Linear Algebra and its Applications 437 (2012) 581–588

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Kleiner's theorem for unitary representations of posets

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ARTICLE INFO

Article history:
Received 26 September 2011
Accepted 22 February 2012
Available online 5 April 2012

Submitted by V. Sergeichuk

AMS classification:
15A63
15A21
16G20

Keywords:
Representations of partially ordered sets
Representation-finite type
Kleiner's theorem

ABSTRACT

A subspace representation of a poset $\mathcal{S} = \{s_1, \ldots, s_t\}$ is given by a system $(V; V_1, \ldots, V_t)$ consisting of a vector space V and its subspaces V_i such that $V_i \subseteq V_j$ if $s_i \prec s_j$. For each real-valued vector $\chi = (\chi_1, \ldots, \chi_t)$ with positive components, we define a unitary χ-representation of $\mathcal{S}$ as a system $(U; U_1, \ldots, U_t)$ that consists of a unitary space U and its subspaces U_i such that $U_i \subseteq U_j$ if $s_i \prec s_j$ and satisfies $\chi_1 P_1 + \cdots + \chi_t P_t = \mathbb{1}$, in which P_i is the orthogonal projection onto U_i.

We prove that $\mathcal{S}$ has a finite number of unitarily nonequivalent indecomposable χ-representations for each weight χ if and only if $\mathcal{S}$ has a finite number of nonequivalent indecomposable subspace representations; that is, if and only if $\mathcal{S}$ contains any of Kleiner's critical posets.

1. Introduction

Kleiner [6] described all partially ordered sets (*posets*) with finite number of nonequivalent indecomposable representations. We extend his description to unitary representations of posets.

The notion of poset representations was introduced by Nazarova and Roiter [11] (see also [2,14]). A *matrix representation* of a finite poset $\mathcal{S} = \{s_1, \ldots, s_t\}$ over a field $\mathbb{F}$ is a block matrix $\mathcal{A} = [A_1 | \ldots | A_t]$ over $\mathbb{F}$. Two representations $\mathcal{A} = [A_1 | \ldots | A_t]$ and $\mathcal{B} = [B_1 | \ldots | B_t]$ are *equivalent* if $\mathcal{A}$ can be reduced to $\mathcal{B}$ by elementary row transformations, elementary column transformations within A_i, and additions of linear combinations of columns of A_i to columns of A_j if $s_i \prec s_j$. The *direct sum* of $\mathcal{A}$ and $\mathcal{B}$ is the representation

$$\mathcal{A} \oplus \mathcal{B} := \left[\begin{array}{cc|cc|c|cc} A_1 & 0 & A_2 & 0 & \ldots & A_t & 0 \\ 0 & B_1 & 0 & B_2 & \ldots & 0 & B_t \end{array}\right].$$

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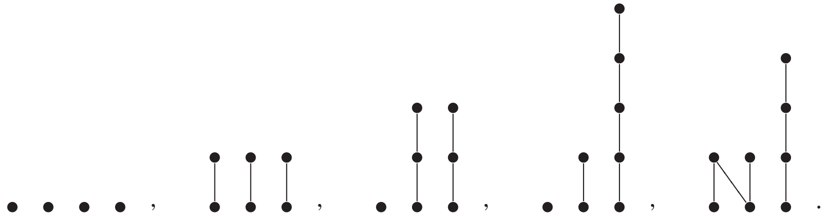
http://dx.doi.org/10.1016/j.laa.2012.02.030

A representation is called *indecomposable* if it is not equivalent to a direct sum of two representations. It is sufficient to classify only indecomposable representations since each representation is equivalent to a direct sum of indecomposable representations, uniquely determined up to isomorphism of summands.

Kleiner [6] (see also [2, Theorem 5.1] and [14, Theorem 10.1]) proved that a poset $\mathcal{S}$ has only a finite number of nonequivalent indecomposable representations if and only if it does not contain a full poset whose Hasse diagram is one of the form



(1)

An equivalent definition of poset representations can be given in terms of subspaces. A *subspace representation* of $\mathcal{S} = \{s_1, \dots, s_t\}$ is a tuple $\mathcal{V} = (V; V_1, \dots, V_t)$, in which V is a vector space over $\mathbb{F}$ and $V_1, \dots, V_t$ are its subspaces such that $V_i \subseteq V_j$ if $s_i \prec s_j$ (that is each representation is a homomorphism from $\mathcal{S}$ to the poset of all subspaces of V). Two subspace representations $\mathcal{V} = (V; V_1, \dots, V_t)$ and $\mathcal{W} = (W; W_1, \dots, W_t)$ are *equivalent* if there exists a linear bijection $g : V \to W$ such that $g(V_i) = W_i$ for all i. For each subspace representation $\mathcal{V} = (V; V_1, \dots, V_t)$, one can construct a matrix representation $\mathcal{A} = [A_1| \dots |A_t]$ in such a way that (i) for each i the columns of all A_j with $s_j \preceq s_i$ generate the subspace V_i and (ii) two subspace representations are equivalent if and only if the corresponding matrix representations are equivalent; see [14, Chapter 3].

From now on, all representations that we consider are over the field $\mathbb{C}$ of complex numbers. By a *unitary representation of dimension d*, we mean a subspace representation $\mathcal{U} = (U; U_1, \dots, U_t)$ in which U is a unitary space of dimension d. Two unitary representations $\mathcal{U} = (U; U_1, \dots, U_t)$ and $\mathcal{V} = (V; V_1, \dots, V_t)$ of a poset $\mathcal{S}$ are *unitarily equivalent* if there exists a unitary bijection $\varphi : U \to V$ such that $\varphi(U_i) = V_i$ for all i. The *orthogonal sum* of unitary representations $\mathcal{U}$ and $\mathcal{V}$ is the unitary representation

$$\mathcal{U} \perp \mathcal{V} := (U \perp V; U_1 \perp V_1, \dots, U_t \perp V_t),$$

in which $U \perp V$ denotes the orthogonal sum of U and V. A unitary representation is called *orthogonally indecomposable* if it is not equivalent to an orthogonal sum of two unitary representations.

Note that the problem of classifying unitary representations is hopeless even for the poset $\mathcal{S} = \{s_1, s_2, s_3 \mid s_1 \prec s_2\}$ since by [10, Theorem 4] it contains the problem of classifying an operator on a unitary space, and hence it contains the problem of classifying any system of operators on unitary spaces [10,13]. The classification becomes possible for a broader class of posets if we impose additional conditions on unitary representations.

We denote the orthogonal projection onto a subspace $M \subset U$ by P_M and the set of positive real numbers by $\mathbb{R}_+$. We say that a unitary representation $\mathcal{U} = (U; U_1, \dots, U_t)$ is a *representation of weight* $\chi = (\chi_1, \dots, \chi_t) \in \mathbb{R}_+^t$ (or χ*-representation*) if

$$\chi_1 P_{U_1} + \dots + \chi_t P_{U_t} = \mathbb{1}; \tag{2}$$

such relations appear in many areas of mathematics, see for example [1,7,9,15,16] and references therein.

Our goal is to prove that Kleiner's theorem holds for χ-representations too:

Theorem 1. *The following conditions are equivalent for each finite poset $\mathcal{S}$ with t elements:*

(i) *For each $\chi \in \mathbb{R}_+^t$, $\mathcal{S}$ has only a finite number of indecomposable unitarily nonequivalent χ-representations.*

(ii) *For each* $\chi \in \mathbb{R}_+^t$ *and* $d \in \mathbb{N}$, $\mathcal{S}$ *has only a finite number of indecomposable unitarily nonequivalent* χ*-representations of dimension d.*

(iii) $\mathcal{S}$ *does not contain a full poset whose Hasse diagram is one of the form* (1).

2. Preliminaries

In what follows we suppose that the elements of a poset $\mathcal{S}$ are numbered from 1 to $|\mathcal{S}|$. A poset is called *primitive* and is denoted by $(t_1, \dots, t_s)$ if it is the disjoint (cardinal) sum of linearly ordered sets of orders t_i. The diagrams (1) and corresponding posets are called *critical*. The poset which corresponds to the last diagram in the list (1) is denoted by $(N, 4)$. To simplify the notation we denote a subspace representation $(V; V_1, \dots, V_t)$ of $\mathcal{S}$ by $(V; V_i)_{i\in\mathcal{S}}$. The similar notation will be used for unitary representations and weights.

A subspace representation $\mathcal{V} = (V; V_i)_{i\in\mathcal{S}}$ is called *schurian* if all its endomorphisms are trivial; that is, the ring $\mathrm{End}(\mathcal{V}) := \{g \in M_{\dim V}(\mathbb{F}) \mid g(V_i) \subseteq V_i,\ i \in \mathcal{S}\}$ is isomorphic to $\mathbb{F}$. Any schurian representation is indecomposable.

Any unitary representation $\mathcal{U} = (U; U_i)_{i\in\mathcal{P}}$ can be viewed as a subspace representation; the forgetful map is denoted by F. If $\mathcal{U}$ is an indecomposable χ-representation, then $F(\mathcal{U})$ is schurian (see [9, Theorem 1]).

Lemma 2. *Let* $P_i, Q_i \in M_n(\mathbb{C})$, $i = 1, \dots, m$ *be orthogonal projections such that*

$$\chi_1 P_1 + \cdots + \chi_m P_m = \chi_1 Q_1 + \cdots + \chi_m Q_m \tag{3}$$

for $(\chi_1, \dots, \chi_m)$ *with positive real* χ_i. *Let there exist a diagonal matrix* $D = \mathrm{diag}(r_1, \dots, r_n)$ *with positive components such that* $P_i D Q_i = D Q_i$ *for all i. Then* $r_1 = \cdots = r_n$ *and* $P_i = Q_i$ *for all i.*

Proof. Write $P_i = [p_{k,l}^{(i)}]$, $Q_i = [q_{k,l}^{(i)}]$, $P_i D Q_i = [t_{k,l}^{(i)}]$, where $t_{k,l}^{(i)} = \sum_{j=1}^n r_j p_{k,j}^{(i)} \overline{q_{l,j}^{(i)}}$. Without losing generality, we may assume that $r_1 = \max\{r_1, \dots, r_n\}$. Since $D \sum_{i=1}^m \chi_i Q_i = \sum_{i=1}^m \chi_i P_i D Q_i$, we have

$$r_1 \sum_{i=1}^m \chi_i q_{1,1}^{(i)} = \sum_{i=1}^m \chi_i \sum_{j=1}^n r_j p_{1,j}^{(i)} \overline{q_{1,j}^{(i)}} \leqslant r_1 \left| \sum_{i=1}^m \chi_i \sum_{j=1}^n p_{1,j}^{(i)} \overline{q_{1,j}^{(i)}} \right|. \tag{4}$$

For

$$x := [\sqrt{\chi_1} p_{1,1}^{(1)}, \dots, \sqrt{\chi_1} p_{1,n}^{(1)}, \dots, \sqrt{\chi_m} p_{1,1}^{(m)}, \dots, \sqrt{\chi_m} p_{1,n}^{(m)}]^T \in \mathbb{C}^{nm},$$
$$y := [\sqrt{\chi_1} q_{1,1}^{(1)}, \dots, \sqrt{\chi_1} q_{1,n}^{(1)}, \dots, \sqrt{\chi_m} q_{1,1}^{(m)}, \dots, \sqrt{\chi_m} q_{1,n}^{(m)}]^T \in \mathbb{C}^{nm},$$

we have

$$(x, y) = \sum_{i=1}^{nm} x_i \bar{y}_i = \sum_{i=1}^m \chi_i \sum_{j=1}^n p_{1,j}^{(i)} \overline{q_{1,j}^{(i)}}.$$

Note that

$$|x|^2 = \sum_{i=1}^{nm} x_i \bar{x}_i = \sum_{i=1}^m \chi_i \sum_{j=1}^n p_{1,j}^{(i)} \overline{p_{1,j}^{(i)}} = \sum_{i=1}^m \chi_i p_{1,1}^{(i)}.$$

Similarly, $|y|^2 = \sum_{i=1}^m \chi_i q_{1,1}^{(i)}$. By (3), we have $\sum_{i=1}^m \chi_i p_{1,1}^{(i)} = \sum_{i=1}^m \chi_i q_{1,1}^{(i)}$, hence $|x| = |y|$. By the Cauchy–Schwartz inequality,

$$\left| \sum_{i=1}^m \chi_i \sum_{j=1}^n p_{1,j}^{(i)} \overline{q_{1,j}^{(i)}} \right| \leqslant \sqrt{\sum_{i=1}^m \chi_i p_{1,1}^{(i)}} \sqrt{\sum_{i=1}^m \chi_i q_{1,1}^{(i)}} = \sum_{i=1}^m \chi_i q_{1,1}^{(i)}.$$

Comparing this inequality with (4), we have $r_1 = \cdots = r_n$ and $P_i = Q_i$ for all i. □

Theorem 3. *Two χ-representations $\mathcal{U} = (U; U_i)_{i\in\mathcal{S}}$ and $\mathcal{U}' = (U'; U_i')_{i\in\mathcal{S}}$ are unitarily equivalent if and only if the corresponding subspace representations $F(\mathcal{U})$ and $F(\mathcal{U}')$ are equivalent.*

Proof. If $\mathcal{U}$ is unitarily equivalent to $\mathcal{U}'$, then $F(\mathcal{U})$ is equivalent to $F(\mathcal{U}')$. Let us prove the converse statement. $F(\mathcal{U})$ is equivalent to $F(\mathcal{U}')$ if and only if there exists an invertible $g : U \to U'$ such that

$$g^{-1}P_{U_i'}gP_{U_i} = P_{U_i}, \quad gP_{U_i}g^{-1}P_{U_i'} = P_{U_i'}, \quad i \in \mathcal{S}.$$

Let $g = \varphi\psi D\psi^*$ be the polar decomposition of g, where $\varphi : U \to U'$ and $\psi : U \to U$ are unitary maps and D is a positively defined diagonal operator. Then

$$(\psi D^{-1}\psi^*\varphi^*)P_{U_i'}(\varphi\psi D\psi^*)P_{U_i} = P_{U_i}, \quad i \in \mathcal{S}.$$

Hence $(\psi^*\varphi^*P_{U_i'}\varphi\psi)D(\psi^*P_{U_i}\psi) = D(\psi^*P_{U_i}\psi)$ for all i. Since $\mathcal{U}$ and $\mathcal{U}'$ are χ-representations,

$$\sum_{i\in\mathcal{S}} \chi_i(\psi^*\varphi^*P_{U_i'}\varphi\psi) = I, \quad \sum_{i\in\mathcal{S}} \chi_i(\psi^*P_{U_i}\psi) = I.$$

Lemma 2 ensures that $\psi^*\varphi^*P_{U_i'}\varphi\psi = \psi^*P_{U_i}\psi$ for all i. Therefore, $\varphi^*P_{U_i'}\varphi = P_{U_i}$ for all i, and so $\mathcal{U}$ is unitarily equivalent to $\mathcal{U}'$. $\square$

Remark 4. By similar argumentation, one can show that χ-representation $\mathcal{U}$ is orthogonally indecomposable if and only $F(\mathcal{U})$ is indecomposable. The connection between usual and orthoscalar representations of quivers was established [9, Theorem 1] in the same way as in Theorem 3.

A representation $\mathcal{V} = (V; V_i)_{i\in\mathcal{S}}$ of weight $\chi = (\chi_i)_{i\in\mathcal{S}}$ is called χ*-stable* if $\sum_{i\in\mathcal{S}} \chi_i \dim V_i = \dim V$ and

$$\sum_{i\in\mathcal{S}} \chi_i \dim(V_i \cap M) < \dim M$$

for any proper subspace $0 \neq M \subset V$.

Lemma 5. *If $\mathcal{U} = (U; U_i)_{i\in\mathcal{S}}$ is an indecomposable χ-representation, then $F(\mathcal{U})$ is χ-stable.*

Proof. Equating the traces of both sides in (2), we obtain $\sum_{i\in\mathcal{S}} \chi_i \dim U_i = \dim U$. If M is any proper subspace of U, then $\sum_{i\in\mathcal{S}} \chi_i P_{U_i}P_M = P_M$. Equating the traces of both sides in the last equality, we get

$$\sum_{i\in\mathcal{S}} \chi_i \mathrm{tr}(P_{U_i}P_M) = \dim M.$$

By [4, Theorem 2], $\mathrm{tr}(P_{M_1\cap M_2}) \leqslant \mathrm{tr}(P_{M_1}P_{M_2})$ for each two subspaces M_1 and M_2, and so

$$\sum_{i\in\mathcal{S}} \chi_i \mathrm{tr}(P_{U_i\cap M}) \leqslant \sum_{i\in\mathcal{S}} \chi_i \mathrm{tr}(P_{U_i}P_M) = \dim M.$$

It remains to prove that the last inequality is strict. Indeed, assume that $\mathrm{tr}(P_{U_i\cap M}) = \mathrm{tr}(P_{U_i}P_M)$ for all i. Then each P_{U_i} commutes with P_M. Hence the subspace M is invariant with respect to the projections P_{U_i} and the representation $\mathcal{U}$ is decomposable. This contradicts the assumption. $\square$

The converse statement to Lemma 5 also holds: if a representation $\mathcal{V} = (V; V_i)_{i \in \mathcal{S}}$ is χ-stable, then one can choose a scalar product in V in such a way that $\mathcal{V}$ becomes a χ-representation; see [5, Theorem 3.5]. Using results from [5,7,15], one can prove the following theorem.

Theorem 6. *An indecomposable unitary representation $\mathcal{U}$ is a χ-representation if and only if the corresponding subspace representation $F(\mathcal{U})$ is χ-stable.*

3. Proof of Theorem 1

The implication (i) $\Rightarrow$ (ii) is trivial.

(iii) $\Rightarrow$ (i). Assume that the Hasse diagram of $\mathcal{S}$ does not contain any of critical diagrams (1). If $\mathcal{S}$ has an infinite number of indecomposable unitarily nonequivalent χ-representations for some weight χ, then by Theorem 3 it has an infinite number of nonequivalent indecomposable subspace representations. By Kleiner's theorem, $\mathcal{S}$ contains a critical diagram; a contradiction.

(ii) $\Rightarrow$ (iii). We say that a poset $\mathcal{S}$ is *unitary representation-infinite* if there exist $d \in \mathbb{N}$ and $\chi^{\mathcal{S}} \in \mathbb{R}_+^{|S|}$ such that $\mathcal{S}$ has an infinite number of indecomposable unitarily nonequivalent $\chi^{\mathcal{S}}$-representations of dimension d. Our aim is to prove that critical posets are unitary representation-infinite.

One can show that critical primitive posets are unitary representation-infinite using [1,8,12]. Namely, there exists a correspondence between the χ-representations of a given poset $\mathcal{S}$ and the representations of a certain $*$-algebra $\mathcal{A}_{\Gamma,\omega}$ associated with a star-shaped graph Γ, which is determined by the Hasse diagram of $\mathcal{S}$, and the parameter ω is determined by the weight χ. If Γ is an extended Dynkin graph (which corresponds to some primitive critical $\mathcal{S}$), then one can choose the parameter ω such that $\mathcal{A}_{\Gamma,\omega}$ has an infinite number of unitarily nonequivalent irreducible representations. The complete description of such representations was given in [1,8,12] (see also Remark 11). But we use another method that handles both primitive and non-primitive cases.

Denote by $e_i^{(n)}$ (or e_i if no confusion can arise) the n-dimensional vector in which the ith coordinate is 1 and the others are 0. Denote by $e_{i_1 \ldots i_k}$ the vector $e_{i_1} + \cdots + e_{i_k}$ and by $\langle x_1, \ldots, x_m \rangle$ the vector space spanned by $x_1, \ldots, x_m \in \mathbb{C}^n$.

For each critical poset $\mathcal{S}$, we define a family of its subspace representations $\mathcal{V}_\lambda(\mathcal{S})$ that depend on a complex parameter $\lambda \in \mathbb{C}$.

- If $\mathcal{S} = (1, 1, 1, 1)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space $\mathbb{C}^2$ and its subspaces

$$\langle e_1 \rangle \qquad \langle e_2 \rangle \qquad \langle e_1 + e_2 \rangle \qquad \langle e_1 + \lambda e_2 \rangle$$

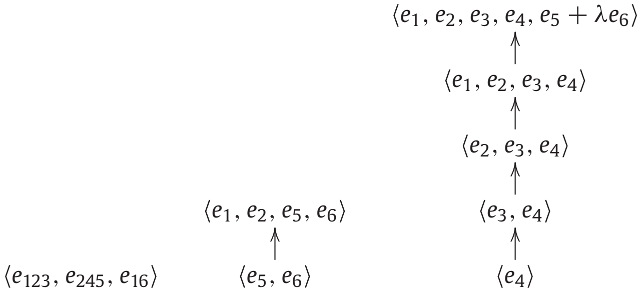
- If $\mathcal{S} = (2, 2, 2)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space $\mathbb{C}^3$ and its subspaces

$$\begin{array}{ccc} \langle e_{123}, e_1 + \lambda e_3 \rangle & \langle e_1, e_2 \rangle & \langle e_2, e_3 \rangle \\ \uparrow & \uparrow & \uparrow \\ \langle e_{123} \rangle & \langle e_1 \rangle & \langle e_3 \rangle \end{array}$$

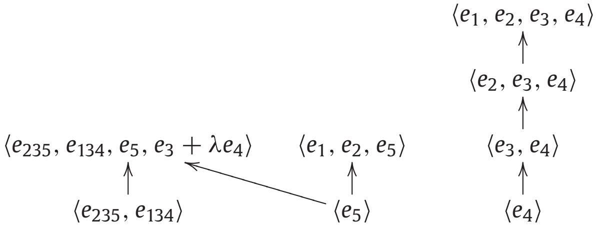
- If $\mathcal{S} = (1, 3, 3)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space $\mathbb{C}^4$ and its subspaces

$$\begin{array}{ccc} & \langle e_1, e_4, e_2 + \lambda e_3 \rangle & \langle e_1, e_2, e_3 \rangle \\ & \uparrow & \uparrow \\ & \langle e_1, e_4 \rangle & \langle e_2, e_3 \rangle \\ & \uparrow & \uparrow \\ \langle e_{123}, e_{24} \rangle & \langle e_4 \rangle & \langle e_3 \rangle \end{array}$$

- If $\mathcal{S} = (1, 2, 5)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space $\mathbb{C}^6$ and its subspaces



- If $\mathcal{S} = (N, 4)$, then $\mathcal{V}_\lambda(\mathcal{S})$ consists of the space $\mathbb{C}^5$ and its subspaces



Denote by $V_\lambda^{\mathcal{S}}$ the only subspace from $\mathcal{V}_\lambda(\mathcal{S})$ that depends on the parameter λ and denote by a the element from $\mathcal{S}$ that corresponds to $V_\lambda^{\mathcal{S}}$. Deleting $V_\lambda^{\mathcal{S}}$ from $\mathcal{V}_\lambda(\mathcal{S})$, we obtain the subspace representation $\mathcal{V}(\mathcal{S}_a) = (V^{\mathcal{S}}; V_i^{\mathcal{S}})_{i \in \mathcal{S}_a}$ of primitive poset $\mathcal{S}_a := \mathcal{S} \setminus \{a\}$.

Proposition 7. *$\mathcal{V}_\lambda(\mathcal{S})$ is not equivalent to $\mathcal{V}_\mu(\mathcal{S})$ if $\lambda \neq \mu$ for each critical poset $\mathcal{S}$. All subspace representation $\mathcal{V}(\mathcal{S}_a)$ are schurian.*

Proof. This proposition is proved by straightforward computations. □

Let $\mathcal{S}$ be a critical poset. The poset $\mathcal{S}_a$ is primitive and does not contain any of the critical posets, its subspace representation $\mathcal{V}(\mathcal{S}_a)$ is schurian. By [3, Proposition 3.1], there exists a weight which we denote by χ^a, such that $\mathcal{V}(\mathcal{S}_a)$ is χ^a-stable. Write

$$R := \min\left\{ \dim M - \sum_{i \in \mathcal{S}_a} \chi_i^a \dim(V_i^{\mathcal{S}} \cap M) \,\middle|\, M \text{ is a proper subspace of } V^{\mathcal{S}} \right\}.$$

The subspace representation $\mathcal{V}(\mathcal{S}_a)$ is χ^a-stable, hence $R > 0$. Let ε be such that $R > \varepsilon > 0$. Write $T := 1 + (R - \varepsilon)(\dim V^{\mathcal{S}})^{-1}$ and

$$\chi^{\mathcal{S}} = (\chi_i^{\mathcal{S}})_{i \in \mathcal{S}}, \quad \chi_i^{\mathcal{S}} := \begin{cases} \chi_i^a \cdot T^{-1}, & \text{if } i \in \mathcal{S}_a, \\ (R - \varepsilon) \cdot (\dim V_\lambda^{\mathcal{S}})^{-1} \cdot T^{-1}, & \text{if } i = a. \end{cases}$$

Proposition 8. *The subspace representations $\mathcal{V}_\lambda(\mathcal{S})$ are $\chi^{\mathcal{S}}$-stable for all λ and $\mathcal{S}$.*

Proof. Note that

$$\begin{aligned} \sum_{i \in \mathcal{S}} \chi_i^{\mathcal{S}} \dim V_i^{\mathcal{S}} &= T^{-1} \sum_{i \in \mathcal{S}_a} \chi_i^a \dim V_i^{\mathcal{S}} + \chi_a^{\mathcal{S}} \dim V_\lambda^{\mathcal{S}} \\ &= T^{-1} \dim V^{\mathcal{S}} + (1 - T^{-1}) \dim V^{\mathcal{S}} = \dim V^{\mathcal{S}}. \end{aligned}$$

Let M be any proper subspace of $V^{\mathcal{S}}$. Then

$$\sum_{i\in\mathcal{S}} \chi_i^{\mathcal{S}} \dim(V_i^{\mathcal{S}} \cap M) = T^{-1} \sum_{i\in\mathcal{S}_a} \chi_i^{a} \dim(V_i^{\mathcal{S}} \cap M) + \chi_a^{\mathcal{S}} \dim(V_\lambda^{\mathcal{S}} \cap M)$$
$$\leqslant T^{-1}\left(\dim M - R + (R-\varepsilon)(\dim V_\lambda^{\mathcal{S}})^{-1} \dim(V_\lambda^{\mathcal{S}} \cap M)\right)$$
$$\leqslant T^{-1}(\dim M - \varepsilon) < \dim M.$$

Hence $\mathcal{V}_\lambda(\mathcal{S})$ is $\chi^{\mathcal{S}}$-stable. □

Proposition 9. *Critical posets are unitary representation-infinite.*

Proof. By Proposition 7 and Proposition 8, any critical poset $\mathcal{S}$ has an infinite number of nonequivalent $\chi^{\mathcal{S}}$-stable subspace representations. By Theorem 6, $\mathcal{S}$ has an infinite number of indecomposable unitarily nonequivalent $\chi^{\mathcal{S}}$-representations. □

Proposition 10. *If a poset $\mathcal{S}$ contains a critical poset (as a full subposet), then $\mathcal{S}$ is unitary representation-infinite.*

Proof. Suppose that $\mathcal{S}$ contains a critical poset $\mathcal{S}_c$. By Proposition 9, there exists a weight χ^c such that $\mathcal{S}_c$ has an infinite number of indecomposable unitarily nonequivalent χ^c-representations of dimension d. Define the following subset of $\mathcal{S}$:

$$\mathcal{S}_{\max} := \{a \in \mathcal{S} \mid b \prec a \text{ for some } b \in \mathcal{S}_c\}.$$

For each χ^c-representation $\mathcal{U} = (U; U_i)_{i\in\mathcal{S}_c}$, define the unitary representation $\mathcal{U}' = (U; U_i')_{i\in\mathcal{S}}$ of $\mathcal{S}$ as follows:

$$U_i' := \begin{cases} 0, & \text{if } i \notin \mathcal{S}_{\max} \cup \mathcal{S}_c, \\ U_i, & \text{if } i \in \mathcal{S}_c, \\ U, & \text{if } i \in \mathcal{S}_{\max}. \end{cases}$$

It is easy to check that $\mathcal{U}'$ is χ'-representation, in which $\chi' = (\chi_i')_{i\in\mathcal{S}}$ is defined by

$$\chi_i' := \begin{cases} \chi_i^c \cdot (1 + |\mathcal{S}_{\max}|)^{-1}, & \text{if } i \in \mathcal{S}^c, \\ (1 + |\mathcal{S}_{\max}|)^{-1}, & \text{otherwise.} \end{cases}$$

Hence $\mathcal{S}$ is unitary representation-infinite. □

The implication (ii) ⇒ (iii) follows from Proposition 10. This finishes the proof of Theorem 1.

Remark 11. Define the following weights:

$$\chi^{(1,1,1,1)} := \frac{1}{2}(1, 1, 1, 1),$$
$$\chi^{(2,2,2)} := \frac{1}{3}(1, 1, 1, 1, 1, 1),$$
$$\chi^{(1,3,3)} := \frac{1}{4}(2, 1, 1, 1, 1, 1, 1),$$
$$\chi^{(1,2,5)} := \frac{1}{6}(3, 2, 2, 1, 1, 1, 1, 1),$$
$$\chi^{(N,4)} := \frac{1}{5}(2, 1, 1, 2, 1, 1, 1, 1).$$

Each weight $\chi^{\mathcal{S}}$ obtained from the minimal imaginary root of the quadratic form related to a critical poset $\mathcal{S}$. We checked (describing all possible subdimension vectors) that the representations $\mathcal{V}_\lambda(\mathcal{S})$ are $\chi^{\mathcal{S}}$-stable for any $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Hence they give rise to an infinite family of nonequivalent $\chi^{\mathcal{S}}$-representations. For primitive $\mathcal{S}$ one can obtain the precise description of projections for such representations using the results from [1,8,12]. The description in the case $\mathcal{S} = (N, 4)$ is unknown.

Acknowledgments

We would like to thank S.A. Kruglyak, V.L. Ostrovskii, and L.B. Turowska for useful discussions and advises and also V.V. Sergeichuk for numerous helpful remarks. The second author was partially supported by DFG Grant SCHM1009/4-1 and Fapesp Grant 2010/15781-0.

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