ be essentially semi-strongly sub-pseudo monotone with respect to $\sum$, the set Ω be the ω-limit set of some precompact positive semi-orbit of Φ, and $(\alpha_1, \alpha_2) \in D_F$ with $(I_1(\alpha_1), I_2(\alpha)) > \Omega$. If there exists $i \in \{1, 2\}$ such that $I_i(\alpha_i) \gg q_i$ for all $q \in \Omega$ and $\alpha_i = i_F(\alpha_i)$, then there exists $q \in \Omega$ such that $(I_1(\alpha_1), I_2(\alpha_2)) \gg q$.

(H_3) Let the semiflow Φ be essentially semi-strongly sup-pseudo monotone with respect to $\sum$, and assume that $(\alpha_1, \alpha_2) \in D_F$ and $\alpha_i = s_F(\alpha_i)$ for some $i \in \{1, 2\}$. If $x \in X$ with $(I_1(\alpha_1), I_2(\alpha_2)) \leqslant x$ and $I_i(\alpha_i) \ll x_i$, then there exists $T > 0$ such that $(I_1(\alpha_1), I_2(\alpha_2)) \ll \Phi_T(x)$.

(H_4) Let the semiflow Φ be essentially semi-strongly sub-pseudo monotone with respect to $\sum$, and assume that $(\alpha_1, \alpha_2) \in D_F$ and $\alpha_i = i_F(\alpha_i)$ for some $i \in \{1, 2\}$. If $x \in X$ with $(I_1(\alpha_1), I_2(\alpha_2)) \geqslant x$ and $I_i(\alpha_i) \gg x_i$, then there exists $T > 0$ such that $(I_1(\alpha_1), I_2(\alpha_2)) \gg \Phi_T(x)$.

Remark 2.3. By the invariance of ω-limit set, we know that (H_3) implies (H_1), and (H_4) implies (H_2).

Lemma 2.1. *Suppose that (H_1) holds, and that $x \in X$ is a given point such that $O(x)$ is precompact. Let $A_x = \{(\alpha, \beta) \in D_F : (I_1(\alpha), I_2(\beta)) \leqslant \omega(x)\}$. Then A_x contains the maximum element $(\alpha^*, \beta^*) \in A_x$, which satisfies that $(I_1(\alpha^*), I_2(\beta^*)) \in \widehat{D_F} \cap \omega(x)$ and that for any $q \in \omega(x) \setminus (I_1(\alpha^*), I_2(\beta^*))$, we have either*

$$I_1(\alpha^*) \ll q_1 \quad \textit{and} \quad I_2(\beta^*) = q_2$$

or

$$I_1(\alpha^*) = q_1 \quad \textit{and} \quad I_2(\beta^*) \ll q_2.$$

Proof. We first prove that A_x contains the maximum element.

By the compactness of $\omega(x)$ and the definition of I_i, there exist $\alpha', \beta' \in R^1$ such that

$$(I_1(\alpha'), I_2(\alpha')) \leqslant \omega(x) \leqslant (I_1(\beta'), I_2(\beta')).$$

Let

$$A'_x = \{(\alpha, \beta) \in A_x : \alpha' \leqslant \alpha \leqslant \beta', \alpha' \leqslant \beta \leqslant \beta'\}.$$

We will show that A_x' contains the maximum element. Since A_x' is a compact subset in R^2, it follows that A_x' must contain the maximal element (α^*, β^*). We claim that (α^*, β^*) is the maximum element of A_x'. By way of contradiction, we assume that, without loss of generality, there exists $(\alpha^{**}, \beta^{**}) \in A_x'$ such that $\alpha^{**} > \alpha^*$ and $\beta^{**} < \beta^*$. Then, from the fact that F is nondecreasing, it follows that $F(\alpha^{**}) = F(\beta^*)$ and hence, $(\alpha^{**}, \beta^*) \in D_F$. By the choice of α^{**} and β^*, we have $(\alpha^{**}, \beta^*) \in A_x'$. This contradicts the fact that (α^*, β^*) is the maximal element of A_x', and thus, the claim is proved. Therefore, by the definition of A_x', (α^*, β^*) is also the maximum element of A_x.

In the remainder of the proof, we first prove that for any $q \in \omega(x)$, one has $((I(\alpha^*), I(\beta^*)), q) \notin \operatorname{Int} R$. Otherwise, $(I(\alpha^*), I(\beta^*)) \ll q$. Thus, by the definition of $\omega(x)$, there exists $t_1 > 0$ such that

$$(I_1(\alpha^*), (I_2(\beta^*)) \ll \Phi_{t_1}(x).$$

Again, by the definition of D_F, there exist α', β' such that

$$(I_1(\alpha^*), (I_2(\beta^*)) \ll (I_1(\alpha'), I_2(\beta')) \ll \Phi_{t_1}(x).$$

Since the semiflow Φ is sup-pseudo monotone with respect to $\widehat{D_F}$, we have

$$(I_1(\alpha'), I_2(\beta')) \leqslant \omega(x),$$

a contradiction to the definition of (α^*, β^*).

We next prove that $(I_1(\alpha^*), I_2(\beta^*)) \in \omega(x)$.

Otherwise, $(I_1(\alpha^*), I_2(\beta^*)) < \omega(x)$. From the above discussion and the fact that the semiflow Φ is essentially semi-strongly sup-pseudo monotone with respect to $\widehat{D_F}$, it follows that there exists $T = T_{(\alpha^*, \beta^*)} > 0$ such that for any $q \in \omega(x)$, we have either

$$I_1(\alpha^*) \ll (\Phi_T(q))_1 \quad \text{and} \quad I_2(\alpha^*) = (\Phi_T(q))_2$$

or

$$I_1(\alpha^*) = (\Phi_T(q))_2 \quad \text{and} \quad I_2(\alpha^*) \ll (\Phi_T(q))_2.$$

Let

$$A_1 = \{q \in \omega(x) : I_1(\alpha^*) \ll q_1\} \quad \text{and} \quad A_2 = \{q \in \omega(x) : I_2(\beta^*) \ll q_2\}.$$

By the above discussion and the invariance of $\omega(x)$, we have $A_1 \cup A_2 = \omega(x)$ and $A_1 \cap A_2 = \phi$. Owing to the compactness of $\omega(x)$, A_1 and A_2 are closed sets. Again, since $\omega(x)$ is connected, it follows that either $A_1 = \phi$ or $A_2 = \phi$. Without loss of generality, we assume that $A_1 = \omega(x)$. We want to show that

$$\alpha^* = s_F(\alpha^*).$$

Otherwise,

$$\alpha^* < s_F(\alpha^*).$$

By the definition of A_1, there exists $\alpha^{**} > \alpha^*$ such that

$$(\alpha^{**}, \beta^*) \in D_F \quad \text{and} \quad (I_1(\alpha^{**}), I_2(\beta^*)) \leqslant A_1 \equiv \omega(x).$$

This contradicts the definition of (α^*, β^*). Thus, by (H_1), there exists $q \in \omega(x)$ such that $(I_1(\alpha^*), I_2(\beta^*)) \ll q$, a contradiction to the above discussion. Therefore, we obtain

$$(I_1(\alpha^*), I_2(\beta^*)) \in \omega(x).$$

Assume that $q \in \omega(x)\backslash\{(I(\alpha^*), I(\beta^*))\}$. From the above discussion and the fact that the semiflow Φ is essentially semi-strongly sup-pseudo monotone with respect to $\widehat{D_F}$, it follows easily that either

$$I_1(\alpha^*) \ll q_1 \quad \text{and} \quad I_2(\beta^*) = q_2$$

or

$$I_1(\alpha^*) = q_1 \quad \text{and} \quad I_2(\beta^*) \ll q_2.$$

This completes the proof. $\square$

Remark 2.4. By Remark 2.3, if assumption (H_3) is satisfied, the result of Lemma 2.1 continues to hold.

Lemma 2.2. *Suppose that (H_2) holds, and that $x \in X$ is a given point such that $O(x)$ is precompact. Let $A_x = \{(\alpha, \beta) \in D_F : (I_1(\alpha), I_2(\beta)) \geqslant \omega(x)\}$. Then A_x contains the minimum element $(\alpha^*, \beta^*) \in A_x$, which satisfies that $(I_1(\alpha^*), I_2(\beta^*)) \in \widehat{D_F} \cap \omega(x)$ and that for any $q \in \omega(x) \setminus (I_1(\alpha^*), I_2(\beta^*))$, we have either*

$$I_1(\alpha^*) \gg q_1 \quad \textit{and} \quad I_2(\beta^*) = q_2$$

or

$$I_1(\alpha^*) = q_1 \quad \textit{and} \quad I_2(\beta^*) \gg q_2.$$

Proof. Let

$$R_i' = \{(x_i, y_i) \in X_i \times X_i : (y_i, x_i) \in R_i\}$$

and $I_i'(\alpha) = I_i(-\alpha)$ for $\alpha \in R^1$. Replace R_i and I_i by R_i' and I_i', respectively, where $i = 1, 2$. The conclusion follows immediately from Lemma 2.1. $\square$

Remark 2.5. By Remark 2.3, if assumption (H_4) is satisfied, the result of Lemma 2.2 continues to hold.

Theorem 2.1. *Let F be a constant function and the semiflow Φ be essentially semi-strongly sup-pseudo (or sub-pseudo) monotone with respect to $\sum$. Suppose that $x \in X$ is a given point such that $O(x)$ is precompact. Then there exist $\alpha^*, \beta^* \in R^1$ such that*

$$\omega(x) = \{(I_1(\alpha^*), I_2(\beta^*))\}.$$

Proof. Without loss of generality, we assume that the semiflow Φ is essentially semi-strongly sup-pseudo monotone with respect to $\sum$. Using the fact that F is a constant function, we have $s_F(\alpha) = +\infty$ for all $\alpha \in R^1$. It then follows that Φ satisfies (H_1), and hence Lemma 2.1 implies that there exist $\alpha^*, \beta^* \in R^1$ such that

$$(I_1(\alpha^*), I_2(\beta^*)) \in \omega(x) \quad \text{and} \quad (I_1(\alpha^*), I_2(\beta^*)) \leqslant \omega(x).$$

Now we will show that $\omega(x) \setminus \{(I_1(\alpha^*), I_2(\beta^*))\} = \phi$. Otherwise, by Lemma 2.1, we may assume, without loss of generality, that there exists $q \in \omega(x)$ such that $q_1 \gg I_1(\alpha^*)$ and $q_2 = I_2(\beta^*)$. Choose $\beta' < \beta^*$ and $\alpha' > \alpha^*$ such that

$$q_1 \gg I_1(\alpha').$$

Then,

$$q \gg (I_1(\alpha'), I_2(\beta')).$$

Since F is a constant function, it follows that $(\alpha', \beta') \in D_F$. By the definition of $\omega(x)$, there exists $t_1 > 0$ such that

$$\Phi_{t_1}(x) \geqslant (I_1(\alpha'), I_2(\beta')).$$

Hence, from the fact that the semiflow Φ is sup-pseudo monotone with respect to $\sum$, it follows that there exists $t_2 > t_1$ such that

$$\Phi_t(x) \geqslant (I_1(\alpha'), I_2(\beta')) \quad \text{for all} \quad t \geqslant t_2.$$

Therefore, we have

$$\omega(x) \geqslant (I_1(\alpha'), I_2(\beta')).$$

But this contradicts the choice of (α^*, β^*). This completes the proof. □

Theorem 2.2. *Let the function F be strictly increasing and assume that either* (H_3) *or* (H_4) *holds. Suppose that* $x \in X$ *is a given point such that* $O(x)$ *is precompact. Then there exist* $\alpha^* \in R^1$ *such that* $\omega(x) = \{(I_1(\alpha^*), I_2(\alpha^*))\}$.

Proof. Without loss of generality, we assume that (H_3) is satisfied. By Remark 2.4 and the fact that F is strictly increasing, there exists $\alpha^* \in R^1$ such that

$$(I_1(\alpha^*), I_2(\alpha^*)) \in \omega(x) \quad \text{and} \quad (I_1(\alpha^*), I_2(\alpha^*)) \leqslant \omega(x).$$

Now we will show that $\omega(x) = \{(I_1(\alpha^*), I_2(\alpha^*))\}$. Otherwise, without loss of generality, we may assume that there exists $q \in \omega(x)$ such that $I_1(\alpha^*) \ll q_1$ and $I_2(\alpha^*) = q_2$. It follows from (H_3) that there exists $T > 0$ such that

$$(I_1(\alpha^*), I_2(\alpha^*)) \ll \Phi_T(q) \in \omega(x),$$

which is a contradiction to Remark 2.4. This completes the proof. □

Generally speaking, assumption (H_i) does not imply that the ω-limit set of precompact orbits is a singleton, where $i = 1, 2, 3, 4$. But, if both (H_1) and (H_2) hold, then we can get the following:

Theorem 2.3. *Let* (H_1) *and* (H_2) *hold. Suppose that* $x \in X$ *is a given point such that* $O(x)$ *is precompact. Then there exists* $(\alpha^*, \beta^*) \in R^2$ *such that*

$$\omega(x) = \{(I_1(\alpha^*), I_2(\beta^*))\}.$$

Proof. By Lemmas 2.1 and 2.2, there exist $\alpha^*, \beta^*, \alpha^{**}, \beta^{**} \in R^1$ such that

$$(I_1(\alpha^*), I_2(\beta^*)), (I_1(\alpha^{**}), I_2(\beta^{**})) \in \omega(x) \text{ and}$$
$$(I_1(\alpha^*), I_2(\beta^*)) \leqslant \omega(x) \leqslant (I_1(\alpha^{**}), I_2(\beta^{**})).$$

Hence, $\alpha^* = \alpha^{**}$ or $\beta^* = \beta^{**}$. Without loss of generality, we assume that $\alpha^* = \alpha^{**}$. If $\beta^* = \beta^{**}$, then the proof is complete. Otherwise, $\beta^* < \beta^{**}$. We next distinguish two cases to finish the proof.

Case 1: $\alpha^* > \beta^*$.

Choose $\alpha' \in R^1$ such that $\beta^* < \alpha' < min\{\alpha^*, \beta^{**}\}$. Then

$$(I_1(\alpha'), I_2(\alpha')) \ll (I_1(\alpha^{**}), I_2(\beta^{**})) \in \omega(x).$$

By the definition of $\omega(x)$ and the fact that the semiflow Φ is sup-pseudo monotone with respect to $\sum$, we have

$$(I_1(\alpha'), I_2(\alpha')) \leqslant \omega(x),$$

a contradiction to the choice of (α^*, β^*).

Case 2: $\alpha^* \leqslant \beta^*$.

Choose $\alpha \in R^1$ such that $\beta^* < \alpha' < \beta^{**}$. Then

$$(I_1(\alpha^*), I_2(\beta^*)) \ll (I_1(\alpha'), I_2(\alpha')).$$

Thus, we have, by the definition of $\omega(x)$ and the fact that the semiflow Φ is sub-pseudo monotone with respect to $\sum$,

$$\omega(x) \leqslant (I_1(\alpha'), I_2(\alpha')).$$

This is a contradiction to the choice of $(\alpha^{**}, \beta^{**})$. The proof is complete. □

Remark 2.6. The result of Theorem 2.3 continues to hold if we replace (H_1) by (H_3) or replace (H_2) by (H_4) in Theorem 2.3.

The following example is given to illustrate that if exactly one of assumptions (H_1) and (H_2) holds, then the result of Theorem 2.3 does not necessarily continue to hold.

Example 2.1. Let $t_{2k} = 2k(k+1)$ and $t_{2k+1} = 2(k+1)^2$, where k is a nonnegative integer. Clearly, $t_0 = 0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{t \longrightarrow +\infty} t_k = +\infty$. Define the function $a : R^1_+ \to R^1$ by setting

$$a(t) = \begin{cases} \frac{t}{(k+1)^2} - \frac{2k}{k+1}, & t_{2k} \leqslant t \leqslant \frac{t_{2k}+t_{2k+1}}{2}, \\ -\frac{t}{(k+1)^2} + 2, & \frac{t_{2k}+t_{2k+1}}{2} \leqslant t \leqslant \frac{t_{2k+1}+t_{2k+2}}{2}, \\ \frac{t}{(k+1)^2} - \frac{2(k+2)}{k+1}, & \frac{t_{2k+1}+t_{2k+2}}{2} \leqslant t \leqslant t_{2k+2}. \end{cases}$$

Then $a(t)$ is continuous on R^1_+ and satisfies that

(i) $0 \leqslant a(t) \leqslant \frac{1}{k+1}$ for $t \in [t_{2k}, t_{2k+1}]$, and $-\frac{1}{k+1} \leqslant a(t) \leqslant 0$ for $t \in [t_{2k+1}, t_{2k+2}]$, where k is a nonnegative integer;

(ii) $\int_{t_{2k}}^{t_{2k+1}} a(s)\,\mathrm{d}s = 1$ and $\int_{t_{2k+1}}^{t_{2k+2}} a(s)\,\mathrm{d}s = -1$, where k is a nonnegative integer;

Define the mappings $f, h, g : R^1 \to R^1$ by

$$f(x) = \begin{cases} x + \frac{1}{e} - 4, & x \leqslant -\frac{1}{e}, \\ -\frac{4}{\sqrt{-\ln(-x)}}, & -1 < x < 0, \\ 0, & 0 \leqslant x \leqslant 1, \\ x - 1, & x \geqslant 1; \end{cases}$$

$$g(x) = \begin{cases} a(0), & x \leqslant -1, \\ a(-\ln(-x)), & -1 < x < 0, \\ f(x), & x \geqslant 0; \end{cases}$$

and

$$h(x) = \begin{cases} x, & x \leqslant 0, \\ 0, & 0 \leqslant x \leqslant 1, \\ x - 1, & x \geqslant 1. \end{cases}$$

We can observe that $f, g, h \in C(R^1)$, f and h are nondecreasing, $g \geqslant f$, and for any $x \in R^1$, there exist $\varepsilon > 0$ and $L > 0$ such that $-f(y) + f(x) \geqslant -L(y - x)$ for any $y \in [x, x + \varepsilon]$.

Let us now consider the following system:

$$\begin{cases} x_1'(t) = -f(x_1(t)) + g(x_2(t-1)), \\ x_2'(t) = -h(x_2(t)) + h(x_1(t-1)). \end{cases} \tag{2.1}$$

In this example, for the sake of convenience, we introduce the following notation:

Let $X_1 = X_2 = C([-1, 0], R^1)$ be the Banach spaces equipped with supremun norms, and define $X_1^+ = X_2^+ = C([-1, 0], R_+^1)$. Then X_i^+ induces a closed partial ordered relation on X_i, where $i = 1, 2$. For any $\alpha \in R^1$, let us define $\widehat{\alpha}(\theta) = \alpha$, $\theta \in [-1, 0]$. It follows that $\widehat{\alpha} \in X_i$. Define $I_i(\alpha) = \widehat{\alpha}$, $\alpha \in R^1$, $i = 1, 2$. Assume that $\varphi \in X = X_1 \times X_2$ and use $x_t(\varphi)$ to denote the solution of (2.1) with the initial data $x_0(\varphi) = \varphi$. Using a similar argument as that of Lemma 3.3 below, we know that $x_t(\varphi)$ exists and is unique on R_+^1. Let $\Phi_t(\varphi) = x_t(\varphi)$, $t \in R_+^1$, $\varphi \in X$. Then Φ is a semiflow on X.

Now we want to show that Φ actually satisfies (H_1). For that purpose, we will first prove the following several claims.

Claim 1. *If* $(\alpha, \beta) \in D_f$ *and* $\varphi \in X$ *with* $(I_1(\alpha), (I_2(\beta)) \leqslant \varphi$, *then*

$$(I_1(\alpha), I_2(\beta)) \leqslant x_t(\varphi) \quad \textit{for} \quad t \in R_+^1.$$

Note that $D_f = D_h$. It is easily verified that Claim 1 is true.

Claim 2. *If $\alpha < 0$ and $\varphi \in X$ with $(I_1(\alpha), I_2(\alpha)) < \varphi$, then*

$$(I_1(\alpha), I_2(\alpha)) \ll x_t(\varphi) \quad \textit{for} \quad t \geqslant 3.$$

Indeed, we may assume that there exists $\theta_1 \in (-1, 0]$ such that $\varphi_1(\theta_1) > \alpha$. Let $t_1 = 1 + \theta_1$. Then $x_2(t_1, \varphi) > \alpha$. Otherwise, by Claim 1, $x_2(t_1, \varphi) = \alpha$. It follows from Claim 1 that $x_2'(t_1, \varphi) = 0$. On the other hand, from (2.1), we have

$$\begin{aligned} x_2'(t_1, \varphi) &= -h(\alpha) + h(\varphi_1(\theta_1)) \\ &> -h(\alpha) + h(\alpha) = 0, \end{aligned}$$

which yields a contradiction. Thus, from (2.1), we obtain

$$\begin{aligned} x_2'(t, \varphi) &= -h(x_2(t, \varphi)) + h(x_1(t-1, \varphi)) \\ &\geqslant -h(x_2(t, \varphi)) + h(\alpha) \\ &\geqslant -(x_2(t, \varphi) - \alpha). \end{aligned}$$

It follows that

$$x_2(t, \varphi) \geqslant \alpha + (x_2(t_1, \varphi) - \alpha)e^{t-t_1} \quad \text{for all} \quad t \geqslant t_1.$$

Hence, $x_2(t, \varphi) > \alpha$ for all $t \geqslant t_1$.

We will show that $x_1(t, \varphi) > \alpha$ for all $t \geqslant t_1 + 1$. Otherwise, $t_2 = \inf\{t \geqslant t_1 + 1 : x_1(t, \varphi) = \alpha\} < +\infty$. Using a similar argument as above, we can know that $x_1(t_1 + 1, \varphi) > \alpha$. Thus, we obtain that $t_2 > t_1 + 1$, $x_1(t_2, \varphi) = \alpha$ and $x_1'(t_2, \varphi) = 0$. Again from (2.1), we have

$$g(x_2(t_2 - 1, \varphi)) = f(x_1(t_2, \varphi)) = f(\alpha).$$

Since $x_2(t_2 - 1, \varphi) > \alpha$, it follows that $g(x_2(t_2 - 1, \varphi)) > f(\alpha)$, which yields a contradiction. Therefore, the Claim 2 is true.

Claim 3. *If $\alpha > 1$ and $\varphi \in X$ with $(I_1(\alpha), I_2(\alpha)) < \varphi$, then $x_4(\varphi) \gg (I_1(\alpha), I_2(\alpha))$.*

Claim 4. *If $\alpha, \beta \in [0, 1]$ and $\varphi \in X$ with $(I_1(\alpha), I_2(\beta)) \leqslant \varphi$, then one of the following holds*:

(i) $(I_1(\alpha), I_2(\beta)) = x_4(\varphi)$;
(ii) $x_4(\varphi) \gg (I_1(\alpha), I_2(\beta))$;
(iii) $x_1(t, \varphi) = \alpha$ *for* $t \in [3, 4]$, *and* $x_2(t, \varphi) > \beta$ *for* $t \in [3, 4]$;
(iv) $x_1(t, \varphi) > \alpha$ *for* $t \in [3, 4]$, *and* $x_2(t, \varphi) = \beta$ *for* $t \in [3, 4]$.

Moreover, we have the following:

(i) *If* $\alpha = 1$ *and* $\varphi_1(\theta) > 1$ *for all* $\theta \in [-1, 0]$, *then* $x_4(\varphi) \gg (I_1(\alpha), I_2(\beta))$;
(ii) *If* $\beta = 1$ *and* $\varphi_2(\theta) > 1$ *for all* $\theta \in [-1, 0]$, *then* $x_4(\varphi) \gg (I_1(\alpha), I_2(\beta))$.

Remark 2.7. Arguing as that in the proof of Lemma 3.4 below, we can prove the Claims 3 and 4.

From the above Claims 1–4, we can know that Φ satisfies (H_1) but does not satisfy (H_2). In fact, let

$$x_1(t) = \begin{cases} \int_0^t a(s)\,\mathrm{d}s, & t \geqslant 0, \\ 0, & -1 \leqslant t \leqslant 0 \end{cases}$$

and

$$x_2(t) = -e^{-t-1}, \qquad t \geqslant -1.$$

Then we can verify that $x(t) = (x_1(t), x_2(t))$ satisfies (2.1). Since $\lim_{k\to+\infty} x_1(t_{2k+1}) = 1$ and $\lim_{k\to+\infty} x_1(t_{2k}) = 0$, it follows that $x(t)$ does not tend to a constant vector as $t \longrightarrow \infty$. Therefore, assumption (H_1) cannot guarantee that the result of Theorem 2.3 remains valid.

Let $\mathbf{Z}$ be a topological space endowed with a closed partial ordered relation $R_{\mathbf{Z}} \subseteq \mathbf{Z} \times \mathbf{Z}$. For any $z', z'', z''' \in \mathbf{Z}$ and any subset $A \subseteq \mathbf{Z}$, the following notations will be used: $z' \leqslant z''$ iff $(z', z'') \in R_{\mathbf{Z}}$, $z' < z''$ iff $(z', z'') \in R_{\mathbf{Z}}$ and $z' \neq z''$, $z' \ll z''$ iff $(z', z'') \in Int\, R_{\mathbf{Z}}$, $A \ll z'''$ iff $a \ll z'''$ for $a \in A$, $z''' \ll A$ iff $z''' \ll a$ for $a \in A$, $A \leqslant z'''$ ($A < z'''$) iff $a \leqslant z'''$ ($a < z'''$) for $a \in A$, $z''' \leqslant A$ ($z''' < A$) iff $z''' \leqslant a$ ($z''' < a$) for $a \in A$.

Assume that Φ is a semiflow on $\mathbf{Z}$ and the mapping $I : R^1 \to \mathbf{Z}$ is continuous and satisfies that

$$I(\alpha) \ll I(\beta) \quad \text{for any} \quad \alpha < \beta$$

and that for any $z \in \mathbf{Z}$, there exist $\alpha', \beta' \in R^1$ such that

$$I(\alpha') \leqslant z \leqslant I(\beta').$$

It is further assumed that $\sum_{\mathbf{Z}}$ is a subset of $\mathbf{Z}$ and $I(R^1) \subseteq \sum_{\mathbf{Z}}$.

Definition 2.4. The semiflow Φ is said to be essentially strongly sup-pseudo (sub-pseudo) monotone with respect to $\sum_{\mathbf{Z}}$ if the semiflow Φ is sup-pseudo (sub-pseudo) monotone with respect to $\sum_{\mathbf{Z}}$ and for any $e \in \sum_{\mathbf{Z}}$, there exists $T = T_e > 0$ such that for any $z \in \mathbf{Z}$ with $e \leqslant z$ ($e \geqslant z$), we have either $\Phi_T(z) = e$ or $e \ll \Phi_T(z)$ (either $\Phi_T(z) = e$ or $e \gg \Phi_T(z)$).

Theorem 2.4. *Let the semiflow Φ be essentially strongly sup-pseudo (or sub-pseudo) monotone with respect to $\sum_{\mathbf{Z}}$. Suppose that $z \in \mathbf{Z}$ is a given point such that $O(z)$ is precompact. Then there exists $\alpha^* \in R^1$ such that*

$$\omega(z) = \{I(\alpha^*)\}.$$

Proof. Without loss of generality, we assume that the semiflow Φ is essentially strongly sup-pseudo monotone with respect to $\sum_{\mathbf{Z}}$. Let $X_1 = \mathbf{Z}$, $R_1 = R_2$, $X_2 = R^1$ and $R_2 = \{(\alpha, \beta) \in R^2 : \beta - \alpha \geqslant 0\}$. Also, let $\Psi_t(x_1, x_2) = (\Phi_t(x_1), x_2)$ for $t \in R^1_+$, $x_1 \in X_1$, $x_2 \in X_2$. It follows that Ψ is a semiflow on $X_1 \times X_2$. Let $\sum = \sum_{\mathbf{Z}} \times R^1$. Then the semiflow Ψ is essentially semi-strongly sup-pseudo monotone with respect to $\sum$. Suppose that $I_1(\alpha) = I(\alpha)$ and $I_2(\alpha) = \alpha$, where $\alpha \in R^1$. Let $F \equiv 0$. Then $\widehat{D_F} \subseteq \sum$. Thus, by Theorem 2.1, there exist $\alpha^*, \beta^* \in R^1$ such that

$$\bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \Psi_t(z, 0)} = \{(I(\alpha^*), I(\beta^*))\}.$$

By the definition of Ψ, we have

$$\bigcap_{t \geqslant 0} \overline{\bigcup_{t \geqslant s} \Phi_t(z)} = \{I(\alpha^*)\},$$

that is, $\omega(z) = \{I(\alpha^*)\}$. This completes the proof. $\square$

Theorem 2.4 improves and extends the convergence principle of [7]. To see this, we state the convergence principle of [7] and use Theorem 2.4 to prove it. Suppose that $X \subseteq C(M, R^1)$ has a topology making its inclusion into $C(M, R^1)$ continuous, where M is a compact topological space. For any $u, v \in X$, the following notations will be used: $u \preccurlyeq v$ iff $u(x) \leqslant v(x)$ for any $x \in M$, $u \prec v$ iff $u \preccurlyeq v$ and $u \neq v$, $u \prec\prec v$ iff $u(x) < v(x)$ for any $x \in M$. For any $\alpha \in R^1$, let us define $\widehat{\alpha}(x) = \alpha$, $x \in M$. Let Φ be a semiflow on X. Moreover, we introduce the following assumptions:

(C_1) If $u \in X$ and $\alpha, \beta \in R^1$ with $\widehat{\alpha} \preccurlyeq u \preccurlyeq \widehat{\beta}$, then $\widehat{\alpha} \preccurlyeq \Phi_t(u) \preccurlyeq \widehat{\beta}$ for all $t \geqslant 0$.
(C_2) There exists $T > 0$ such that for any $u \in X$ and $\alpha \in R^1$ with $u \prec \widehat{\alpha}$ ($\widehat{\alpha} \prec u$), we have $\Phi_T(u) \prec\prec \widehat{\alpha}$ ($\widehat{\alpha} \prec\prec \Phi_T(u)$).

Corollary 2.1. *Let (C_1) and (C_2) hold. Then each precompact orbit tends to a constant function.*

Proof. Let $I(\alpha) = \widehat{\alpha}$ for all $\alpha \in R^1$, and $R = \{(u, v) \in X \times X : u(x) \leqslant v(x) \text{ for } x \in M\}$. If $u, v \in X$ with $u \prec\prec v$, then $(u, v) \in Int\, R$, since $X \subseteq C(M, R^1)$ has a topology making its inclusion into $C(M, R^1)$ continuous, where $Int\, R$ denotes the interior of R in $X \times X$. It follows from assumptions (C_1) and (C_2) that Φ is a essentially strongly pseudo monotone semiflow on X. Thus, by Theorem 2.4, we can conclude that the conclusion of Corollary 2.1 is true.

Remark 2.8. In fact, if exactly one of assumptions (C_1) and (C_2) is satisfied, then the conclusion of Corollary 2.1 continues to hold. We refer to [7] for a detailed description of the applications of Corollary 2.1 to neutral functional differential equation and semilinear parabolic partial differential equation with Neumamn boundary condition.

Remark 2.9. Let J be a subinterval of R^1 such as [0,1], [0,1) and so forth. We assume that the map $I_i : J \to X_i$ is continuous and satisfies that $I_i(\alpha_i) \ll I_i(\alpha_i')$ for all $\alpha_i' > \alpha_i$ and that for any $x_i \in X_i$, there exist $\alpha_i, \alpha_i' \in J$ such that $I_i(\alpha_i) \leqslant x_i \leqslant I_i(\alpha_i')$, where $i = 1, 2$. Let $F : R^1 \to R^1$ be continuous and nondecreasing. Also, let

$$\begin{aligned} s_F^J(\alpha) &= \sup\{\beta \in J : F(\beta) = F(\alpha)\}, \\ i_F^J(\alpha) &= \inf\{\beta \in J : F(\beta) = F(\alpha)\}, \\ D_F^J &= \{(\alpha, \beta) \in J \times J : F(\alpha) = F(\beta)\}, \\ \widehat{D_F^J} &= \{(I_1(\alpha), I_2(\beta)) \in X : (\alpha, \beta) \in D_F\}. \end{aligned}$$

Assume that $\widehat{D_F^J} \subseteq \sum \subseteq X$. If $s_F(\alpha_i)$, $i_F(\alpha_i)$ and D_F in (H_1)–(H_4) are replaced by the above $s_F^J(\alpha_i)$, $i_F^J(\alpha_i)$ and D_F^J, respectively, then the results of Lemmas 2.1–2.2 and Theorems 2.1–2.3 continue to hold. Clearly, Theorem 2.4 can also be improved in a similar way.

3. Applications to delay differential equations

As some applications of the convergence principles in Section 2, we consider several systems of delay differential equations.

3.1. Consider the following system of delay differential equations

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -F_1(x_1(t)) + F_1(x_2(t - r_2)), \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -F_2(x_2(t)) + F_2(x_1(t - r_1)), \end{cases} \tag{3.1}$$

where $r_1, r_2 > 0$ are constants and $F_1, F_2 \in C(R^1)$ is nondecreasing.

System (3.1) can be used to model a compartmental system with two pipes (see [6]). Let $\tau = \min\{r_1, r_2\}$ and $r = \max\{r_1, r_2\}$.

Lemma 3.1. *Let $F \in C(R^1)$ be nondecreasing on R^1. For any constants K, t_0 and x_0, the initial value problem*

$$\begin{cases} x'(t) = -F(x(t)) + K, \\ x(t_0) = x_0 \end{cases} \tag{3.2}$$

exists a unique solution $x(t, t_0, x_0)$ on $[t_0, \infty)$.

Proof. From the Peano theorem, we know that the solutions of the initial value problem (3.2) locally exist. Again, since F is nondecreasing, it follows from [2] that right-hand solutions of the initial value problem (3.2) are also unique. Hence, $x(t, t_0, x_0)$ exists and is unique on $[t_0, \eta)$ for some positive constant η, where $[t_0, \eta)$ denotes the maximal right-interval of existence of $x(t, t_0, x_0)$. We will show that $\eta = +\infty$. Otherwise, $\eta < +\infty$ and $\overline{\lim}_{t \longrightarrow \eta^-} |x(t, t_0, x_0)| = +\infty$. We next distinguish several cases to finish the proof. $\square$

Case 1: There exists $t_1 \in [t_0, \eta)$ such that $-F(x(t_1, t_0, x_0)) + K = 0$. Let

$$\widetilde{x}(t) = \begin{cases} x(t, t_0, x_0) & \text{for } t_0 \leqslant t \leqslant t_1, \\ x(t_1, t_0, x_0) & \text{for } t \geqslant t_1. \end{cases}$$

It follows that $\widetilde{x}(t)$ satisfies (3.2) and hence, $x(t, t_0, x_0) \equiv \widetilde{x}(t)$, which contradicts $\eta < +\infty$.

Case 2: $-F(x(t, t_0, x_0)) + K < 0$ for $t \in [t_0, \eta)$. Then $x(t, t_0, x_0)$ is strictly decreasing on $[t_0, \eta)$ and thus, $x(t, t_0, x_0) \leqslant x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. It follows that $-F(x(t, t_0, x_0)) + K \geqslant -F(x(t_0, t_0, x_0)) + K$ for all $t \in [t_0, \eta)$, and hence, $x(t, t_0, x_0) \geqslant (K - F(x(t_0, t_0, x_0)))t + x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. Therefore, $\lim_{t \longrightarrow \eta^-} |x(t, t_0, x_0)| < +\infty$, which yields a contradiction.

Case 3: $-F(x(t, t_0, x_0)) + K > 0$ for $t \in [t_0, \eta)$. Then $x(t, t_0, x_0)$ is strictly increasing on $[t_0, \eta)$ and thus, $x(t, t_0, x_0) \geqslant x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. It follows that $-F(x(t, t_0, x_0)) + K \leqslant -F(x(t_0, t_0, x_0)) + K$ for all $t \in [t_0, \eta)$, and hence, $x(t, t_0, x_0) \leqslant (K - F(x(t_0, t_0, x_0)))t + x(t_0, t_0, x_0)$ for all $t \in [t_0, \eta)$. Therefore, $\lim_{t \longrightarrow \eta^-} |x(t, t_0, x_0)| < +\infty$, which yields a contradiction.

The proof of the lemma is complete. $\square$

Lemma 3.2. *Let s be a given positive constant, $g \in C([t_0, t_0 + s], R^1)$, $F \in C(R^1)$ and F be nondecreasing on R^1. Then the initial value problem*

$$\begin{cases} x'(t) = -F(x(t)) + d(t), \\ x(t_0) = x_0 \end{cases}$$

exists a unique solution $x(t, t_0, x_0)$ on $[t_0, t_0 + s]$.

Proof. Lemma 3.2 follows by applying the standard technique of differential inequalities and Lemma 3.1. $\square$

Lemma 3.3. *Let $x(t, \varphi)$ be the solution of* (3.1) *with the initial value $\varphi \in C = C([-r, 0], R^2)$. Then $x_t(\varphi)$ exists and is unique on R^1_+.*

Proof. We only need to prove that $x_t(\varphi)$ exists and is unique on $[0, \tau]$. We now show that $x_1(t, \varphi)$ exists and is unique on $[0, \tau]$. Let

$$g_1(t) = F_1(\varphi_2(t - r_2)), \quad t \in [0, \tau].$$

Obviously, $g_1 \in C([0, \tau], R^1)$. From Lemma 3.2, we know that $x_1(t, \varphi)$ exists and is unique on $[0, \tau]$. Similarly, we can show that $x_2(t, \varphi)$ exists and is unique on $[0, \tau]$. The proof is now complete. $\square$

Lemma 3.4. *Let $F_1, F_2 \in C(R^1)$ be nondecreasing on R^1. Then there exists a nondecreasing function $F \in C(R^1)$ such that $D_F = D_{F_1} \bigcap D_{F_2}$. Moreover, we have the following*:

(i) *If $\alpha^* \in R^1$ with $\alpha^* = s_F(\alpha^*)$, then there exists $i \in \{1, 2\}$ such that $\alpha^* = s_{F_i}(\alpha^*)$;*
(ii) *If $\alpha \in R^1$ with $\alpha^* = i_F(\alpha^*)$, then there exists $i \in \{1, 2\}$ such that $\alpha^* = i_{F_i}(\alpha^*)$.*

Proof. Let $F(x) = F_1(x) + F_2(x)$, for $x \in R^1$. It is easily verified that $F \in C(R^1)$ is nondecreasing on R^1 and $D_F = D_{F_1} \bigcap D_{F_2}$. Next, we will show conclusion (i). The proof of conclusion (ii) can be dealt with similarly and thus, it is omitted. Suppose, by contradiction, that there exists $\alpha^* \in R^1$ such that $\alpha^* = s_F(\alpha^*)$, $\alpha^* < s_{F_1}(\alpha^*)$ and $\alpha^* < s_{F_2}(\alpha^*)$. Setting $\beta^* = \min\{s_{F_1}(\alpha^*), s_{F_2}(\alpha^*)\}$, we can conclude from the definitions of s_{F_1} and s_{F_2} that

$$F_1(\alpha^*) = F_1(\beta^*) \quad \text{and} \quad F_2(\alpha^*) = F_2(\beta^*).$$

Hence, $F(\alpha^*) = F(\beta^*)$. But the definition of s_F implies that $F(\alpha^*) < F(\beta^*)$, which yields a contradiction. This completes the proof. $\square$

In this subsection, we introduce the following notation:

Let $C_1 = C([-r_1, 0], R^1)$ and $C_2 = C([-r_2, 0], R^1)$ be the Banach spaces equipped with supremun norms, and define $C_1^+ = C([-r_1, 0], R^1_+)$ and $C_2^+ = C([-r_2, 0], R^1_+)$. Then C_i^+ induces a closed partial ordered relation on C_i, where $i = 1, 2$. Define $I_i : R^1 \longrightarrow C_i$ by setting $I_i(\alpha)(\theta) = \alpha, \alpha \in R^1$, $\theta \in [-r_i, 0]$, $i = 1, 2$. Assume that $\varphi \in C = C_1 \times C_2$ and use $x_t(\varphi)$ to denote the solution of (3.1) with the initial data $x_0(\varphi) = \varphi$. By Lemma 3.3, we know that $x_t(\varphi)$ exists and is unique on R^1_+. Let $\Phi_t(\varphi) = x_t(\varphi)$, $t \in R^1_+$, $\varphi \in C$. Then Φ is a semiflow on C.

Define

$$D = \{(\alpha, \beta) \in R^2 : F_i(\alpha) = F_i(\beta), i = 1, 2\} \quad \text{and} \quad \widehat{D} = \{\hat{x} \in C : x \in D\}.$$

By Lemma 3.4, we know that $D = D_F$ and $\widehat{D} = \widehat{D_F}$.

To proceed further, we assume the following hypotheses are satisfied:

(C_1) For any $\alpha \in R^1$, there exist $\varepsilon > 0$ and $L > 0$ such that $-F_i(x)+F_i(\alpha) \geqslant -L(x-\alpha)$ for any $x \in [\alpha, \alpha+\varepsilon]$, where $i = 1, 2$.

(C_2) For any $\alpha \in R^1$, there exist $\varepsilon > 0$ and $L > 0$ such that $-F_i(x)+F_i(\alpha) \leqslant -L(x-\alpha)$ for any $x \in [\alpha-\varepsilon, \alpha]$, where $i = 1, 2$.

Lemma 3.5. *Let $\varphi \in C$ and $d \in D$ with $\varphi \geqslant \hat{d}$. Then $x_t(\varphi) \geqslant \hat{d}$ for all $t \geqslant 0$. Furthermore, we have one of the following*:

(i) $x_t(\varphi) = \hat{d}$ *for* $t \geqslant 5r$;
(ii) $x_t(\varphi) \gg \hat{d}$ *for* $t \geqslant 5r$;
(iii) $x_1(x, \varphi) > d_1$ *and* $x_2(t, \varphi) = d_2$ *for* $t \geqslant 5r$;
(iv) $x_2(t, \varphi) = d_1$ *and* $x_2(t, \varphi) > d_2$ *for* $t \geqslant 5r$.

Proof. Since F_1 and F_2 are nondecreasing, it follows from [19, Proposition 1.1] that

$$x_t(\varphi) \geqslant \hat{d} \quad \text{for all} \quad t \geqslant 0.$$

We next distinguish four cases to finish the proof.

Case 1: $x_t(\varphi) = \hat{d}$ for any $t \in [0, 4r]$. Then, we have $x_t(\varphi) \equiv \hat{d}$ for all $t \geqslant r$.

Case 2: $x_1(t, \varphi) = d_1$ for any $t \in [0, 4r]$ and $x_2(t_2, \varphi) > d_2$ for some $t_2 \in [0, 4r]$.

From (3.1) and the above discussion, we obtain

$$\begin{aligned} \frac{\mathrm{d}x_2(t, \varphi)}{\mathrm{d}t} &= -F_2(x_2(t, \varphi)) + F_2(x_1(t - r_1, \varphi)) \\ &\geqslant -F_2(x_2(t, \varphi)) + F_2(d_1) \\ &= -F_2(x_2(t, \varphi)) + F_2(d_2). \end{aligned}$$

Now, we will prove that $x_2(t, \varphi) > d_2$ for all $t \geqslant t_2$. Otherwise, $t_3 = \inf\{t \geqslant t_2 : x_2(t, \varphi) = d_2\} < +\infty$. Hence, $t_3 > t_2$ and $x_2(t_3, \varphi) = d_2$. By assumption (C_1), there exist $\delta > 0$ and $L > 0$ such that $t_3 - \delta > t_2$ and $-F_2(x_2(t, \varphi)) + F_2(d_2) \geqslant -L(x_2(t, \varphi) - d_2)$ for all $t \in [t_3 - \delta, t_3]$. So, we have $x_2(t_3, \varphi) \geqslant d_2 + (x(t_3 - \delta) - d_2)e^{-L\delta}$. Therefore, $x_2(t_3, \varphi) > d_2$, which yields a contradiction.

Next, we will show that $x_1(t, \varphi) = d_1$ for $t \in [0, 4r + \tau]$. Indeed, from (3.1), it follows that

$$x_2'(t, \varphi) = -F_2(x_2(t, \varphi)) + F_2(d_2) \text{ for } t \in [r_1, 4r].$$

Thus, $x_2'(t, \varphi) \leqslant 0$ for $t \in [r_1, 4r]$. Again from (3.1), we have

$$x_1'(t, \varphi) = -F_1(x_1(t, \varphi)) + F_1(x_2(t - r_2, \varphi)) \quad \text{for} \quad t \geqslant 0.$$

It follows that

$$F_1(d_1) = F_1(x_2(t - r_2, \varphi)) \text{ for } \quad t \in [0, 4r].$$

Thus,

$$F_1(x_2(t, \varphi)) \leqslant F_1(d_1) \text{ for } \quad t \in [0, 4r].$$

Therefore, from (3.1), we obtain

$$x_1'(t, \varphi) \leqslant -F(d_1) + F_1(x_2(t - r_2, \varphi)) \text{ for } \quad t \in [r_2, 4r + r_2],$$

that is,

$$x_1'(t, \varphi) \leqslant 0 \quad \text{for} \quad t \in [r_2, 4r + r_2].$$

Hence, from $x_t(\varphi) \geqslant \hat{d}$ and $x_1(r_2, \varphi) = d_1$, we have

$$x_1(t, \varphi) = d_1 \quad \text{for } t \in [r_2, 4r + r_2].$$

Therefore,

$$x_1(t, \varphi) = d_1 \quad \text{for} \quad t \in [0, 4r + \tau].$$

So, by induction, we get $x_1(t, \varphi) = d_1$ for all $t \geqslant 0$, and thus, conclusion (iv) is established.

Case 3: $x_1(t_1, \varphi) > d_1$ for some $t_1 \in [0, 4r]$ and $x_2(t, \varphi) = d_2$ for all $t \in [0, 4r]$.

Using a similar argument as that of Case 2, we can prove that conclusion (iii) is true.

Case 4: $x_1(t_1 \varphi) > d$ and $x_2(t_2, \varphi) > d_2$ for some $t_1, t_2 \in [0, 4r]$.

Using a similar argument as that of Case 2, we can prove that conclusion (ii) is true. □

Arguing as in the proof of Lemma 3.5, we can get the following result:

Lemma 3.6. *Let $\varphi \in C$ and $d \in D$ with $\varphi \leqslant \hat{d}$. Then $x_t(\varphi) \leqslant \hat{d}$ for all $t \geqslant 0$. Furthermore, we have one of the following*:

(i) $x_t(\varphi) = \hat{d}$ *for* $t \geqslant 5r$;
(ii) $x_t(\varphi) \ll \hat{d}$ *for* $t \geqslant 5r$;
(iii) $x_1(x, \varphi) < d_1$ *and* $x_2(t, \varphi) = d_2$ *for* $t \geqslant 5r$;
(iv) $x_2(t, \varphi) = d_1$ *and* $x_2(t, \varphi) < d_2$ *for* $t \geqslant 5r$.

Lemma 3.7. *Suppose that $A \subseteq C$ is a compact subset such that $x_t(A) = A$ for $t \geqslant 0$. Let $(\alpha^*, \beta^*) \in D_F$ with $\widehat{(\alpha^*, \beta^*)} \leqslant A$. Then we have the following*:

(i) *If $A = \{\varphi \in A : \alpha^* < \varphi_1(\theta)$ for any $\theta \in [-r, 0]\}$ and $\alpha^* = s_F(\alpha^*)$, then there exists $\varphi^* \in A$ such that $\widehat{(\alpha^*, \beta^*)} \ll \varphi^*$;*

(ii) *If $A = \{\varphi \in A : \beta^* < \varphi_2(\theta)$ for any $\theta \in [-r, 0]\}$ and $\beta^* = s_F(\alpha^*)$, then there exists $\varphi^* \in A$ such that $\widehat{(\alpha^*, \beta^*)} \ll \varphi^*$.*

Proof. We will only prove conclusion (i). The proof of conclusion (ii) is similar. By Lemma 3.4, there exists some $i \in \{1, 2\}$ such that

$$\alpha^* = s_{F_i}(\alpha^*).$$

We next distinguish two cases to finish the proof.

Case 1: $\alpha^* = s_{F_2}(\alpha^*)$.

Let $\varphi \in A$ and $x_i(t) = x_i(t, \varphi), i \in \{1, 2\}$. By the invariance of A, we have that

$$x_1(t) > \alpha^* \quad \text{for all} \quad t \geqslant -r_1.$$

From (3.1), one obtains

$$\begin{aligned} x_2'(t) &= -F_2(x_2(t)) + F_2(x_1(t - r_1)) \\ &> -F_2(x_2(t)) + F_2(\alpha^*) \\ &= F_2(x_2(t)) + F_2(\beta^*). \end{aligned}$$

Hence, $x_2(t) > \beta^*$ for $t \geqslant 0$. Therefore, we obtain $x_r(\varphi) \gg \widehat{(\alpha^*, \beta^*)}$.

Case 2: $\alpha^* = s_{F_1}(\alpha^*)$ and $\alpha^* < s_{F_2}(\alpha^*)$. Suppose that conclusion (i) is not true. Then, by Lemma 3.5 and the invariance of A, we have

$$\varphi_2(\theta) = \beta^* \quad \text{for} \quad \theta \in [-r_2, \theta] \quad \text{and} \quad \varphi \in A.$$

Let $\alpha^{**} = sup\{\varphi_1(\theta) : \varphi \in A,\ \theta \in [-r_1, 0]\}$. By the invariance and compactness of A, there exists φ^{**} such that $\alpha^{**} = \varphi_1^{**}(0)$. Again, by the invariance of A, there exists $\varphi \in A$ such that $x_r(\varphi) = \varphi^{**}$. Let $y_i(t) = x_i(t, \varphi), i = 1, 2$. Then, by the Fermat's theorem, we get $y_1'(r) = 0$. From (3.1), it follows that $-F_1(y_1(r)) + F_1(y_2(r - r_2)) = 0$. That is, $F_1(\beta^*) = F_1(y_1(r))$. That is, $F_1(\alpha^*) = F_1(y_1(r))$. On the other hand, $y_1(r) = \varphi_1^{**}(0) = \alpha^{**} > \alpha^*$, which contradicts the choice of α^*. This completes the proof. □

Using a similar argument as that in the proof of Lemma 3.7, we can obtain the following:

Lemma 3.8. *Suppose that* $A \subseteq C$ *is a compact subset such that* $x_t(A) = A$ *for* $t \geqslant 0$. *Let* $(\alpha^*, \beta^*) \in D_F$ *with* $\widehat{(\alpha^*, \beta^*)} \geqslant A$. *Then we have the following*:

(i) *If* $A = \{\varphi \in A : \alpha^* > \varphi_1(\theta)$ *for any* $\theta \in [-r, 0]\}$ *and* $\alpha^* = i_F(\alpha^*)$, *then there exists* $\varphi^* \in A$ *such that* $\widehat{(\alpha^*, \beta^*)} \gg \varphi^*$;

(ii) *If* $A = \{\varphi \in A : \beta^* > \varphi_2(\theta)$ *for any* $\theta \in [-r, 0]\}$ *and* $\beta^* = i_F(\alpha^*)$, *then there exists* $\varphi^* \in A$ *such that* $\widehat{(\alpha^*, \beta^*)} \gg \varphi^*$.

Theorem 3.1. *Let* $\varphi \in C$. *Then there exist* $\alpha^*, \beta^* \in R^1$ *such that* $\lim_{t\to\infty} x(t, \varphi) = (\alpha^*, \beta^*)$.

Proof. Let Φ be the solution semiflow generated by system (3.1). By Lemmas 3.5 and 3.6, we know that all orbits of Φ are bounded, and are thus precompact. Lemmas 3.5–3.8 implies that assumptions (H_1) and (H_2) are satisfied. It then from Theorem 2.3 that Theorem 3.1 holds. This completes the proof. $\square$

3.2. Consider a class of so-called pseudo cooperative and irreducible systems. More precisely, we consider the following system:

$$x'(t) = f(x_t), \tag{3.3}$$

where $f \in C(U, R^n)$, $U \subseteq C([-r, 0], R^n)$, $r > 0$.

In this subsection, we introduce the following notation. Let $C = C([-r, 0], R^n)$ be the Banach space endowed with the usual supremum norm. Define $C_+ = C([-r, 0], R^n_+)$, where R^n_+ denotes the set of all nonnegative vectors in R^n. For $x \in R^n$, we write $\widehat{x}$ for the element of C satisfying $\widehat{x}(\theta) = x$, $\theta \in [-r, 0]$. We tacitly assume that the initial value problem (3.3) globally exists a unique solution, denoted by $x_t(\varphi)(x(t, \varphi))$, satisfying $x_0(\varphi) = \varphi \in U$. Set $N = \{1, 2, \ldots, n\}$. For any $x, y \in R^n$, the following notations will be used: $x \leqslant y$ iff $y - x \in R^n_+$, $x < y$ iff $x \leqslant y$ and $x \neq y, x \ll y$ iff $y - x \in Int\, R^n_+$. For any $\varphi, \psi \in C, \varphi \leqslant \psi$ iff $\psi - \varphi \in C_+, \varphi < \psi$ iff $\varphi \leqslant \psi$ and $\varphi \neq \psi, \varphi \ll \phi$ iff $\psi - \varphi \in Int\, C_+$. Let $E_+ = \{\hat{x} \in U : f(\hat{x}) \geqslant 0\}$ and $E_- = \{\hat{x} \in U : f(\hat{x}) \leqslant 0\}$. It is easy to observe that $E_+ \cap E_-$ is the set of equilibria of system (3.3).

Assume that $\hat{e} \in E_+$, we introduce the following assumptions:

(P_e^+) If $\varphi \in U$ with $\varphi \geqslant \hat{e}$, then $f_i(\varphi) \geqslant \alpha_i(\varphi)(\varphi_i(0) - e_i)$, where $i \in N$ and $\alpha_i : U \to R^1$ is continuous.

(I_e^+) Assume that $\varphi \in U$ with $\varphi \geqslant \widehat{e}$. Denote $D^+ = \{i \in N : \varphi_i(\theta) > e_i,\ \theta \in [-r, 0]\}$ and $D = \{i \in N : \varphi_i(\theta) = e_i,\ \theta \in [-r, 0]\}$. If $D^+ \bigcup D = N, D^+ \neq \phi$ and $D \neq N$, then there exists $i \in N \setminus D^+$ such that $f_i(\varphi) > 0$.

Assume that $\hat{e} \in E_-$, then we make the following assumptions:

(P_e^-) If $\varphi \in U$ with $\varphi \leqslant \hat{e}$, then $f_i(\varphi) \leqslant \alpha_i(\varphi)(\varphi_i(0) - e_i)$, where $i \in N$ and $\alpha_i : U \to R^1$ is continuous.

(I_e^-) Assume that $\varphi \in U$ with $\varphi \leqslant \widehat{e}$. Denote $D^+ = \{i \in N : \varphi_i(\theta) < e_i,\ \ \theta \in [-r, 0]\}$ and $D = \{i \in N : \varphi_i(\theta) = e_i,\ \ \theta \in [-r, 0]\}$. If $D^+ \bigcup D = N, D^+ \neq \phi$ and $D \neq N$, then there exists $i \in N \setminus D^+$ such that $f_i(\varphi) < 0$.

Lemma 3.9. *Let $\hat{e} \in E_+$ and (P_e^+) hold. If $\varphi \in U$ with $\varphi \leqslant \widehat{e}$, then $x_t(\varphi) \leqslant \widehat{e}$ for all $t \geqslant 0$. Moreover, if $\varphi_i(0) > e_i$ for some $i \in N$, then $x_i(t, \varphi) > e_i$ for all $t \geqslant 0$.*

Proof. From (P_e^+) and Remark 2.1, Chapter 5 of Smith [21], we obtain that $x_t(\varphi) \geqslant \hat{e}$ for $t \geqslant 0$. Again, from (P_e^+), we get

$$f_i(x_t(\varphi)) \geqslant \alpha_i(x_t(\varphi))(x_i(t, \varphi) - e_i) \quad \text{for} \quad t \geqslant 0.$$

Thus, from (3.3), it follows that

$$\frac{\mathrm{d}(x_i(t, \varphi) - e_i))}{\mathrm{d}t} \geqslant \alpha_i(x_t(\varphi))(x_i(t, \varphi) - e_i) \quad \text{for} \quad t \geqslant 0.$$

Therefore,

$$(x_i(t, \varphi) - e_i) \geqslant e^{\int_0^t \alpha_i(x_s(\varphi))\,\mathrm{d}s}(\varphi_i(0) - e_i) > 0 \quad \text{for} \quad t \geqslant 0,$$

that is,

$$x_i(t, \varphi) > e_i \quad \text{for} \quad t \geqslant 0.$$

This completes the proof. $\square$

Lemma 3.10. *Let $\hat{e} \in E_+$ and assume that (P_e^+) and (I_e^+) are satisfied. If $\varphi \in U$ with $\varphi \geqslant \hat{e}$, then either*

$$x_t(\varphi) \gg \hat{e} \quad \textit{for} \quad t \geqslant (n+2)r$$

or

$$x_t(\varphi) = \hat{e} \quad \textit{for} \quad t \geqslant (n+2)r.$$

Proof. We distinguish two cases to finish the proof.

Case 1: $x(t, \varphi) = e$ for all $t \in [0, r]$.

It follows that $f(\widehat{e}) = 0$. Hence, $x_t(\varphi) = \widehat{e}$ for $t \geqslant r$.

Case 2: $x(t_1, \varphi) > e$ for some $t_1 \in [0, r]$.

Let $M_t = \{i \in N : x_i(t, \varphi) > e_i\}$, $t \geqslant 0$. It follows that $M_{t_1} \neq \phi$. Thus, by Lemma 3.9, it follows that

$$M_s \subseteq M_t, \quad 0 \leqslant s \leqslant t.$$

Claim. *If* $t^* \in R^1_+$ *and* $M_{t^*} \notin \{\phi, N\}$, *then* $M_{t^*} \neq M_{t^*+r}$.

If the claim is not true, then $M_t = M_{t^*}$ for all $t \in [t^*, t^*+r]$. It follows from (I_e^+) that there exists $i \in N \setminus M_{t^*+r}$ such that $f_i(x_{t^*+r}(\varphi)) > 0$. Thus, from (3.3), we get

$$x_i'(t^*+r, \varphi) = f_i(x_{t^*+r}(\varphi)) > 0.$$

Therefore, there exists $\varepsilon > 0$ such that

$$\frac{\mathrm{d}(x_i(t,\varphi) - e_i)}{\mathrm{d}t} > 0 \quad \text{for} \quad t \in [t^*+r-\varepsilon, t^*+r].$$

Since $x_t(\varphi) \geqslant \widehat{e}$ for any $t \geqslant 0$, we have $x_i(t^*+r, \varphi) > e_i$. So, it follows that $i \in M_{t^*+r}$, which yields a contradiction. This completes the proof of the claim.

Now, we will show that $M_{t_1+(n-1)r} = N$. Otherwise, by the above claim, we have

$$\phi \neq M_{t_1} \subseteq M_{t_1+r} \subseteq \cdots \subseteq M_{t_1+(n-1)r} \subseteq M_{t_1+nr} \text{ and } M_{t_1+ir} \neq M_{t_1+(i-1)r},$$
$$i = 1, 2, \ldots, n.$$

But this contradicts $M_t \subseteq N$ for $t \geqslant 0$. This completes the proof. $\square$

Arguing as in the proof of Lemma 3.10, we can get the following result:

Lemma 3.11. *Let* $\hat{e} \in E_-$ *and assume that* (P_e^-) *and* (I_e^-) *are satisfied. If* $\varphi \in U$ *with* $\varphi \leqslant \hat{e}$, *then either*

$$x_t(\varphi) \ll \hat{e} \quad \textit{for} \quad t \geqslant (n+2)r$$

or

$$x_t(\varphi) = \hat{e} \quad \textit{for} \quad t \geqslant (n+2)r.$$

Assume that the mapping $I : R^1 \to U$ *is continuous and satisfies that*

(i) $I(\alpha) \ll I(\beta)$, *for* $\alpha < \beta$;
(ii) *For any* $\varphi \in U$, *there exist* $\alpha^*, \beta^* \in R^1$ *such that*

$$I(\alpha^*) \leqslant \varphi \leqslant I(\beta^*).$$

Definition 3.1. System (3.3) is said to be sup-pseudo cooperative and irreducible with respect to I if $I(R^1) \subseteq E_+$ and for any $\hat{e} \in I(R^1)$, assumptions (P_e^+) and (I_e^+) are satisfied. System (3.3) is said to be sub-pseudo cooperative and irreducible with respect to I if $I(R^1) \subseteq E_-$ and for any $\hat{e} \in I(R^1)$, assumptions (P_e^-) and (I_e^-) are satisfied.

Theorem 3.2. *Let system* (3.3) *be sup-pseudo* (*sub-pseudo*) *cooperative and irreducible with respect to I. If $\varphi \in U$ is given such that $O(\varphi)$ is precompact, then there exists $\alpha^* \in R^1$ such that*

$$\omega(\varphi) = \{I(\alpha^*)\}.$$

Proof. Without loss of generality, we assume that system (3.3) is sup-pseudo cooperative and irreducible with respect to I. Let $\Phi_t(\varphi) = x_t(\varphi)$, $t \in R^1_+$, $\varphi \in U$. Then, by Lemma 3.10, the semiflow Φ is essentially strongly sup-pseudo monotone with respect to $I(R^1)$. Theorem 3.2 follows immediately from Theorem 2.4. □

Example 3.1. Consider the following compartmental system with three pipes [6]:

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -F_1(x_1(t)) + G_1(x_2(t-r_2)), \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -F_2(x_2(t)) + G_2(x_3(t-r_3)), \\ \dfrac{\mathrm{d}x_3(t)}{\mathrm{d}t} = -F_3(x_3(t)) + G_3(x_1(t-r_1)), \end{cases} \tag{3.4}$$

where r_i is a positive constant, $F_i,\ G_i \in C(R^1)$, and F_i is strictly increasing on R^1, $i = 1, 2, 3$.

Corollary 3.1. *Assume one of the following conditions is satisfied*:

(i) *$G_i \geqslant F_i$ and for any $\alpha \in R^1$, there exists a continuous function $L : [\alpha, \infty) \to R^1_+$ such that $F_i(x) - F_i(\alpha) \leqslant L(x)(x - \alpha)$ for all $x \in [\alpha, \infty)$;*

(ii) *$G_i \leqslant F_i$ and for any $\alpha \in R^1$, there exists a continuous function $L : (-\infty, \alpha] \to R^1_+$ such that $F_i(x) - F_i(\alpha) \geqslant L(x)(x - \alpha)$ for all $x \in (-\infty, \alpha]$.*

Then each bounded solution of system (3.4) *tends to a constant as $t \longrightarrow \infty$.*

Proof. Without loss of generality, we assume that condition (i) is satisfied. Let $r = \max\{r_1, r_2, r_3\}$ and $X = C([-r, 0], R^3)$. Define the mappings $g : X \to R^3$ and $I : R^1 \to X$ as

$$g_i(\varphi) = -F_i(\varphi_i(0)) + G_i(\varphi_{(i+1) \bmod 3}(-r_{(i+1) \bmod 3})), \quad \varphi \in X,$$

and

$$(I(\alpha))(\theta) = (\alpha, \alpha, \alpha), \quad \alpha \in R^1,\ \theta \in [-r, 0].$$

Then, from condition (i), we can see that g is sup-pseudo cooperative and irreducible with respect to I. Therefore our conclusion follows from Theorem 3.2. □

Remark 3.1. If G_i is not strictly increasing for some $i \in \{1, 2, 3\}$, then system (3.4) in Corollary 3.1 is not cooperative and irreducible in the sense of Smith [21].

3.3. Consider the following well-known system of delay differential equations

$$x'(t) = F(x(t), x(t-r)), \tag{3.5}$$

where $r > 0$ is a constant and $F : R^2 \longrightarrow R^1$ is continuous.

System (3.5), based on certain conditions, have been widely studied by many researchers (see, for example, [3,4,6,7]). In this subsection, we introduce the following notations and assumptions. Let $C = C([-r, 0], R^1)$ be the Banach space of continuous mappings from $[-r, 0]$ into R^1, equipped with the usual supremum norm. Define

$$C_+ = C([-r, 0], R^1_+).$$

Then C_+ is an order cone in C, and thus, induces a partial order relation "$\leqslant$", which can be defined as that in Section 3.2. For $\varphi \in C$, by $x_t(\varphi)$ we denote a solution of (3.5) with the initial data $x_0(\varphi) = \varphi$. We assume that $x_t(\varphi)$ exists and is unique on R^1_+ for each $\varphi \in C$.

We need the following assumptions:

(H_+) For $\alpha \in R^1$, $M > 0$, there exist $\varepsilon = \varepsilon(\alpha, M) > 0$ and $L = L(\alpha, M) > 0$ such that $F(x, y) \geqslant -L(x - \alpha)$ for any $x \in [\alpha, \alpha + \varepsilon]$ and $y \in [\alpha, \alpha + M]$.

(H_-) For $\alpha \in R^1$, $M > 0$, there exist $\varepsilon = \varepsilon(\alpha, M) > 0$ and $L = L(\alpha, M) > 0$ such that $F(x, y) \leqslant -L(x - \alpha)$ for any $x \in [\alpha - \varepsilon, \alpha]$ and $y \in [\alpha - M, \alpha]$.

Lemma 3.12. *Let (H_+) hold and assume that $\varphi \in C$ and $\alpha \in R^1$ with $\varphi \geqslant \widehat{\alpha}$. Then either*

$$x_t(\varphi) \gg \widehat{\alpha} \quad \textit{for} \quad t \geqslant 2r$$

or

$$x_t(\varphi) = \widehat{\alpha} \quad \textit{for} \quad t \geqslant 2r.$$

Proof. Define $f : C \to R^1$ as $f(\psi) = F(\psi(0), \psi(-r))$. It then follows from ($H_+$) that for any $\alpha \in R^1$ with $\varphi \geqslant \widehat{\alpha}$ and $\varphi(0) = \alpha$, we obtain $f(\varphi) \geqslant 0$. Hence, by Remark 2.1 in Chapter 5 of Smith [21], we get $x_t(\varphi) \geqslant \widehat{\alpha}$ for all $t \geqslant 0$. We next distinguish two cases to finish the proof.

Case 1: $x(t, \varphi) = \alpha$, $t \in [0, r]$.

For this case, we have $F(\alpha, \alpha) = 0$, and hence $x(t, \varphi) = \alpha$ for all $t \geqslant 0$.

Case 2: $x(t_1, \varphi) > \alpha$ for some $t_1 \in [0, r]$.

We will show that $x(t,\varphi) > \alpha$ for all $t \geqslant t_1$. Otherwise, we have $t_2 = \inf\{t \geqslant t_1 : x(t,\varphi) = \alpha\} < +\infty$. Hence, $t_2 > t_1$ and $x(t_2,\varphi) = \alpha$. By (H_+) and the above discussion, there exist $\varepsilon > 0$ and $L > 0$ such that $t_2 - \varepsilon > t_1$ and

$$F(x(t,\varphi), x(t-r,\varphi)) \geqslant -L(x(t,\varphi) - \alpha) \quad \text{for all} \quad t \in [t_2 - \varepsilon, t_2].$$

From (3.5), we obtain

$$x'(t,\varphi) \geqslant -L(x(t,\varphi) - \alpha) \text{ for all } t \in [t_2 - \varepsilon, t_2].$$

Thus,

$$x(t,\varphi) \geqslant \alpha + (x(t_2 - \varepsilon, \varphi) - \alpha)e^{L(t_2 - t - \varepsilon)}.$$

It follows that

$$x(t_2,\varphi) \geqslant \alpha + (x(t_2 - \varepsilon, \varphi) - \alpha)e^{-L\varepsilon}.$$

Therefore, we obtain $x(t_2,\varphi) > \alpha$, which yields a contradiction. This completes the proof. □

Arguing as in the proof of Lemma 3.12, we can get the following result:

Lemma 3.13. *Let (H_-) hold and assume that $\varphi \in C$ and $\alpha \in R^1$ with $\varphi \leqslant \widehat{\alpha}$. Then either*

$$x_t(\varphi) \ll \widehat{\alpha} \quad \textit{for} \quad t \geqslant 2r$$

or

$$x_t(\varphi) = \widehat{\alpha} \quad \textit{for} \quad t \geqslant 2r.$$

Theorem 3.3. *If either (H_+) or (H_-) holds, then each bounded solution of system* (3.5) *tends to a constant as $t \longrightarrow \infty$.*

Proof. Without loss of generality, we may assume that (H_+) holds. Then by Lemma 3.11, the semiflow generated by (3.5) satisfies the conditions of Theorem 2.4, and thus the conclusion of the theorem is true. □

Example 3.2. As an application of Theorem 3.3, we consider the following scalar delay differential equation:

$$x'(t) = -F(x(t)) + G(x(t-r)), \tag{3.6}$$

where r is a positive constant, $F, G \in C(R^1)$, and F is nondecreasing on R^1. In the case where $G \equiv F$, Eq. (3.6) has been used as a model for some population growth, the spread of epidemics, and the dynamics of capital stocks (see [3,4,6] for more details).

Corollary 3.2. *Assume one of the following conditions is satisfied*:

(i) $G \geqslant F$ *and for any* $\alpha \in R^1$, *there exist* $\varepsilon > 0$ *and* $L > 0$ *such that* $-F(x) + F(\alpha) \geqslant -L(x-\alpha)$ *for all* $x \in [\alpha, \alpha+\varepsilon]$;
(ii) $G \leqslant F$ *and for any* $\alpha \in R^1$, *there exist* $\varepsilon > 0$ *and* $L > 0$ *such that* $-F(x) + F(\alpha) \leqslant -L(x-\alpha)$ *for all* $x \in [\alpha-\varepsilon, \alpha]$.

Then each bounded solution of Eq. (3.6) *tends to a constant as* $t \longrightarrow \infty$.

Proof. Without loss of generality, we assume that assumption (i) is satisfied. Clearly, by assumption (i) and the fact that F is nondecreasing, we know that (H_+) holds. Therefore, Theorem 3.3 can then be applied to get the result of the corollary.

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