An overshoot approach to recurrence and transience of Markov processes

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Abstract

We develop criteria for recurrence and transience of one-dimensional Markov processes which have jumps and oscillate between $+\infty$ and $-\infty$. The conditions are based on a Markov chain which only consists of jumps (overshoots) of the process into complementary parts of the state space.

In particular, we show that a stable-like process with generator $-(-\Delta)^{\alpha(x)/2}$ such that $\alpha(x) = \alpha$ for $x < -R$ and $\alpha(x) = \beta$ for $x > R$ for some $R > 0$ and $\alpha, \beta \in (0, 2)$ is transient if and only if $\alpha + \beta < 2$, otherwise it is recurrent.

As a special case, this yields a new proof for the recurrence, point recurrence and transience of symmetric $\alpha$-stable processes.

Keywords: Markov processes with jumps; Recurrence; Transience; Stable-like processes

1. Introduction

The recurrence and transience of Markov processes has been studied by various authors and various techniques. First and foremost, there are the potential theoretic approach (see [8] for a unification of the criteria) and the Markov chain approach by Meyn and Tweedie [11]. For Feller processes there have been several attempts to classify their behavior based on the generator or the associated Dirichlet form; see Chapter 6 of Jacob [9] and the references given therein.

In one dimension a transient process either drifts to infinity (i.e., $\lim_{t \to \infty} X_t = +\infty$ or $= -\infty$) or it may be oscillating: $\limsup_{t \to \infty} X_t = +\infty$ and $\liminf_{t \to \infty} X_t = -\infty$. An
oscillating process may be recurrent, transient or neither of those (cf. Sections 2 and 4 for the definitions).

Consider for example a stable-like process, i.e., a Markov process with generator $-\Delta^{\alpha(x)/2}$ and symbol $|\xi|^{\alpha(x)}$, respectively (see [2] for a construction). These processes are well studied, but their recurrence and transience behavior is in general unknown. Besides symmetric $\alpha$-stable Lévy processes the only processes of this type treated in the literature are processes where $\alpha(\cdot)$ is periodic [7] or related processes where the generator is a symmetric Dirichlet form [18,19]. The initial motivation for this paper was to treat the non-symmetric case. But in the following we develop a more general framework.

In Section 2 we introduce a “local” notion of recurrence and transience for which we will give sufficient conditions in Section 3. Afterward in Section 4 the local notions are linked to the (global) recurrence and transience of processes. In particular, conditions which imply the recurrence–transience dichotomy are given. Furthermore we present a result which allows us to compare the behavior of Markov processes which coincide outside some compact ball.

This paper closes with an application to stable and stable-like processes.

2. Recurrence and transience

We consider time homogeneous strong Markov processes $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbb{P}_x)$ with càdlàg paths on $\mathbb{R}^d$ ($d \in \mathbb{N}$), where the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions. Note that $(\theta_t)_{t \geq 0}$ is the family of shift operators on $\Omega$, i.e., $X_s(\theta_t(\omega)) = X_{t+s}(\omega)$ for $\omega \in \Omega$.

To simplify notation we denote such a process by $(X_t)_{t \geq 0}$. The state space $\mathbb{R}^d$ will be equipped with the Borel-$\sigma$-algebra $\mathcal{B}(\mathbb{R}^d)$ and sets will be elements of $\mathcal{B}(\mathbb{R}^d)$ if not stated otherwise. For a set $A$ the first entrance time is defined, with the convention $\inf \emptyset = \infty$, by

$$\tau_A := \inf \{t \geq 0 \mid X_t \in A\}. $$

Note that $\tau_A$ is a stopping time for any $A \in \mathcal{B}(\mathbb{R}^d)$, since the process is right continuous and adapted, hence progressive. Furthermore for any stopping time $\sigma$ also

$$\tau_{A,\sigma} := \inf \{t \geq \sigma \mid X_t \in A\}$$

is a stopping time since

$$\{\tau_{A,\sigma} \leq t\} = \bigcup_{s \in \mathbb{Q} \cap [0,t]} \{X_s \in A\} \cap \{\sigma \leq s\} \in \mathcal{F}_t$$

(compare [6, Chapter 2, Prop. 1.5]).

Now we define a pointwise (local) notion of recurrence and transience.

**Definition 2.1.** Let $(X_t)_{t \geq 0}$ be $\mathbb{R}^d$-valued process and $b \in \mathbb{R}^d$. With respect to $(X_t)_{t \geq 0}$ the point $b$ is called

- **recurrent** if
  \[ \mathbb{P}_b(\forall T > 0 \ \exists t > T : X_t = b) = 1, \]

- **left limit recurrent** if
  \[ \mathbb{P}_b(\forall T > 0 \ \exists t > T : X_{t^-} = b) = 1, \]
• **locally recurrent** if
  \[ P_b(\liminf_{t \to \infty} |X_t - b| = 0) = 1, \]
  
  • **locally transient** if
  \[ P_b(\liminf_{t \to \infty} |X_t - b| = 0) < 1, \]
  
  • **transient** if
  \[ P_b(\liminf_{t \to \infty} |X_t - b| = \infty) = 1. \]

**Remark 2.2.** The notion of *local* is meant in a spatial sense, as opposed to a temporal sense. One would get the latter by transferring the definition of (deterministic) locally recurrent functions (e.g. [5]) to processes.

Note that only for left limit recurrence we need that the paths have left limits, the right continuity is not necessary for these definitions. The reason for introducing left limit recurrence at all, is that our method will not allow us to prove recurrence for points but at most left limit recurrence. Nevertheless we have the following lemma to conclude the recurrence for a point.

**Lemma 2.3.** Let \((X_t)_{t \geq 0}\) be quasi-left continuous, i.e., for every increasing sequence of stopping times \(\sigma_n\) with limit \(\sigma\):

\[ X_{\sigma_n} \xrightarrow{n \to \infty} X_\sigma \quad \text{a.s. on } \{\sigma < \infty\}. \]

Then the following implication holds:

\( b \) is left limit recurrent \( \Rightarrow \) \( b \) is recurrent.

**Proof.** Define \(\sigma_0 := k \in \mathbb{N}\) and for \(n \in \mathbb{N}\)

\[ \sigma_n := \inf \left\{ t \geq \sigma_{n-1} \mid |X_t - b| < \frac{1}{n} \right\} \quad \text{and} \quad \sigma := \lim_{n \to \infty} \sigma_n. \]

Clearly, \((\sigma_n)_{n \in \mathbb{N}}\) is increasing. Thus \(\sigma\) is well defined and

\[ P_b(\sigma < \infty) = 1, \]

since \(b\) is left limit recurrent. Note that \(\sigma_n\) might be constant for \(n\) large, but in this case the process is already in \(b\). In general, by the quasi-left continuity,

\[ P_b \left( X_\sigma = \lim_{n \to \infty} X_{\sigma_n} = b \right) = 1. \]

This shows that \(b\) is recurrent, since \(k\) was arbitrary. \( \square \)

Further simple consequences of **Definition 2.1** are that (left limit) recurrence implies local recurrence and that we have the dichotomy

\[ b \text{ is either locally recurrent or locally transient.} \quad (2.1) \]

A process \((X_t)_{t \geq 0}\) is point recurrent if and only if all \(b \in \mathbb{R}^d\) are recurrent. The other common (global) notions for recurrence and transience of processes do not have such a simple relation to the above local notions. The details will be given in Section 4.
3. Overshoots and Markov processes

In this section we treat for simplicity the case \( d = 1 \); see Remark 3.5 for the extension to higher dimensions. Let \( (X_t)_{t \geq 0} \) be a process on \( \mathbb{R} \) satisfying

\[
\limsup_{t \to \infty} X_t = \infty \quad \text{and} \quad \liminf_{t \to \infty} X_t = -\infty \quad \text{a.s.} \tag{3.1}
\]

Further assume that there exists some \( b \in \mathbb{R} \) such that for the stopping times

\[
\tau^b := \inf\{t \geq 0 \mid X_t < b\} \quad \text{and} \quad \sigma^b := \inf\{t \geq 0 \mid X_t \geq b\}
\]

the process satisfies

\[
\mathbb{P}_x(X_{\tau^b} = b) = 0 \quad \text{for all} \quad x > b,
\]

\[
\mathbb{P}_x(X_{\sigma^b} = b) = 0 \quad \text{for all} \quad x < b,
\tag{3.2}
\]

i.e., the process almost surely enters \((-\infty, b]\) and \([b, \infty)\) not by hitting \( b \). The distributions of \( X_{\tau^b} \) and \( X_{\sigma^b} \) are called overshoot distributions.

**Remark 3.1.** Note that assumption (3.2) is not equivalent to assuming that the process is non-creeping (non-creeping at level \( b \) means that it does not enter \((b, \infty)\) continuously; cf. [3, p. 174]). For example, consider a compound Poisson process on \( \mathbb{R} \) with jump distribution \( \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{1} \). The process started in 0 is non-creeping, but it hits \( b = 1 \) with probability one.

Now define \( \sigma_0 := 0 \) and for each \( n \in \mathbb{N} \) set

\[
\tau_n := \inf\{t \geq \sigma_{n-1} \mid X_t < b\},
\]

\[
\sigma_n := \inf\{t \geq \tau_n \mid X_t > b\}.
\]

Note that \( \sigma_1 \) is always the first time of passing \( b \) from below. On the contrary, \( \tau_1 \) is for the process started in \( x > b \) the first time of passing \( b \) from above, but \( \tau_1 = 0 \) for \( x < b \).

These stopping times have the following properties.

**Proposition 3.2.** Let \( x \neq b \), then

(i) \( \mathbb{P}_x(\tau_n < \infty) = 1 \) and \( \mathbb{P}_x(\sigma_n < \infty) = 1 \) for all \( n \in \mathbb{N} \),

(ii) \( \{X_{\tau_n} < b\} \subset \{\sigma_n > \tau_n\} \),

(iii) \( \mathbb{P}_x(X_{\tau_n} < b) = 1 \) implies \( \mathbb{P}_x(X_{\sigma_n} > b) = 1 \),

(iv) \( \mathbb{P}_x(X_{\sigma_n} > b, X_{\tau_n} < b, \forall n \in \mathbb{N}) = 1 \),

(v) \( \mathbb{P}_x(\sigma_{n-1} < \tau_n < \sigma_n, \forall n \in \mathbb{N}) = 1 \).

**Proof.** (i) By (3.1) the process will pass \( b \) infinitely often almost surely, i.e., \( \tau_n \) and \( \sigma_n \) are finite almost surely.

(ii) Let \( \omega \in \{X_{\tau_n} < b\} \). Then, by the right continuity (since \( (X_t)_{t \geq 0} \) is càdlàg), there exists an \( \varepsilon_\omega > 0 \) such that \( X_{\tau_n + \varepsilon_\omega} \) \( (\omega) < b \). Thus \( \sigma_n(\omega) \geq \tau_n(\omega) + \varepsilon_\omega \), i.e.,

\[
\sigma_n(\omega) > \tau_n(\omega).
\]

(iii) First note that \( \mathbb{P}_x(X_{\tau_n} < b) = 1 \) implies by (ii) that \( \mathbb{P}_x(\sigma_n > \tau_n) = 1 \), and \( \tau_n \) is a finite stopping time by (i). By the right continuity \( \{X_{\sigma_1} = b\} \) contains all paths which enter \((b, \infty)\) continuously from \( b \) and \( \{X_{\sigma_1} = b\} \) contains all paths which enter \([b, \infty)\) at \( b \). Thus \( \{X_{\sigma_1} = b\} \subset \{X_{\sigma_1} = b\} \), i.e.,

\[
\mathbb{P}_y(X_{\sigma_1} = b) \leq \mathbb{P}_y(X_{\sigma_1} = b) = 0,
\]
which implies $\mathbb{P}_y(X_{\sigma_1} > b) = 1$. Now for $y < b$ the strong Markov property (note that: $\sigma_n = \sigma_1 \circ \theta_{\tau_n}$) yields

$$
\mathbb{P}_x(X_{\sigma_n} > b | X_{\tau_n} = y) = \mathbb{P}_y(X_{\sigma_1} > b) = 1.
$$

Then

$$
\mathbb{P}_x(X_{\sigma_n} > b) = \int_{(-\infty,b]} \mathbb{P}_x(X_{\sigma_n} > b | X_{\tau_n} = y) \mathbb{P}_x(X_{\tau_n} \in dy) = \int_{(-\infty,b]} 1 \mathbb{P}_x(X_{\tau_n} \in dy) = 1.
$$

(iv) The following statements can be proved analogously to (ii) and (iii):

(ii*) $\{X_{\sigma_n} > b\} \subset \{\tau_{n+1} > \sigma_n\}$.

(iii*) $\mathbb{P}_x(X_{\sigma_n} > b) = 1$ implies $\mathbb{P}_x(X_{\tau_{n+1}} < b) = 1$.

Obviously we have for $x < b$

$$
\mathbb{P}_x(X_{\tau_1} < b) = 1,
$$

and this also holds for $x > b$ by (iii*). Thus by repeated applications of (iii) resp. (iii*)

$$
\mathbb{P}_x(X_{\tau_n} < b) = \mathbb{P}_x(X_{\sigma_n} > b) = 1 \quad \text{for each } n \in \mathbb{N}.
$$

Finally,

$$
\mathbb{P}_x(X_{\tau_n} < b, X_{\sigma_n} > b, \forall n \in \mathbb{N}) = 1,
$$

since the set is a countable intersection of sets of measure one.

(v) This is a consequence of (ii), (ii*) and (iv). \quad \square

Now define for $x > b$ on the set $\{\sigma_{n-1} < \tau_n < \sigma_n, \forall n \in \mathbb{N}\}$, which has probability one by Proposition 3.2(v), the sequence $(Y_n)_{n \geq 0}$ by

$$
Y_n := X_{\sigma_n}, \quad (3.3)
$$

and note that the strong Markov property of $(X_t)_{t \geq 0}$ implies that $(Y_n)_{n \geq 0}$ is a Markov chain on $(b, \infty)$. The sequence of stopping times $(\sigma_n)_{n \in \mathbb{N}}$ is strictly increasing, but it is possibly bounded. Thus the Markov chain captures only the first countably many overshoots of the process $(X_t)_{t \geq 0}$ which jump across level $b$ from below (i.e., overshoots starting in $(-\infty, b)$).

Nevertheless this Markov chain can be used to determine the local recurrence/transience behavior of $(X_t)_{t \geq 0}$ by the following theorem.

**Theorem 3.3.** Let $(X_t)_{t \geq 0}$ be a time homogeneous strong Markov process with càdlàg paths satisfying (3.1) and (3.2) for some $b \in \mathbb{R}$, and let, for each $x > b$, $(Y_n)_{n \geq 0}$ be the corresponding Markov chain defined by (3.3).

(i) If $\mathbb{P}_x(\lim_{n \to \infty} Y_n = \infty) = 1$ for all $x > b$ and there exist $r, R > 0$ and $c < 1$ such that

$$
\sup_{y \in [b-r, b+r]} \mathbb{P}_y(X_{\sigma_1} > b + R) < c, \quad (3.4)
$$

then $b$ is locally transient.

(ii) If $\mathbb{P}_x(\liminf_{n \to \infty} Y_n = b) = 1$ for all $x > b$, then $b$ is locally recurrent.

(iii) If $\mathbb{P}_x(\lim_{n \to \infty} Y_n = b) = 1$ for all $x > b$ and there exist $r', R' > 0$ and $c < 1$ such that

$$
\sup_{y \geq b + r'} \mathbb{P}_y(X_{\sigma_1} < b + R') < c, \quad (3.5)
$$

then $b$ is left limit recurrent.
**Remark 3.4.** Roughly speaking, condition (3.4) ensures that the overshoots represent the whole process, whereas condition (3.5) ensures that the limit $b$ is reached in finite time. The following two examples show that these conditions cannot be removed.

1. Let $(N_t)_{t \geq 0}$ be a Poisson process, and $(\tilde{X}_n)_{n \geq 0}$ be a Markov chain with transition probabilities defined by

$$
\mathbb{P}(\tilde{X}_1 \in dy \mid \tilde{X}_0 = x) = \begin{cases} 
\delta_{\frac{1}{x}}(dy) & \text{for } |x| > 1, \\
\delta_{\frac{1}{x}+|x|}(dy) & \text{for } 0 < |x| \leq 1, \\
\delta_1(dy) & \text{for } x = 0.
\end{cases}
$$

This Markov chain is in fact deterministic and, when started in 0, the chain moves as

$$0, 1, -2, -\frac{1}{2}, 3, \frac{1}{3}, -4, -\frac{1}{4}, \ldots.$$

Now, the chain subordinated by the Poisson process is a càdlàg time homogeneous strong Markov process $(X_t)_{t \geq 0}$ satisfying (3.1) and (3.2) for $b = 0$. Furthermore 0 is locally recurrent and thus not locally transient. The associated chain of overshoots is deterministic, especially for $x \in (0, 1)$:

$$Y_0 = x, \quad Y_1 = \frac{1}{x} + 2 \quad \text{and for } n \in \mathbb{N}, Y_n = \frac{1}{x} + 2n,$$

i.e., $\lim_{n \to \infty} Y_n = \infty$ and

$$\forall R, r > 0 : \sup_{y \in [-r, r], y \neq 0} \mathbb{P}_y(X_{\sigma_{1}} > R) \geq \sup_{y \in (0, r]} \mathbb{P}_y(Y_1 > R) = 1.$$

2. Changing the definition of the transition probabilities to

$$
\mathbb{P}(\tilde{X}_1 \in dy \mid \tilde{X}_0 = x) = \begin{cases} 
\delta_{\frac{1}{x}}(dy) & \text{for } |x| > 1, \\
\delta_{\frac{1}{x}+|x|}(dy) & \text{for } 0 < |x| \leq 1, \\
\delta_1(dy) & \text{for } x = 0,
\end{cases}
$$

yields that the chain started in 0 moves as

$$0, 1, 2, -\frac{1}{2}, -3, \frac{1}{3}, 4, -\frac{1}{4}, -5, \ldots.$$

As in the previous example, the chain subordinated by the Poisson process is a càdlàg time homogeneous strong Markov process $(X_t)_{t \geq 0}$ satisfying (3.1) and (3.2) for $b = 0$. Furthermore 0 is locally recurrent, but 0 is not left limit recurrent (in finite time). For $x > 1$ the associated jump chain is

$$Y_0 = x, \quad Y_1 = \frac{1}{x+1} \quad \text{and in general } Y_n = \frac{1}{x+2n-1},$$

i.e., $\lim_{n \to \infty} Y_n = 0$ and $\forall R, r > 0 : \sup_{y \geq r} \mathbb{P}_y(X_{\sigma_{1}} < R) = \sup_{y \geq r} \mathbb{P}_y(Y_1 < R) = 1$.

**Proof of Theorem 3.3.** Throughout the proof let $b \in \mathbb{R}$ be fixed, and $(X_t)_{t \geq 0}$ be a time homogeneous strong Markov process with càdlàg paths satisfying (3.1) and (3.2). Furthermore, for each $x > b$ let $(Y_n)_{n \geq 0}$ be the corresponding Markov chain defined by (3.3).

(i) We assume that $\mathbb{P}_x(\lim_{n \to \infty} Y_n = \infty) = 1$ for all $x > b$ and fix $r, R > 0$ and $c < 1$ such that (3.4) holds, i.e., $\sup_{y \in B} \mathbb{P}_y(X_{\sigma_{1}} > b + R) < c$ with $B := [b - r, b + r] \setminus \{b\}$.

Assumption (3.1) ensures that $(X_t)_{t \geq 0}$ does not explode in finite time. Therefore $\infty = \lim_{n \to \infty} Y_n = \lim_{n \to \infty} X_{\sigma_{n}}$ a.s. implies that $\sigma_n \to \infty$ almost surely.
Now fix $\varepsilon > 0$. Then for all $x > b$ there exists a $N = N(x) > 0$ such that
\[ \forall n \geq N : \mathbb{P}_x(X_{\sigma_n} > R + b) \geq 1 - \varepsilon, \]
since $\lim_{n \to \infty} X_{\sigma_n} = \infty$.

Let $n \geq N$ and define $\nu_n$ as the time of the first visit to $B$ after time $\sigma_n$, i.e.,
\[ \nu_n := \inf\{ t \geq \sigma_n \mid X_t \in B \} \]
and $\sigma_k$ be the time of the first jump into $(b, \infty)$ after $\nu_n$, i.e.,
\[ k := \inf\{ l \in \mathbb{N} \mid \sigma_l > \nu_n \}. \]

Now suppose $b$ is locally recurrent, i.e., $\mathbb{P}_b(\liminf_{t \to \infty} |X_t - b| = 0) = 1$. Note that an overshoot hits $b$ with probability zero, since (3.2) holds. Thus the local recurrence of $b$ implies that there exists an $x > b$ such that $\mathbb{P}_x(\nu_n < \infty) = 1$ and $\mathbb{P}_x(X_{\nu_n} \in B) = 1$.

For this $x$ we find $\mathbb{P}_x(k < \infty) = 1$ and $\sigma_k = \sigma_1 \circ \theta_{\nu_n}$, where $\theta_{\nu_n}$ is the shift operator corresponding to $\nu_n$. Then the strong Markov property yields
\[ 1 - \varepsilon \leq \mathbb{P}_x(X_{\sigma_k} > R + b) \]
\[ = \int_B \mathbb{P}_x(X_{\sigma_k} > R + b | X_{\nu_n} = y) \mathbb{P}_x(X_{\nu_n} \in dy) \]
\[ = \int_B \mathbb{P}_y(X_{\sigma_1} > R + b) \mathbb{P}_x(X_{\nu_n} \in dy) \]
\[ \leq \sup_{y \in B} \mathbb{P}_y(X_{\sigma_1} > R + b) < c < 1, \]
which is a contradiction, since $\varepsilon$ was arbitrary and $c < 1$ was fixed. Thus $b$ is locally transient.

(ii) Let $\liminf_{t \to \infty} Y_n = b$ almost surely. If $\sigma_n \to \infty$ a.s. the statement is obvious. In general let $\varepsilon > 0$, $T > 0$ and
\[ \eta_T := \inf\{ t \geq T \mid X_t \in [b + 1, \infty) \}. \]
By (3.1) we have $\mathbb{P}_y(\eta_T < \infty) = 1$ for all $y \in \mathbb{R}$. Thus for $x > b$ the strong Markov property yields
\[ \mathbb{P}_x(\exists t > T : |X_t - b| < \varepsilon) \geq \mathbb{P}_x(\exists t > 0 : |X_{t+\eta_T} - b| < \varepsilon) \]
\[ = \int_{[b+1,\infty)} \mathbb{P}_x(\exists t > 0 : |X_{t+\eta_T} - b| < \varepsilon | \eta_T = y) \mathbb{P}_x(X_{\eta_T} \in dy) \]
\[ = \int_{[b+1,\infty)} \mathbb{P}_y(\exists t > 0 : |X_t - b| < \varepsilon) \mathbb{P}_x(X_{\eta_T} \in dy) \]
\[ \geq \int_{[b+1,\infty)} \mathbb{P}_y(\exists n \in \mathbb{N} : |Y_n - b| < \varepsilon) \mathbb{P}_x(X_{\eta_T} \in dy) = 1. \]

Since $T$ and $\varepsilon$ were arbitrary this implies that $b$ is locally recurrent.

(iii) Let $\lim_{n \to \infty} Y_n = b$ almost surely for all $x > b$ and fix $r'$, $R' > 0$ and $c < 1$ such that (3.5) holds, i.e., $\sup_{y \geq b + r'} \mathbb{P}_y(X_{\sigma_1} < b + R') < c$.
If $(\sigma_n)_{n \in \mathbb{N}}$ is a.s. bounded then $b$ is reached at least as left limit once in finite time, since $\lim_{n \to \infty} Y_n = b$. In this case the same argument as in part (ii) yields that $b$ is left limit recurrent.
Thus it is sufficient to prove that \((\sigma_n)_{n \in \mathbb{N}}\) is a.s. bounded. To show this, we set \(\sigma_\infty := \lim_{n \to \infty} \sigma_n\), and let \(\varepsilon > 0\). Then, since \(\lim_{n \to \infty} Y_n = b\) a.s., for each \(x > b\) there exists \(N = N(x) > 0\) such that for all \(n \geq N\)

\[
P_x(X_{\sigma_n} < b + R') \geq 1 - \varepsilon.
\]

Now let \(n \geq N\) and define \(\nu_n\) as the time of the first visit to \((b + R', \infty)\) after time \(\sigma_n\), i.e.,

\[
\nu_n := \inf\{t \geq \sigma_n | X_t \geq b + R'\}
\]

and

\[
k := \inf\{l \in \mathbb{N} | \sigma_l > \nu_n\}.
\]

Note that \(\nu_n\) is almost surely finite by (3.1), and \(k < \infty\) if and only if \(\sigma_\infty > \nu_n\). Now assume that \(P_x(\sigma_\infty = \infty) =: p > 0\). On \(\{\sigma_k > \nu_n\}\) the stopping time \(\sigma_k\) is the time of the first jump into \((b, \infty)\) after \(\nu_n\), i.e., on this set \(\sigma_k = \sigma_1 \circ \theta_{\nu_n}\) holds, and we have \(P_x(\sigma_k > \nu_n) \geq p\) or equivalently \(P_x(\sigma_k \leq \nu_n) \leq 1 - p\). Furthermore, \(1_{\{\sigma_k > \nu_n\}}\) is \(F_{\nu_n}\) measurable and the strong Markov property by conditioning on \(F_{\nu_n}\) (the \(\sigma\)-algebra associated with \(\nu_n\)) together with assumption (3.5) yields

\[
1 - \varepsilon \leq P_x(X_{\sigma_k} < b + R') = P_x(X_{\sigma_k} < b + R', \sigma_k \leq \nu_n) + P_x(X_{\sigma_k} < b + R', \sigma_k > \nu_n)
\]

\[
\leq P_x(\sigma_k \leq \nu_n) + E_x(E_x(1_{\{X_{\sigma_k} < b + R'\}}1_{\{\sigma_k > \nu_n\}} | F_{\nu_n}))
\]

\[
= P_x(\sigma_k \leq \nu_n) + E_x(1_{\{\sigma_k > \nu_n\}}P_{X_{\nu_n}}(X_{\sigma_1} < b + R'))
\]

\[
< P_x(\sigma_k \leq \nu_n) + c(1 - P_x(\sigma_k \leq \nu_n)) \leq (1 - p)(1 - c) + c < 1,
\]

which is a contradiction, since \(\varepsilon\) was arbitrary. Thus \(p = 0\), i.e., \((\sigma_n)_{n \in \mathbb{N}}\) is a.s.

bounded. \(\square\)

**Remark 3.5.** In order to use this approach for \(d > 1\) one has to replace \((-\infty, b]\) and \([b, \infty)\) by parts of the state space separated by a \((d - 1)\)-dimensional hyperplane. Furthermore (3.1) has to be reformulated, such that it ensures that the process passes the hyperplane infinitely often and reaches an arbitrary large distance to the hyperplane. Then analogous to (3.2) it has to be required that the up/down shoots with respect to the hyperplane do not hit it. With this an analogue to Proposition 3.2 holds. Also an analogous result to Theorem 3.3 can be proved. For part (i) condition (3.4) has to be defined with respect to the hyperplane and the limit of the distance of the overshoots to the hyperplane should become arbitrary large with probability 1, part (ii) for \(b \in \mathbb{R}^d\) is analogous to the one-dimensional case and part (iii) requires again a reformulation of (3.5) in terms of the hyperplane.

But note that for \(d > 1\) the set of cases where the theorem does not lead to a conclusion will be considerably larger than in one dimension, since the transience part only considers deviations which are (in a sense) orthogonal to the hyperplane.

4. Recurrence and transience of processes

In this section we will link local recurrence and local transience to the notion of recurrence and transience for processes, as used by Meyn and Tweedie e.g. in [11] (our presentation is partly motivated by [16]). Note that all results of this section would also hold if we weaken our assumption on the processes from càdlàg to only right continuous.
In the following $\lambda$ denotes the Lebesgue measure.

**Definition 4.1.** A process $(X_t)_{t \geq 0}$ on $\mathbb{R}^d$ is called

- **$\lambda$-irreducible** if
  \[ \lambda(A) > 0 \Rightarrow \mathbb{E}_x \left( \int_0^\infty 1_A(X_t) \, dt \right) > 0 \quad \text{for all } x, \]

- **recurrent** with respect to $\lambda$ if
  \[ \lambda(A) > 0 \Rightarrow \mathbb{E}_x \left( \int_0^\infty 1_A(X_t) \, dt \right) = \infty \quad \text{for all } x, \]

- **Harris recurrent** with respect to $\lambda$ if
  \[ \lambda(A) > 0 \Rightarrow \mathbb{P}_x \left( \int_0^\infty 1_A(X_t) \, dt = \infty \right) = 1 \quad \text{for all } x, \]

- **transient** if there exists a countable cover of $\mathbb{R}^d$ with sets $A_j$ such that for each $j$ there is a finite constant $M_j > 0$ such that:
  \[ \mathbb{E}_x \left( \int_0^\infty 1_{A_j}(X_t) \, dt \right) < M_j, \]

- a **$T$-model** if for some probability measure $\mu$ on $[0, \infty)$ there exists a kernel $T(x, A)$ with $T(x, \mathbb{R}^d) > 0$ for all $x$ such that the function $x \mapsto T(x, A)$ is lower semi-continuous for all $A \in \mathcal{B}(\mathbb{R}^d)$ and
  \[ \int_0^\infty \mathbb{E}_x \left( 1_A(X_t) \right) \mu(dt) \geq T(x, A) \]
  holds for all $x, A \in \mathcal{B}(\mathbb{R}^d)$.

We start with the recurrence–transience dichotomy for $\lambda$-irreducible $T$-models, which was essentially proved in [16].

**Theorem 4.2.** Let $(X_t)_{t \geq 0}$ be a $\lambda$-irreducible $T$-model, then it is either Harris recurrent or transient.

**Proof.** The statement was proved in Prop. 3.1 in [16] under the additional assumption that the process $(X_t)_{t \geq 0}$ satisfies a certain stochastic differential equation. But this assumption was only due to the topic of the paper, and it was not required in the proof. The main idea of the proof is that by [11,10] (see also [17]) a $\lambda$-irreducible $T$-model is either transient or Harris recurrent with respect to $\phi = \mu R$, where $\mu$ is a non-trivial measure and $R$ is a kernel which satisfies

\[ \lambda(A) > 0 \Rightarrow R(x, A) > 0 \quad \forall x \in \mathbb{R}^d. \]

Thus $(X_t)_{t \geq 0}$ is also Harris recurrent with respect to $\lambda$. \qed

Now we can state the main theorem of this section which links the local notions introduced in Section 2 to the stability of the process.

**Theorem 4.3.** Let $(X_t)_{t \geq 0}$ on $\mathbb{R}^d$ be a $\lambda$-irreducible $T$-model, then

(i) $\exists b \in \mathbb{R}^d$ which is locally recurrent $\iff (X_t)_{t \geq 0}$ is Harris recurrent.

(ii) $\exists b \in \mathbb{R}^d$ which is locally transient $\iff (X_t)_{t \geq 0}$ is transient.
Proof. Let \((X_t)_{t \geq 0}\) be a \(\lambda\)-irreducible \(T\)-model. This is, by Theorem 4.2, either Harris recurrent or transient. Thus it is enough to prove the equivalence in (i) since local recurrence and local transience are also complementary.

We follow \([17]\) and call a point \(b \in \mathbb{R}^d\) topologically recurrent if \(\mathbb{E}_x \left( \int_0^\infty 1_A(X_t) \, dt \right) = \infty\) for all neighborhoods \(A\) of \(b\), and we call \(b\) reachable if for every \(x\) and every neighborhood \(A\) of \(b\) the probability \(\mathbb{P}_x(\tau_A < \infty)\) is positive. Note that for a \(\lambda\)-irreducible process each point is reachable. Thus we can use Thm. 4.1 of \([17]\), which states that for a \(\lambda\)-irreducible \(T\)-model \((X_t)_{t \geq 0}\), for which \(b\) is reachable, we have:

\[b\] is topologically recurrent \(\Leftrightarrow (X_t)_{t \geq 0}\) is recurrent.

Now assume that \(b\) is locally recurrent. For any neighborhood \(A\) of \(b\) we find an open ball with center \(b\) and radius \(\varepsilon > 0\) such that \(B_\varepsilon(b) \subset A\). The local recurrence implies that the process hits \(B_\varepsilon(b)\) with probability one, also after arbitrary large times, i.e., for all \(R > 0\)

\[\mathbb{P}_b \left( \exists t > R : X_t \in B_\varepsilon(b) \right) = 1.\]

Furthermore since \(X_t\) is right continuous the average time spent in \(B_\varepsilon(b)\) after hitting \(B_\varepsilon(b)\) is positive, i.e.,

\[0 < \inf_{y \in B_\varepsilon(b)} \mathbb{E}_y(\tau_{\mathbb{R}^d \setminus B_\varepsilon(b)}).\]

Thus we get

\[\mathbb{E}_b \left( \int_0^\infty 1_A(X_t) \, dt \right) \geq \mathbb{E}_b \left( \int_0^\infty 1_{B_\varepsilon(b)}(X_t) \, dt \right) \geq \infty,\]

i.e., \(b\) is topological recurrent. Therefore \((X_t)_{t \geq 0}\) is recurrent. By the dichotomy we get that in fact \((X_t)_{t \geq 0}\) is Harris recurrent, since it is not transient.

On the other hand, let \((X_t)_{t \geq 0}\) be Harris recurrent. Thus

\[\mathbb{P}_x \left( \int_0^\infty 1_A(X_t) \, dt = \infty \right) = 1 \quad \text{for all } x \text{ and all } A \text{ with } \lambda(A) > 0\]

holds and especially the path returns into \(B_\varepsilon(b)\) for any \(\varepsilon > 0\) after any time, i.e., \(b\) is locally recurrent. \(\square\)

We further recall the following theorem, which provides some way to check that \((X_t)_{t \geq 0}\) is a \(T\)-model.

**Theorem 4.4** *(Thms. 5.1 and 7.1 in [17]).*

(i) \((X_t)_{t \geq 0}\) is a \(T\)-model, if every compact set \(C\) is petite, i.e., there exists a probability measure \(\mu\) on \([0, \infty)\) and a non-trivial measure \(\nu\) on \(\mathbb{R}^d\) such that

\[\int_0^\infty \mathbb{E}_x(1_A(X_t)) \, \mu(dt) \geq \nu(A) \quad \text{for all } x \in C \text{ and all } A.\]

(ii) Let \((X_t)_{t \geq 0}\) be \(\lambda\)-irreducible and \(x \mapsto \mathbb{E}_x(f(X_t))\) be continuous for all continuous and bounded functions \(f\), then \((X_t)_{t \geq 0}\) is a \(T\)-model.

Part (ii) in particular shows that every \(\lambda\)-irreducible \(C_b\)-Feller process is a \(T\)-model, and note that [15] gives necessary and sufficient conditions for a \(C_\infty\)-Feller process to be also \(C_b\)-Feller.

The following theorem is useful for applications. It gives sufficient criteria for a process to be a \(\lambda\)-irreducible \(T\)-model.
Theorem 4.5. Let \((X_t)_{t \geq 0}\) be a process on \(\mathbb{R}^d\) and denote its transition probabilities by
\[ P_t(x, A) := \mathbb{P}_x(X_t \in A). \]

Then
(i) \((X_t)_{t \geq 0}\) is \(\lambda\)-irreducible if
\[ \lambda(A) > 0 \Rightarrow P_t(x, A) > 0 \quad \text{for all } t > 0, x \in \mathbb{R}^d, \] (4.1)
(ii) \((X_t)_{t \geq 0}\) is a \(\lambda\)-irreducible T-model if (4.1) holds and there exists a compact set \(K \subset [0, \infty]\) and a non-trivial measure \(\nu\) such that for all compact sets \(C \subset \mathbb{R}^d\)
\[ \inf_{t \in K} \inf_{x \in C} P_t(x, A) \geq \nu(A) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^d). \] (4.2)

Further, a special case of (ii):
(iii) \((X_t)_{t \geq 0}\) is a \(\lambda\)-irreducible T-model if the transition probability \(P_t(x, .)\) is the sum of a, possibly trivial, discrete measure and a measure which has a (sub)probability density \(\tilde{p}_t(x, y)\) with respect to \(\lambda\) such that
\[ \tilde{p}_t(x, y) > 0 \quad \text{for all } x, y \in \mathbb{R}^d, t > 0, \] (4.3)
\[ \inf_{t \in [1, 2]} \inf_{x \in C} \tilde{p}_t(x, y) > 0 \quad \text{for all } y \in \mathbb{R}^d \text{ and all compact sets } C. \] (4.4)

Proof. Let \((X_t)_{t \geq 0}\) be a process on \(\mathbb{R}^d\) with transition probabilities \(P_t(x, .)\).
Assume that (4.1) holds and let \(A\) be such that \(\lambda(A) > 0\). Then
\[ \mathbb{P}_x(\tau_A < \infty) \geq P_t(x, A) > 0 \quad \text{for any } t > 0, \]
and Prop. 2.1 in [11] states that, under the above condition, the process \((X_t)_{t \geq 0}\) is \(\phi\)-irreducible with
\[ \phi(.) := \iint_{[0, \infty)} e^{-t} P_t(x, .) \, dt \, \lambda(dx). \]
Clearly, for \(A \in \mathcal{B}(\mathbb{R}^d)\) with \(\lambda(A) > 0\) we have
\[ \int_{[0, \infty)} e^{-t} P_t(x, A) \, dt > 0. \]
Therefore \(\phi\) is equivalent to \(\lambda\), i.e., \((X_t)_{t \geq 0}\) is \(\lambda\)-irreducible.

If in addition (4.2) holds, then Theorem 4.4 part (i) with \(\mu(dt) = e^{-t} \, dt\) implies that \((X_t)_{t \geq 0}\) is a T-model.

For part (iii) note that (4.3) implies that (4.1) holds and (4.4) implies that (4.2) holds with \(\nu\) being a subprobability measure with density \(\inf_{t \in [1, 2]} \inf_{x \in C} \tilde{p}_t(x, .)e^{-2}. \)

We give a further characterization of recurrence and transience in this context, which shows that it is in fact enough to know the behavior of the process outside some compact set.

Theorem 4.6. Let \((X_t)_{t \geq 0}\) be \(\lambda\)-irreducible T-model, \(R\) be some positive constant and \(\overline{B_R(0)}\) denote the closed ball centered at 0 with radius \(R\), then
(i) \( \forall x : \mathbb{P}_x \left( \tau_{\partial R(0)} < \infty \right) = 1 \iff (X_t)_{t \geq 0} \) is Harris recurrent.

(ii) \( \exists x : \mathbb{P}_x \left( \tau_{\partial R(0)} < \infty \right) < 1 \iff (X_t)_{t \geq 0} \) is transient.

**Proof.** Let \( (X_t)_{t \geq 0} \) be a \( \lambda \)-irreducible \( T \)-model. Then, by Thm. 5.1 in [17], every compact set is petite (compare with Theorem 4.4 part (i)). Furthermore Thm. 3.3 in [10] states that the facts that \( \mathbb{P}_x \left( \tau_{\partial R(0)} < \infty \right) = 1 \) for all \( x \) and \( \bar{B}_R(0) \) is petite imply that \( (X_t)_{t \geq 0} \) is Harris recurrent. Thus we have shown the implication \( \Rightarrow \) in (i).

For \( \Rightarrow \) in (ii) note that \( \lambda(B_R(0)) > 0 \). Thus \( (X_t)_{t \geq 0} \) cannot be Harris recurrent and the dichotomy implies that it is transient.

Harris recurrence and transience are complementary. Also the left-hand sides of (i) and (ii) are complementary. Thus the implications \( \Leftarrow \) in (i) and (ii) hold. \( \square \)

In fact Theorem 4.6 shows that \( \lambda \)-irreducible \( T \)-models which coincide outside a ball have the same recurrence and transience behavior, respectively.

**Corollary 4.7.** Let \( (X_t)_{t \geq 0} \) and \( (Y_t)_{t \geq 0} \) be \( \lambda \)-irreducible \( T \)-models. If there exists an \( R > 0 \) such that

\[
\tau_{\partial R(0)}^X \overset{d}{=} \tau_{\partial R(0)}^Y \quad \text{for all } X_0 = Y_0 = x \in \mathbb{R}^d \setminus \bar{B}_R(0)
\]

then \( (X_t)_{t \geq 0} \) and \( (Y_t)_{t \geq 0} \) have the same recurrence/transience behavior.

Here \( \tau^X \) and \( \tau^Y \) are the entrance times corresponding to \( X_t \) and \( Y_t \), respectively and \( \overset{d}{=} \) denotes equality in distribution.

**Proof.** In the setting of Theorem 4.6 we find

\[
\mathbb{P}_x \left( \tau_{\partial R(0)} = 0 \right) = 1 \quad \text{for all } x \in \bar{B}_R(0).
\]

This shows that \( \mathbb{P}_x \left( \tau_{\partial R(0)} < \infty \right) \) can only be less than 1 for some \( x \in \mathbb{R}^d \setminus \bar{B}_R(0) \), i.e., only the distribution of \( \tau_{\partial R(0)} \) for \( x \in \mathbb{R}^d \setminus \bar{B}_R(0) \) needs to be checked. Thus, the distribution of \( \tau_{\partial R(0)} \) for \( x \in \mathbb{R}^d \setminus \bar{B}_R(0) \) is sufficient to determine the recurrence/transience behavior by Theorem 4.6. \( \square \)

5. \( \alpha \)-stable and stable-like processes

Let \( (X_t)_{t \geq 0} \) be a real-valued symmetric \( \alpha \)-stable process, i.e., it is a Lévy process with characteristic exponent \( |\xi|^{\alpha} \) with \( \alpha \in (0, 2) \). In particular it is a time homogeneous strong Markov process with càdlàg paths. Note that \( (X_t)_{t \geq 0} \) sampled at integer times \( (X_n)_{n \in \mathbb{N}_0} \) is a symmetric random walk, and hence (3.1) holds. Define \( \sigma^b \) and \( \tau^b \) as in Section 3, i.e., for \( b \in \mathbb{R} \)

\[
\tau^b := \inf \{ t \geq 0 \mid X_t \leq b \} \quad \text{and} \quad \sigma^b := \inf \{ t \geq 0 \mid X_t \geq b \}.
\]

In 1958 Ray [13] showed that for \( b > 0 \)

\[
\mathbb{P}_0(X_{\sigma^b} \in dy) = \frac{\sin \left( \frac{\alpha \pi}{2} \right)}{\pi} \frac{1}{y} \left( \frac{b}{y - b} \right)^{\frac{\alpha}{2}} 1_{[b, \infty)}(y) \, dy
\]

and in particular for \( 0 < \alpha < 2 \)

\[
\mathbb{P}_0(X_{\sigma^b} = b) = 0.
\]
The translation invariance of \((X_t)_{t \geq 0}\) yields for all \(b \in \mathbb{R}\)
\[
\mathbb{P}_x(X_{\sigma b} \in dy) = \mathbb{P}_0(X_{\sigma b} - x \in dy)
\]
\[
= \frac{\sin \left( \frac{\alpha \pi}{2} \right)}{\pi} \frac{1}{y-x} \left( \frac{b-x}{y-b} \right) \mathbb{I}_{[b,\infty)}(y) \, dy \quad \text{for } x < b
\]  
(5.1)
and the symmetry yields
\[
\mathbb{P}_x(X_{\tau b} \in dy) = \mathbb{P}_{-x}(-X_{\sigma b} \in dy)
\]
\[
= \frac{\sin \left( \frac{\alpha \pi}{2} \right)}{\pi} \frac{1}{x-y} \left( \frac{b-y}{x-b} \right) \mathbb{I}_{(-\infty,b]}(y) \, dy \quad \text{for } x > b.
\]  
(5.2)

In particular (3.2) is satisfied.

Note that by the translation invariance we have for any \(b \in \mathbb{R}\):

for \(x < 0\): \(\mathbb{P}_x(X_{\sigma b} < r) = \mathbb{P}_{x+b}(X_{\sigma b} < r + b)\),
for \(x > 0\): \(\mathbb{P}_x(X_{\tau b} < r) = \mathbb{P}_{x+b}(X_{\tau b} < r + b)\).

Thus, for simplicity, we will only consider the case \(b = 0\) in what follows and define the upwards-overshoot density \(u\) and the downwards-overshoot density \(v\) for \(\alpha \in (0, 2)\) by

for \(x < 0\): \(u_{\alpha}(x, y) := \frac{\sin \left( \frac{\alpha \pi}{2} \right)}{\pi} \frac{1}{y-x} \left( \frac{-x}{y} \right)^{\frac{\alpha}{2}} \mathbb{I}_{[0,\infty)}(y)\)

for \(x > 0\): \(v_{\alpha}(x, y) := \frac{\sin \left( \frac{\alpha \pi}{2} \right)}{\pi} \frac{1}{x-y} \left( \frac{y}{x} \right)^{\frac{\alpha}{2}} \mathbb{I}_{(-\infty,0]}(y)\).

We will write \(X \sim f\) for a random variable \(X\) with density \(f\).

**Lemma 5.1.** Let \(\alpha, \beta \in (0, 2)\) and \(U \sim u_{\alpha}(-1, \cdot)\) and \(V \sim v_{\beta}(1, \cdot)\) be independent. Then

(i) the overshoot densities satisfy for \(y \in \mathbb{R}\)

for \(x < 0\): \(u_{\alpha}(x, y) = -\frac{1}{x} u_{\alpha} \left( -1, -\frac{y}{x} \right)\)

and

for \(x > 0\): \(v_{\beta}(x, y) = \frac{1}{x} v_{\beta} \left( 1, \frac{y}{x} \right)\),

(ii) for arbitrary probability densities \(f\) on \([0, \infty)\) and \(g\) on \((-\infty, 0]\), and random variables \(F \sim f, G \sim g\) independent of \(V\) and \(U\), respectively, it holds that (for \(s \in \mathbb{R}\))

\[
\mathbb{P}(FV \leq s) = \int_{-\infty}^{s} \int_{-\infty}^{\infty} f(x) v_{\beta}(x, y) \, dx \, dy
\]

and

\[
\mathbb{P}(-GU \leq s) = \int_{-\infty}^{s} \int_{-\infty}^{\infty} g(x) u_{\alpha}(x, y) \, dx \, dy,
\]

(iii) for \(r \in \mathbb{R}\)

\[
\mathbb{E}(U^r) = \begin{cases} 
\frac{\sin \left( \frac{\alpha \pi}{2} \right)}{\pi} & \text{for } \frac{\alpha}{2} - 1 < r < \frac{\alpha}{2}, \\
\sin \left( \frac{(\alpha - 2r)\pi}{2} \right) & \text{otherwise,}
\end{cases}
\]
and

\[ \mathbb{E}((-VU)^r) = \begin{cases} \sin \left( \frac{\alpha \pi}{2} \right) \sin \left( \frac{\beta \pi}{2} \right) & \text{for } \alpha \lor \beta = \frac{\alpha \land \beta}{2} - 1 < r < \frac{\alpha \land \beta}{2}, \\ \sin \left( \frac{(\alpha-2r)\pi}{2} \right) \sin \left( \frac{(\beta-2r)\pi}{2} \right) & \text{otherwise}, \end{cases} \]

(iv) for \( \alpha + \beta \neq 2 \) there exists a moment of a downwards-overshoot followed by an upwards-overshoot which is less than 1, i.e.,

\[ \alpha + \beta < 2 : \exists r < 0 : \mathbb{E}(-VU)_r < 1, \]

\[ \alpha + \beta > 2 : \exists r > 0 : \mathbb{E}(-VU)_r < 1, \]

and for \( \alpha + \beta = 2 \) there is a symmetry:

\[ \forall s : \mathbb{P}(-VU \leq s) = \mathbb{P}((-VU)^{-1} \leq s). \]

**Proof.** (i) For \( x < 0 \)

\[ -\frac{1}{x} u_\alpha \left( -1, -\frac{y}{x} \right) = \frac{\sin \left( \frac{\alpha \pi}{2} \right)}{\pi} \frac{1}{-x \left( -\frac{y}{x} + 1 \right)} \left( \frac{1}{-\frac{y}{x}} \right)^{\alpha} 1_{[0, \infty)} \left( -\frac{y}{x} \right) \]

\[ = \sin \left( \frac{\alpha \pi}{2} \right) \frac{1}{y-x} \left( -\frac{x}{y} \right)^{\alpha} 1_{[0, \infty)} (y) \]

and analogously for \( x > 0 \)

\[ \frac{1}{x} v_\beta \left( 1, \frac{y}{x} \right) = \frac{\sin \left( \frac{\beta \pi}{2} \right)}{\pi} \frac{1}{x \left( 1 - \frac{y}{x} \right)} \left( \frac{1}{-\frac{y}{x}} \right)^{\beta} 1_{(-\infty, 0]} \left( \frac{y}{x} \right) \]

\[ = \sin \left( \frac{\beta \pi}{2} \right) \frac{1}{x-y} \left( -\frac{x}{y} \right)^{\beta} 1_{(-\infty, 0]} (y) \]

(ii) Let \( s \in \mathbb{R} \). Now, by the formula of part (i) and the substitution \( \tilde{y}x = y \), we get

\[ \int_{-\infty}^{s} \int_{-\infty}^{\infty} f(x) v_\beta (x, y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{s} f(x) \frac{1}{x} v_\beta (1, \frac{y}{x}) \, dy \, dx \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1_{(-\infty, s]}(\tilde{y}x) f(x) \frac{1}{x} v_\beta (1, \tilde{y}) x \, d\tilde{y} \, dx \]

\[ = \mathbb{P}(FV \leq s). \]

Similarly, the substitution \( -\tilde{y}x = y \) yields

\[ \int_{-\infty}^{s} \int_{-\infty}^{\infty} g(x) u_\alpha (x, y) \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{s} g(x) \left( -\frac{1}{x} \right) u_\alpha \left( -1, -\frac{y}{x} \right) \, dy \, dx \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1_{(-\infty, s]}(-\tilde{y}x) g(x) \left( -\frac{1}{x} \right) u_\alpha (-1, \tilde{y}) (-x) \, d\tilde{y} \, dx \]

\[ = \mathbb{P}(-GU \leq s). \]
(iii) Note that
\[
\int_0^\infty (y + 1)^{-1} y^{-s} \, dy = B(1 - s, s) = \frac{\Gamma(1 - s)\Gamma(s)}{\Gamma(1)} = \frac{\pi}{\sin(s\pi)} \quad \text{for all } 0 < s < 1
\]
where $B(\cdot, \cdot)$ is the Beta function and the last equality holds by the reflection formula for the Gamma function (e.g. 6.1.17 in [1]). Thus
\[
\mathbb{E}(U'^r) = \int_0^\infty y^r u_\alpha(1, y) \, dy = \frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\pi} \int_0^\infty (y + 1)^{-1} y^{-\frac{\alpha}{2} + r} \, dy
\]
for all $r$ such that $\frac{\alpha}{2} - 1 < r < \frac{\alpha}{2}$. Further for $r \geq \frac{\alpha}{2}$ and $y \geq 1$:
\[
(y + 1)^{-1} y^{-\frac{\alpha}{2} + r} \geq \frac{1}{2} y^{-\frac{\alpha}{2} + r - 1},
\]
which is not integrable on $[1, \infty)$. Thus $\mathbb{E}(U'^r) = \infty$ for $r \geq \frac{\alpha}{2}$. Similarly, for $r \leq \frac{\alpha}{2} - 1$ and $y \leq 1$:
\[
(y + 1)^{-1} y^{-\frac{\alpha}{2} + r} \geq y^{-\frac{\alpha}{2} + r},
\]
which is not integrable on $(0, 1]$. Thus $\mathbb{E}(U'^r) = \infty$ for $r \leq \frac{\alpha}{2} - 1$.

Furthermore, for $y > 0$
\[
v_\beta(1, y) = \frac{\sin\left(\frac{\beta\pi}{2}\right)}{\pi} \frac{1}{y + 1} y^{-\frac{\beta}{2}} 1_{(0, \infty)}(y) = u_\beta(-1, y)
\]
and hence
\[
\mathbb{E}((-V)'^r) = \int_{-\infty}^\infty (-y)^r v_\beta(1, y) \, dy = \int_{-\infty}^\infty \tilde{y}^r u_\beta(-1, \tilde{y}) \, d\tilde{y}.
\]
Finally, the independence of $V, U$ yields
\[
\mathbb{E}((-VU)'^r) = \frac{\sin\left(\frac{\alpha\pi}{2}\right) \sin\left(\frac{\beta\pi}{2}\right)}{\sin\left(\frac{(\alpha - 2r)\pi}{2}\right) \sin\left(\frac{(\beta - 2r)\pi}{2}\right)}
\]
for $r \in \left(\frac{\alpha + \beta}{2} - 1, \frac{\alpha + \beta}{2}\right)$.

(iv) For $r^* = \frac{\alpha + \beta}{4} - \frac{1}{2}$ we find
\[
\mathbb{E}((-VU)'^r) = \frac{\sin\left(\frac{\alpha\pi}{2}\right) \sin\left(\frac{\beta\pi}{2}\right)}{\sin\left(\frac{(\alpha - \beta/4)\pi}{2}\right) \sin\left(\frac{(\beta - \alpha/4)\pi}{2}\right)} = \frac{1 + \cos\left(\frac{\alpha + \beta}{2}\pi\right)}{1 + \cos\left(\frac{\alpha - \beta}{2}\pi\right)},
\]
where we used in the first step translation identities and symmetry of sin and cos. In the last step formula 4.3.31 [1] was used for the numerator and 4.3.25 [1] for the denominator.
Thus $\mathbb{E}((-VU)'^r) < 1$ for $\alpha + \beta \neq 2$. Note that $r^*$ is negative for $\alpha + \beta < 2$ and positive for $\alpha + \beta > 2$. 
Finally for $\alpha + \beta = 2$ we get

$$
\mathbb{P}(-UV \leq s) = \int \int 1_{(-\infty, s]}(-\tilde{x} \tilde{y}) v_2(1, \tilde{x}) u_\alpha(-1, \tilde{y}) \, d\tilde{y} \, d\tilde{x}
$$

$$
= \int \int 1_{(-\infty, s]} \frac{-1}{xy} u_\alpha(-1, \frac{1}{y}) \frac{1}{x^2 y^2} \, dy \, dx
$$

$$
= \int \int 1_{(-\infty, s]} \frac{-1}{xy} \frac{\sin \left( \frac{\alpha \pi}{2} \right)}{\pi} \sin \left( \frac{\beta \pi}{2} \right)

\times \frac{1}{1 + \frac{1}{x} \frac{\beta}{2} - 1 - \frac{1}{y} \frac{\alpha}{2}} \frac{1}{y} (-y)^{\frac{\beta}{2} - 1} \, dy \, dx
$$

$$
= \int \int 1_{(-\infty, s]} \frac{-1}{xy} u_\alpha(1, y) u_\alpha(-1, x) x^{\frac{\beta + \alpha}{2} - 1} (-y)^{\frac{\alpha + \beta}{2} - 1} \, dy \, dx
$$

$$
= \mathbb{P}(-(UV)^{-1} \leq s),
$$

where we used in the second line the substitutions $\tilde{x} = -\frac{1}{x}$ and $\tilde{y} = -\frac{1}{y}$.

The above lemma enables us to prove the main theorem of this section.

**Theorem 5.2.** Let $(X_t)_{t \geq 0}$ be a càdlàg time homogeneous strong Markov process on $\mathbb{R}$ such that (3.1) holds and such that there exist $b \in \mathbb{R}$, $\alpha, \beta \in (0, 2)$ such that

$$
\lim_{t \to 0} \mathbb{E}_x \left( \frac{e^{iX_t \xi} - 1}{t} \right) = \begin{cases} 
-|\xi|^\alpha & \text{for } x > b, \\
-|\xi|^\beta & \text{for } x < b.
\end{cases}
$$

(5.3)

Then

(i) $b$ is left limit recurrent if $\alpha + \beta > 2$,

(ii) $b$ is recurrent if $\alpha + \beta \geq 2$,

(iii) $b$ is transient if $\alpha + \beta < 2$.

**Remark 5.3.** Note that (5.3) does not pose a condition on the symbol of the process started in $b$. In fact, the behavior for $b = x$ can be arbitrary, as long as the process is a càdlàg time homogeneous strong Markov process on $\mathbb{R}$ satisfying (3.1). Furthermore, the existence of such a process is in general non-trivial. Naturally one might consider a stable-like process (in the sense of Bass [2]) with discontinuous $\alpha(\cdot)$, but up to now all stable-like processes studied in the literature have continuous $\alpha(\cdot)$.

The proof of the existence of such a process (and that it is a $\lambda$-irreducible $T$-model) is part of ongoing research and will be postponed to a forthcoming paper. This seems reasonable to us, since the existence of the process can be related to the question of solving SDEs with discontinuous coefficients, and the solution theory for such equations requires tools which go beyond the scope of the present paper.
Proof of Theorem 5.2. Let \((X_t)_{t \geq 0}\) be a c\'adl\'ag time homogeneous strong Markov process on \(\mathbb{R}\) such that (3.1) holds and (5.3) is satisfied for some \(b \in \mathbb{R}\). Then this process satisfies (3.2), since the overshoots across \(b\) coincide with \(\beta\)-stable or \(\alpha\)-stable overshoots, and these satisfy (3.2) as shown at the beginning of this section.

If \(b \neq 0\) consider \((X_t - b)_{t \geq 0}\) for which the properties at 0 correspond to those of \((X_t)_{t \geq 0}\) at \(b\). Thus, without loss of generality, we may assume that \(b = 0\).

For \(x > 0\) let \((Y_n)_{n \geq 0}\) be the overshoot Markov chain corresponding to \((X_t)_{t \geq 0}\) as defined in Section 3. Then for \(s \in \mathbb{R}\)

\[
\mathbb{P}_x(Y_n \leq s) = \int_{-\infty}^{s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_\beta(y, v) u_\alpha(v, u) \mathbb{P}_x(Y_{n-1} \in dy) \, dv \, du
\]

and by Lemma 5.1(ii)

\[
Y_n = Y_1 \prod_{i=1}^{n-1} (-U_i V_i),
\]

where \(U_i \sim u_\alpha(-1, \cdot), V_i \sim v_\beta(1, \cdot)\) and \((U_i)_{i=1,\ldots,n-1}, (V_i)_{i=1,\ldots,n-1}, Y_1\) are independent. In particular for \(r \in \mathbb{R}\)

\[
\mathbb{E}_x(Y_n^r) = \mathbb{E}_x(Y_1^r) \left( \mathbb{E}((-U_1 V_1)^r) \right)^{n-1}.
\]

Furthermore, using the definition of \(Y_1\) and Lemma 5.1(ii) for \(\tilde{V} \sim v_\beta(x, \cdot)\) independent of \(U_1\)

\[
\mathbb{E}_x(Y_1^r) = \mathbb{E}(-\tilde{V}^r) \mathbb{E}(U_1^r)
\]

and

\[
\mathbb{E}(-\tilde{V}^r) = -\int_{\mathbb{R}} \tilde{v}^r v_\beta(x, \tilde{v}) \, d\tilde{v} = -\int_{-\infty}^{\infty} v^r \frac{1}{x} v_\beta \left(1, \frac{v}{x}\right) \, dv = \frac{-x^r}{r} \int_{\mathbb{R}} v^r v_\beta(1, v) \, dv = x^r \mathbb{E}(-V_1^r).
\]

To prove (i), let \(\alpha + \beta > 2\) and choose \(r > 0\), cf. Lemma 5.1(iv), such that

\[
\mathbb{E}((-U_1 V_1)^r) < 1.
\]

Then \(\mathbb{E}_x(Y_1^r) < \infty\) and for all \(\varepsilon > 0\), by the Chebyshev inequality,

\[
\sum_{n=1}^{\infty} \mathbb{P}_x(Y_n \geq \varepsilon) \leq \sum_{n=1}^{\infty} \frac{\mathbb{E}_x(Y_n^r)}{\varepsilon^r} = \frac{\mathbb{E}_x(Y_1^r)}{\varepsilon^r} \sum_{n=1}^{\infty} \mathbb{E}((-U_1 V_1)^r)^{n-1} = \frac{\mathbb{E}_x(Y_1^r)}{\varepsilon^r} \frac{1}{1 - \mathbb{E}((-U_1 V_1)^r)} < \infty.
\]

Thus the Borel–Cantelli Lemma implies that \(Y_n \xrightarrow{n \to \infty} 0\) almost surely. Let \(q \in \left(\frac{\alpha \wedge \beta}{2} - 1, 0\right)\), then \(0 < \mathbb{E}((-U_1 V_1)^q) < \infty\) by Lemma 5.1(iii). With \(R' := \left(2 \mathbb{E}((-U_1 V_1)^q)\right)^{\frac{1}{q}}\) we get

\[
\sup_{y \geq 1} \mathbb{P}_y(X_{\sigma_1} < R') = \sup_{y \geq 1} \mathbb{P}_y(Y_1 < R') = \sup_{y \geq 1} \mathbb{P}_y(Y_1^q > R'^q) \leq \frac{1}{R'^q} \mathbb{E}((-U_1 V_1)^q) \leq \frac{1}{2},
\]

i.e., (3.5) holds. Thus 0 is left limit recurrent by Theorem 3.3(iii).
Analogously, to prove (iii), let $\alpha + \beta < 2$ and choose $r < 0$, cf. Lemma 5.1(iv), such that $\mathbb{E}((-U_1 V_1)^r) < 1$.

Then $\mathbb{E}_x (Y_1^r) < \infty$ and for all $\varepsilon > 0$, by the Chebyshev inequality,

$$\sum_{n=1}^{\infty} \mathbb{P}_x \left( \frac{1}{Y_n} \geq \varepsilon \right) \leq \sum_{n=1}^{\infty} \frac{\mathbb{E}_x (Y_n^{-|r|})}{\varepsilon^{|r|}} = \frac{\mathbb{E}_x (Y_1^r)}{\varepsilon^{|r|}} \sum_{n=1}^{\infty} \mathbb{E}((-U_1 V_1)^r)^{n-1} = \frac{\mathbb{E}_x Y_1^r}{\varepsilon^{|r|}} \frac{1}{\mathbb{E}((-U_1 V_1)^r)} < \infty.$$  

Thus the Borel–Cantelli lemma implies $1 / Y_n \xrightarrow{n \to \infty} 0$ almost surely, i.e., $Y_n \xrightarrow{n \to \infty} \infty$ almost surely. Now let $q \in \left(0, \frac{\alpha + \beta}{2}\right)$, then $0 < \mathbb{E}((-U_1 V_1)^q) < \infty$ and $R := (2\mathbb{E}((-U_1 V_1)^q))^{\frac{1}{q}}$ yields

$$\sup_{y \in (0,1]} \mathbb{P}_y (X_{\sigma_1} > R) = \sup_{y \in (0,1]} \mathbb{P}_y (Y_1 > R) \leq \sup_{y \in (0,1]} \frac{\mathbb{E}_y Y_1^q}{R^q} = \sup_{y \in (0,1]} y^q \frac{\mathbb{E}((-U_1 V_1)^q)}{R^q} = \frac{1}{2}.$$  

Moreover, for $y < 0$

$$\mathbb{P}_y (X_{\sigma_1} > R) = \int_R^\infty u_\alpha (y, z) \, dz = \int_R^\infty \frac{-1}{y} u_\alpha \left(-1, \frac{z}{y}\right) \, dz = \int_{-\infty}^R u_\alpha (-1, \tilde{z}) \, d\tilde{z}$$

and thus

$$\sup_{y \in (-1,0]} \mathbb{P}_y (X_{\sigma_1} > R) = \int_R^\infty u_\alpha (-1, \tilde{z}) \, d\tilde{z} < 1,$$

which is strictly less than 1 since $R > 0$ and $u_\alpha$ is a probability density with $u_\alpha (-1, x) > 0$ for all $x > 0$. Therefore (3.4) holds and 0 is by Theorem 3.3(i) locally transient.

Finally, to prove (iii), let $\alpha + \beta = 2$ and note that

$$\log Y_n = d \log Y_1 + \sum_{i=1}^{n-1} \log (-U_i V_i).$$

By Lemma 5.1(iv) for any $r \in \mathbb{R}$

$$\mathbb{P} (\log (-U_1 V_1) \leq r) = \mathbb{P} (\log ((-U_1 V_1)^{-1}) \leq r) = \mathbb{P} (\log (-U_1 V_1) \geq -r)$$

and thus $\log Y_n$ has the same distribution as a symmetric random walk with initial distribution given by log $Y_1$. Hence

$$\lim_{n \to \infty} \sup \log (Y_n) = +\infty \quad \text{and} \quad \lim_{n \to \infty} \inf \log (Y_n) = -\infty$$

holds and therefore

$$\lim_{n \to \infty} \sup Y_n = +\infty \quad \text{and} \quad \lim_{n \to \infty} \inf Y_n = 0.$$  

Now Theorem 3.3(ii) implies that 0 is locally recurrent. 

The next result for symmetric $\alpha$-stable Lévy processes is well known (e.g. [14]). We just present it with a new proof.
Corollary 5.4. Let \((X_t)_{t \geq 0}\) be a symmetric \(\alpha\)-stable Lévy process with stability index \(\alpha \in (0, 2)\), then \((X_t)_{t \geq 0}\) is

(i) point recurrent, i.e., \(\forall x \in \mathbb{R} : \mathbb{P}_x (\forall t > T : X_t = x)\), if \(\alpha > 1\),
(ii) Harris recurrent if \(\alpha \geq 1\),
(iii) transient if \(\alpha < 1\).

Proof. Let \((X_t)_{t \geq 0}\) be a symmetric \(\alpha\)-stable Lévy process. Then it is a càdlàg time homogeneous strong Markov process on \(\mathbb{R}\), which satisfies (3.1) and (3.2), as shown at the beginning of this section. Now, we just apply Theorem 5.2 for \(\alpha = \beta\) and note that \(b\) can be chosen arbitrarily. Further note that the process is clearly a \(\lambda\)-irreducible \(T\)-model, since it is a \(C_b\)-Feller process with positive transition density. Thus Theorem 4.3 yields the recurrence–transience dichotomy. Furthermore, Lemma 2.3 is applicable since the process is a Hunt process, i.e., in particular it is quasi-left continuous (e.g. Thm. I.9.4 in [4]).

The results of Section 2 show that two \(\lambda\)-irreducible \(C_b\)-Feller processes have the same recurrence (transience) behavior if they have the same generator outside an arbitrary ball. In particular we get the following corollary for stable-like processes.

Corollary 5.5. Assume that the process in Theorem 5.2 exists and is a \(\lambda\)-irreducible \(T\)-model. Let \((X_t)_{t \geq 0}\) be a stable-like process on \(\mathbb{R}\) with symbol \(|\xi|^{\alpha(x)}\) and suppose there exists \(\alpha, \beta \in (0, 2)\) such that for some arbitrary \(R > 0\)
\[
\begin{align*}
\alpha(x) &= \alpha \quad \text{for } x < -R, \\
\alpha(x) &= \beta \quad \text{for } x > R,
\end{align*}
\]
then \((X_t)_{t \geq 0}\) is

• Harris recurrent if and only if \(\alpha + \beta \geq 2\),
• transient if and only if \(\alpha + \beta < 2\).

Proof. The process \((X_t)_{t \geq 0}\) given above is \(\lambda\)-irreducible, since it has a transition density with respect to the Lebesgue measure (cf. [12]), and it is a \(T\)-model, since it is a \(C_b\)-Feller process by Prop. 6.2 in [2].

The process coincides on \(\mathbb{R} \setminus \overline{B_R(0)}\) with the process of Theorem 5.2 and therefore by Corollary 4.7 both processes have the same recurrence/transience behavior. Thus Theorem 5.2 implies the result.

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