Mathematical Modelling, Vol. 7, pp. 197–200, 1986
0270-0255/86 $3.00 + .00

THE BALLOT PROBLEM

BARRY A. CIPRA

Department of Mathematics
St. Olaf College
Northfield, Minnesota 55057

Communicated by M. S. Klamkin

(*Received September* 1985)

Abstract—A simplified proof is given for a result of Klamkin and Rhemtulla on the two-office ballot problem.

INTRODUCTION

In order to average out the effect of ordering on a ballot in an election of officers, it may be desirable to prepare several versions of a ballot with the candidates' names in various orders, so that in the complete set of ballots no candidate occupies any given position in the list more often than any other candidate for the same office. This leads to the problem of determining the minimum number of different ballots required. Klamkin and Rhemtulla[1] solved this for the case of two offices with m and n candidates respectively, showing the minimum number to be $m + n - \gcd(m, n)$, where gcd denotes the greatest common divisor. Their proof, however, was rather complicated. This paper gives a much simpler proof of the same result, which the author believes provides additional insight into the general problem. For more than two offices, the problem is still open.

MAIN RESULT

Consider the two-office ballot problem, in which m candidates are running for office 1, and n candidates are running for office 2. Let R and C be particular candidates for offices 1 and 2, respectively. Given a set of ballots, we can sort them, at least initially, according to the positions of these two candidates. We thus form a matrix of nonnegative integers $A = (a_{ij})$, in which a_{ij} is the number of ballots on which candidate R appears in the ith position and candidate C in the jth.

Now let

$$r_i = \sum_{j=1}^{n} a_{ij} \qquad \text{for } i = 1, \ldots, m$$

and

$$c_j = \sum_{i=1}^{m} a_{ij} \qquad \text{for } j = 1, \ldots, n.$$

Clearly r_i is the total number of times candidate R appears in the ith position, and c_j is the total number of times C appears in the jth.

Suppose now that for each office every candidate appears the same number of times in each given position. It is easy to see then that each given candidate appears the same number of times in every position—i.e. $r_1 = r_2 = \cdots = r_m$ and $c_1 = c_2 = \cdots = c_n$. We will call the matrix A "magic" if it satisfies this condition. (The adjective derives from "magic squares," a well-known subject in recreational number theory. Benjamin Franklin was a connoisseur of magic squares.)

Conversely, given a magic matrix, we can immediately construct a set of ballots satisfying the supposition above by simply using cyclic permutations of the names in the two lists. In this way the problem of finding the minimal number of different ballots required to satisfy the supposition is solved by finding the minimal number of nonzero entries possible in a magic matrix. (Of course we exclude the "null" case, $a_{ij} = 0$ for all i, j.)

We shall call a magic matrix "minimal" if it has a minimal number of nonzero (hence positive) entries. Let $M(m, n)$ denote this number: The number of nonzero entries in an $m \times n$ minimal magic matrix. Our object is to give a simple proof of the following

THEOREM. $M(m, n) = m + n - \gcd(m, n)$.

The proof of the theorem is indeed simple: one merely applies induction to the following lemma and proposition.

Lemma. $M(m, n) = M(n, m)$, and $M(m, m) = m$.

Proposition. If $m < n$, then $M(m, n) = m + M(m, n - m)$.

The lemma is trivial: one merely transposes M to obtain the first statement, and observes that the identity matrix I always provides a minimal magic-square matrix. The rest of our work comes in proving the proposition. We shall do this by proving inequalities $\leq$ and $\geq$ for the relation between $M(m, n)$ and $m + M(m, n - m)$.

Lemma. $M(m, n) \leq m + M(m, n - m)$.

Proof. Let A_1 be a minimal magic matrix of size $m \times n - m$, and let c be the common sum of its columns. Let A be the $m \times n$ matrix formed by adjoining the $m \times m$ scalar matrix cI to the left of A_1; for example, if

$$A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix},$$

then

$$A = \left[\begin{array}{ccc|cc} 3 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 & 2 \\ \hline 0 & 0 & 3 & 1 & 1 \end{array}\right].$$

(The dotted lines are added for emphasis.) The matrix A is clearly magic, and has exactly $m + M(m, n - m)$ nonzero entries. The lemma follows.

We shall call a matrix A "diagonalized" if it has the form $(cI \mid A_1)$ as above. Clearly, if A is magic, then so is A_1. Moreover, if A is minimal, then A_1 is also.

Lemma. $M(m, n) \geq m + M(m, n - m)$.

Proof. It suffices to show that there is always a minimal magic matrix in diagonalized form. We shall prove this by showing how any magic matrix with $m < n$ can be diagonalized by a procedure which does not decrease the number of zero entries. The procedure is based on the following

Sublemma. Suppose A is a magic matrix with $m < n$. Then every row of A contains at least two nonzero entries.

Proof of the sublemma. Suppose there were a row with only one nonzero entry r. Then r is the common sum of all the rows. From the relation $rm = cn$, where c is the common column-sum, we have $c = rm/n < r$, which is contradicted by the column containing the entry r (since negative entries are not permitted).

We can now describe the procedure for diagonalizing A. By permuting rows and columns as necessary, we may assume that $a_{11} \neq 0$. By the sublemma, $a_{1j} \neq 0$ also, for some $j > 1$. Suppose now that $a_{i1} \neq 0$. Consider the submatrix

$$\begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix},$$

and let $\alpha = \min(a_{i1}, a_{1j})$. Define a new matrix $A' = (a'_{ij})$ by

$$\begin{pmatrix} a'_{11} & a'_{1j} \\ a'_{i1} & a'_{ij} \end{pmatrix} = \begin{pmatrix} a_{11} + \alpha & a_{1j} - \alpha \\ a_{i1} - \alpha & a_{ij} + \alpha \end{pmatrix},$$

and $a'_{hk} = a_{hk}$ for all other entries. It is easy to see that A' is again a magic matrix, that A' has at least as many zeros as A, and that $a'_{i1} < a_{i1}$. By repeating this process a finite number of times, we eventually have $a'_{i1} = 0$, and this can be done for all $i = 2, \ldots, m$.

Having "diagonalized" the first column, a straightforward induction argument shows that columns $2, \ldots, m$ can likewise be diagonalized, leaving a diagonalized magic matrix A' having at least as many zeros as A. If A is minimal to start with, then A' is also minimal and we have concluded the proof of the lemma, and thus of the theorem.

REMARKS

1. Nonnegativity of the a_{ij}'s is essential, as the example

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

shows. The proof here used the fact that the a_{ij}'s were integers, but rational entries do not change the result: one merely multiplies through by the greatest common denominator. It is also fairly simple to show that using positive real coefficients cannot lower the minimality number either: The set of magic matrices with non-zero entries in M prescribed positions can be described as the intersection of $m + n - 2$ rationally defined hyperplanes in R^M, so that rational magic matrices are consequently dense in this set.

2. The preliminary discussion, which related the ballot problem for two offices to the problem of finding minimal magic matrices, holds also for the general case with N offices, except we have to deal with N-dimensional arrays of integers in place of ordinary matrices. Unfortunately, the diagonalization procedure does not work in general or, at least, not in such a simple manner. One can still juxtapose a solution to the problem $M(m_1, m_2, \ldots, m_N - m_1)$ (assuming $1 < m_1 < m_2 < \cdots < m_N$) with a "diagonal embedding" of a solution to $M(m_1, m_2, \ldots, m_{N-1})$ to obtain a "magic matrix," but the result will not be minimal.

REFERENCE

1. M. Klamin and A. Rhemtulla, The ballot problem. *Mathematical Modelling* **5**, 1–6 (1984).