Injective Dimension of Semi-Primary Rings*

ABRAHAM ZAKS

Technion, Mathematics Department, Haifa, Israel

Communicated by I. N. Herstein
Received October 10, 1968

In recent works by Eilenberg Nagao and Nakayama [4], Jans and Nakayama [8], Harada [7], and Zaks [11] the problem of characterizing semi-primary rings all of whose residue rings have finite global dimension, was solved. It turns out that if $N$ is the radical of a semi-primary ring, then the ring $R/N^2$ plays a crucial role. It also plays an important role in the study of restricted quasi-Frobenius rings, in works by Faith [5], Levy [9], and Zaks [12].

The purpose of this paper is to investigate the possibility of replacing the global dimension by injective dimension. The ideas beyond the treatment are very much influenced by the ones used in the global dimension case, as for instance starting the study from rings with radical of square zero. The most surprising result seems to be that if $N^a = 0$ and $\text{l. inj. dim } R < \infty$, then $R$ is a ring direct product of a quasi-Frobenius ring and a ring of finite global dimension.

The results obtained for rings with radical of square zero are applied in Section 3 to arbitrary semi-primary rings, and in Section 4 to Noetherian semi-local rings. As a consequence of the general result we deduce the well-known theorem that a Dedekind domain with a finite number of maximal ideals is a principal ideal domain.

We add an Appendix in which we discuss the relation between $\text{l. inj. dim}_R R$ and $\text{r. inj. dim}_R R$ for semi-primary rings and for Noetherian rings. Finally, we introduce an example of an Artinian ring $R$ with $\text{l. inj. dim}_R R = 1$ and $\text{r. inj. dim. } R/N^a = \infty$, that is of course interesting for the comparison of the finite global dimension case and the finite injective dimension case.

1. Preliminaries

A semi-primary ring $R$, is a ring with nilpotent (Jacobson) radical $N$, and such that $S = R/N$ is a semi-simple Artinian ring. For any primitive idempotent $e$ in $R$ we set $eS = eR/eN$, and $Se = Re/Ne$.

* Dedicated to the memory of the late Professor Eri Jabotinsky.
All ideals are left ideals, and all modules are unitary left modules unless otherwise stated.

A Noetherian ring, will denote a left and right Noetherian ring.

A local ring is a ring that has a unique maximal left (right) ideal. This ideal is of course the unique maximal twosided ideal.

A left (right) semi-local ring $R$ is a ring that has a finite number of maximal left (right) ideals, $N_1, \ldots, N_s$, such that each $N_i$ is a twosided ideal. Let $M = N_1 \cap \cdots \cap N_s$ denote the (Jacobson) radical of $R$, then this is equivalent to saying that $R/M$ is a direct sum of division rings. Whence $R$ is a left semi-local ring if and only if it is a right semi-local ring. We thus say that $R$ is a semi-local ring.


An Artinian uniserial ring is a ring in which every component has a unique composition series.

2. $N^2 = 0$

In this section $R$ will stand for a semi-primary ring with radical $N$ of square zero, and $\mathrm{l.\ inj. dim}_R R < \infty$, unless otherwise stated. In particular we have on $N$ an $R$-module structure and an $S$-module structure that naturally coincide.

**Lemma 1.** Let $R$ be a left self-injective ring, then $R$ is a quasi-Frobenius ring.

**Proof.** For every minimal left ideal $I$ in $R$, its injective envelop $E(I)$ in $R$ is a direct summand of $R$. Let $J$ be a minimal left ideal in $R$, then $I$ and $J$ are isomorphic to each other if and only if so are $E(I)$ and $E(J)$. Since $N^2 = 0$, every indecomposable left ideal in $R$ is either a direct summand of $R$, or else it is a minimal left ideal. In particular, let $R = R_{e_1} + \cdots + R_{e_t}$ be a complete decomposition of $R$, then for each $i, 1 \leq i \leq t$, $R_{e_i}$ is either a minimal ideal, or else $0 \subseteq N_{e_i} \subseteq R_{e_i}$ is a Jordan–Hölder series, $R_{e_i}$ being the injective envelop of the minimal left ideal $N_{e_i}$. Thus $R$ is an Artinian ring, whence a quasi-Frobenius ring.

We proceed studying some properties of idempotents. One may verify that the finiteness of the injective dimension is not essential for Lemmas 2 and 3'.
Lemma 2. Let $e, f$ be primitive idempotents in $R$, such that $eNf \neq 0$. If for some positive integer $i$, $\operatorname{Ext}^{i+1}_R(Sf, R) = 0$ then $\operatorname{Ext}^i_R(Se, R) = 0$.

Proof. Since $Nf \neq 0$, and since $Rf$ is an indecomposable ideal, it results that $Sf$ is not a projective module. Consider the exact sequence:

$$0 \rightarrow Nf \rightarrow Rf \rightarrow Sf \rightarrow 0$$

Then $\operatorname{Ext}^{k+1}_R(Sf, R)$ is isomorphic with $\operatorname{Ext}^k_R(Nf, R)$ for all positive integers $k$.

Since $N^2 = 0$, then $Nf$ is isomorphic to a direct sum of modules, $\sum_{u \in U} S\tilde{e}_u$. Given $eNf \neq 0$ there results the existence of $v, v \in U$, such that $S\tilde{e}_v$ is isomorphic to $Se$. In particular $\operatorname{Ext}^j_R(Nf, R) = 0$ implies $\operatorname{Ext}^j_R(Se, R) = 0$ for every positive integer $j$.

The conclusion now follows immediately.

Under the finiteness hypothesis on $\operatorname{l. inj. dim}_R R$, we notice that if $fNe \neq 0$ and $eNf \neq 0$ then Lemma 2 enables us to conclude that $\operatorname{Ext}^i_R(Sf, R) = \operatorname{Ext}^i_R(Se, R) = 0$ for every positive integer $i$. Furthermore, let $R = Re_1 + \cdots + Re_t$ be a complete decomposition for $R$ then:

Lemma 3. Let $e_1, \ldots, e_n$ be such that $e_i Ne_{i+1} \neq 0$ for all $i, i = 1, \ldots, (n - 1)$ and $e_n Ne_1 \neq 0$.

Set $T = (e_1 + \cdots + e_n)R(e_1 + \cdots + e_n)$. If $\operatorname{l. inj. dim}_R R = s < \infty$, then $T$ is a quasi-Frobenius ring, if $e_k Ne_i = 0$ whenever $k > n$ and $i \leq n$.

Proof. Since $T$ is a semi-primary ring, it follows by Lemma 1 that it suffices to prove that $T$ is left self-injective.

Successive application of Lemma 2, yields $\operatorname{Ext}^i_R(Se_i, R) = 0$ for all positive integers $i$ and for all $j, 1 \leq j \leq n$.

Let $\{e_1', \ldots, e_u'\}$ be the maximal subset of $\{e_1, \ldots, e_n\}$ such that $\operatorname{Ext}^i_R(Se_i', R) = 0$ for every positive integer $i$ whenever $j \leq u$.

Let $T' = (e_1' + \cdots + e_u')R(e_1' + \cdots + e_u')$. Assume $T'$ is proved to be a quasi-Frobenius ring, then the vanishing hypothesis on $e_k Ne_i$ imply that $T$ is a direct summand of $T'$, whence a quasi-Frobenius ring.

Therefore we'll be done once we prove the Lemma under the assumption that $\{e_1, \ldots, e_n\}$ is a maximal subset of primitive idempotents with respect to the property:

$$\operatorname{Ext}^i_R(Se_j, R) = 0$$
for every positive integer $i$ and for all $j, j \leq n$. In general the condition $e_i Ne_{i+1} \neq 0$ is no more valid. However, the condition $e_p Ne_i \neq 0$ for some $i \leq n$ and $p, 1 \leq p \leq t$, now implies $p \leq n$. Whence $Te_i = Re_i$ for any $i, i \leq n$. As a consequence the exact sequence:

$$0 \rightarrow Ne_j \rightarrow Re_j \rightarrow Se_j \rightarrow 0$$

can be regarded as an exact sequence of $R$-modules as well as $T$-modules.

For the rest of the proof take a fixed $j$.

If $M$ denotes the radical of $T$, then $Me_j = Ne_j$.

Since $\text{Ext}_R^1 (Se_j, R) = 0$, then the induced map

$$\text{Hom}_R (Re_j, R) \rightarrow \text{Hom}_R (Ne_j, R)$$

is an epimorphism.

Let $F$ be any $T$-homomorphism of $Me_j$ into $Te_j$ then $F$ is an $R$-homomorphism of $Ne_j$ into $R$, regarding $T$ as a subset of $R$. Therefore, there exists an element $r$ in $R$, such that $F(me_j) = me_j r$ for every $m$ in $M$. Since $F(me_j) \in T$ it follows that $me_j r = me_j (e_1 + \cdots + e_n)$, in particular $F$ is induced by $e_j r (e_1 + \cdots + e_n)$ which is an element of $T$. Hence the induced map on $\text{Hom}_R (Te_j, T) \rightarrow \text{Hom}_T (Me_j, T)$ is an epimorphism, thus $\text{Ext}_T^1 (Se_j, T) = 0$.

Since this holds for $j = 1, \ldots, n$, it follows by [1] that $T$ is a self-injective ring, and this completes the proof.

Observe that the same proof applies to:

**Lemma 3'**. Let $e_1, \ldots, e_n$ be mutually orthogonal primitive idempotents in $R$, such that $\text{Ext}_R^i (Se_j, R) = 0$ for every positive integer $i$, and for all $j 1 \leq j \leq n$. Set $T = (e_1 + \cdots + e_n) R (e_1 + \cdots + e_n)$. Then $T$ is a quasi-Frobenius ring, if $e_k Ne_i = 0$ whenever $k > n$ and $i \leq n$.

Since in a quasi-Frobenius ring every component is the injective envelop of its minimal ideal we obtain:

**Corollary 4**. Let $f_1, \ldots, f_s$ be mutually orthogonal primitive idempotents in $T$, such that $Q = (f_1 + \cdots + f_s) T (f_1 + \cdots + f_s)$ is an indecomposable factor of $T$. Then $f_i T f_i$ is a division ring for all $i, 1 \leq i \leq s$, or else $Q$ is isomorphic to an $s \times s$ matrix algebra over a quasi-Frobenius local ring. In case that $f_i T f_i$ is a division ring there results a unique ordering, (up to cyclic permutations), say $h_1, \ldots, h_p$, such that: (i) for every $i, 1 \leq i \leq s$, there exists a unique $j, 1 \leq j \leq p$, such that $T f_i$ is isomorphic to $Th_j$, and (ii) $h_m M h_q \neq 0$ if and only if $q = m + 1$ (setting $h_{p+1} = h_1$), where $M$ denotes the radical of $Q$.

For the rest let $R = Re_1 + \cdots + Re_i$ be a complete decomposition for $R$, $e, e', e''$, will denote primitive idempotents, all idempotents to appear are
presumed to be from the set \( \{e_1, \ldots, e_t\} \), and \( \text{inj. dim}_R R = s < \infty \), unless otherwise specified. Finally, let \( \text{Ext}_R^i (S e_j, R) = 0 \) for all positive integers \( i \), if and only if \( j \leq n \) (including for the moment the possibility of \( n = 0 \) for the vacuous case).

Set \( T = (e_1 + \cdots + e_n) R (e_1 + \cdots + e_n) \). If \( T \neq 0 \) then by Lemma 3, \( T \) is a quasi-Frobenius ring. Furthermore, by Lemma 2 and the assumption \( \text{inj. dim}_R R = s < \infty \) it results that \( e_k Ne_k = 0 \) for \( k > n \), thus:

**Lemma 5.** \( e_k Re_k \) is a division ring for every \( k, k > n \).

**Definition.** \( C(e_j) = \{e_m \mid \text{there exists idempotents } e_1', \ldots, e_h' \text{ such that} \)

\( (i) \) \( e_1' = e_h' = e_j \), \( (ii) \) \( e_i' = e_m \) for some \( i, 1 \leq i \leq h \), and \( (iii) e_u'Re_{u+1}' \neq 0 \) for \( u = 1, \ldots, (h - 1) \). \( C(e_j) \) is the cycle defined by the idempotent \( e_j \). Obviously \( C(e_j) \) contains \( e_j \) and every idempotent \( e' \) such that \( Re' \) is isomorphic to \( Re_j \).

It is important to keep in mind that the cycle defined by an idempotent \( e \), depends in general on the prescribed decomposition of \( R \). It is possible to define the cycles without the restriction on the idempotents to belong to a prescribed decomposition of \( R \), however there is nothing that we know of that can be gained in this way except for complications.

We list some properties of cycles

\( C_1 \) : Let \( e', e'' \in C(e) \). If for some idempotent \( f \) we have \( e' Rf \neq 0 \) and \( f R e'' \neq 0 \), then \( f \in C(e) \).

\( C_2 \) : If \( e \in C(e') \) then \( e' \in C(e) \).

\( C_3 \) : If \( e \notin C(e') \) then \( C(e) \cap C(e') = \emptyset \).

\( C_4 \) : If \( e \in C(e') \) and \( \text{Ext}_R^i (S e', R) = 0 \) for every positive integer \( i \), then \( \text{Ext}_R^i (S e, R) = 0 \) for every positive integer \( i \).

\( C_5 \) : If \( \text{Ext}_R^i (S e', R) = 0 \) for every positive integer \( i \), and if \( e Ne' \neq 0 \) then \( e \in C(e') \).

\( C_6 \) : If \( e \in C(e') \) implies that \( R e \) is isomorphic to \( R e' \), then \( eRe \) is a division ring, or \( \text{Ext}_R^i (S e, R) = 0 \) for every positive integer \( i \).

\( C_7 \) : If \( e \in C(e') \) and \( R e \) is not isomorphic to \( R e' \), then \( f R f \) is a division ring for every \( f \) in \( C(e') \). Furthermore, \( \text{Ext}_R^i (S f, R) = 0 \) for every positive integer \( i \) and for every idempotent \( f \) in \( C(e') \). This results from Corollary 4 after observing that the hypothesis in discussion leads to a situation as described in Lemma 3.

\( C_8 \) : The subring \( T' = \sum f R g \) — the sum taken over all pairs of idempotent \( f, g \) such that \( f \in C(g) \) — is a quasi-Frobenius ring.

**Definition.** \( C(e_0'), \ldots, C(e_m') \) is a connected sequence of cycles of length \( m \).
if there exist idempotents \( f_j \in C(e_j') \) for every \( j, \ 0 \leq j \leq m \), such that \( f_j N f_{j+1} \neq 0 \), and \( f_j \notin C(e_{j+1}) \) for every \( j, j = 0, \ldots, (m - 1) \). We say that \( C(e') \) and \( C(e'') \) are connected if and only if there exists a connected sequence starting at \( C(e') \) and ending at \( C(e'') \).

Some useful properties of connected sequences of cycles are listed below. Most of them are straightforward, and for the rest we state some hints, leaving the details as an exercise for the interested reader.

S1: If \( C(e') \) and \( C(e) \) are not connected then \( e'Re = 0 \) whenever \( e \notin C(e') \).

S2: If \( C(e'_0), \ldots, C(e'_m) \) is a connected sequence of cycles and \( C(e'_0') = C(e'_m') \), then \( m = 0 \).

This is a result of the hypothesis \( \text{I. inj. dim.} R = s < \infty \) together with Lemma 3 and Corollary 4. We may restate this as:

**Proposition 6.** If \( \text{I. inj. dim.} R = s < \infty \), then connected sequences of cycles are bounded in length. We set \( r = \text{Supremum of the lengths of connected cycles} \).

S3: Any cycle \( C(e) \) that may occur in a connected sequence \( C(e'), C(e) \) consists entirely of idempotents \( f \) such that \( Rf \) is isomorphic to \( Re \), and \( fRf \) is a division ring.

From Corollary 4 and Lemma 2 it results that there exists a positive integer \( j \) such that \( \text{Ext}^j_R (Se, R) \neq 0 \). The result follows from C6 and C7.

S4: Let \( C(e'_0), \ldots, C(e'_m) \) be a connected sequence of cycles, if \( m \neq 0 \) then there exist integers \( i_j \) such that \( j \leq i_j \), and \( \text{Ext}^{i_j}_R (Se'_i, R) \neq 0 \). One can get \( i_0 < i_1 < i_2 < \cdots < i_m \).

This is an immediate consequence of Lemma 2, taking into account the previous properties of connected sequences.

We rephrase this property as:

**Proposition 7.** \( r \leq s \)

Also it results by similar arguments that there always exists an idempotent \( e \) for which \( \text{Ext}^i_R (Se, R) = 0 \) for every positive integer \( i \). In particular, for the ring \( T \) defined on top we always have \( n \neq 0 \).

Finally if \( T \) is a semi-simple ring, then one easily verifies that we are reduced to the triangular ring case, and \( \text{gl. dim.} R = s \). The connected sequences of cycles turn out to be exactly the connected sequences of idempotent as introduced by Jans and Nakayama in [8]. For further discussion of this setting we refer to [7] and [II].

In particular, assuming for the rest that \( \text{gl. dim.} R = \infty \), we may equally well require that \( T \) be a quasi-Frobenius ring with nonzero radical.
**Proposition 8.** \( s \leq r \)

**Proof.** There exists as idempotent \( e \) for which \( \text{Ext}_R^s(Se, R) \neq 0 \) and \( \text{Ext}_R^{s+i}(Se, R) = 0 \) for every positive integer \( i \). Consider the exact sequence:

\[
0 \to N_e \to R_e \to S_e \to 0
\]

then \( \text{Ext}_R^j(N_e, R) = \text{Ext}_R^{s+j}(Se, R) \) for every positive integer \( j \). If \( s = 0 \) we are done, otherwise \( \text{Ext}_R^{s-1}(N_e, R) \neq 0 \). Since \( N_e^2 = 0 \), \( N_e \) is isomorphic to a direct sum of minimal modules hence there exists an \( e' \) such that \( \text{Ext}_R^{s-1}(Se', R) \neq 0 \), \( e'N_e \neq 0 \), and \( \text{Ext}_R^{s+k}(Se', R) = 0 \) for every \( k, k \geq 0 \). In particular \( C(e'), C(e) \) is a connected sequence of cycles, since obviously \( e' \notin C(e) \) whenever \( s \neq 0 \).

Proceeding this going down process we end with a connected sequence of cycles \( C(e_0') \cdots C(e_s') \) such that \( \text{Ext}_R^j(Se'_j, R) \) vanish whenever \( i > j \), and \( \text{Ext}_R^j(Se'_j, R) \neq 0 \) for all \( j, j = 0, \ldots, s \).

Collecting Propositions 7 and 8 we get:

**Theorem 9.** Let \( R \) be a semi-primary ring with radical of square zero and let \( \text{inj. dim.}_R R = s < \infty \). Then \( r = s \).

The hypothesis \( \text{inj. dim.}_R R = s < \infty \) is essential even in case \( s = 0 \), since not every Artinian local ring with radical of square zero is a quasi-Frobenius ring. Still for this type of rings we obviously have \( r = 0 \).

We defined in \( R \) the subring \( T \). Another subring of \( R \) consists of the sum \( A = \sum e_i R e_j \)—the sum taken over all pairs of idempotents \( e_i, e_j \) such that \( e_i \) does not belong to \( T \); this necessarily implies that \( e_j \) does not belong to \( T \). In other words, we set \( A \) to be \( (e_{n+1} + \cdots + e_i) R(e_{n+1} + \cdots + e_i) \). Obviously if \( u = t \) we take \( A = 0 \). We have thus defined on \( R \) a splitting, \( R = T + A + A \), where

\[
A = (e_{n+1} + \cdots + e_i) R(e_1 + \cdots + e_n) + (e_1 + \cdots + e_n) R(e_{n+1} + \cdots + e_i).
\]

Since the first of the summands vanishes we have

\[
A = (e_1 + \cdots + e_n) R(e_{n+1} + \cdots + e_i),
\]

or \( AA = AT = 0 \), and \( TA = AA = A \). In particular, on the left hand side of \( A \), the \( R \) and \( T \)-module structure coincide.

Let \( e \) be any idempotent in \( T \), then \( \text{Ext}_R^1(Se, R) = 0 \), which means that the induced map \( \text{Hom}_R(Re, R) \to \text{Hom}_R(Ne, R) \) is an epimorphism.

Assuming that \( T \) is a quasi-Frobenius ring, that is non-semi-simple, one easily verifies that we may presume the existence of a splitting as above with the hypothesis that if \( T \) is a direct product of two subring \( T_1 \) and \( T_2 \), then both \( T_1 \) and \( T_2 \) are quasi-Frobenius rings that are not semi-simple rings. This further assumption enables us the following reasoning: Unless \( A = 0 \)
there exists an element \( a \) in \( A \) and a primitive idempotent \( f \) in \( T \) such that \( Ta = Ra \) is isomorphic to \( Nf \).

Consider the exact sequence:

\[
0 \rightarrow Nf \rightarrow Rf \rightarrow Sf \rightarrow 0
\]

and the isomorphism map \( F : Nf \rightarrow Ra \). Since \( \text{Hom}_R(Rf, R) \rightarrow \text{Hom}_R(Nf, R) \) is an epimorphism, there exists an \( r \) in \( R \), such that \( F(nf) = nf r \subset Ra \). Therefore we may assume that \( r \in rA \), but then \( fr \in A \subset N \), whence \( nf r \subset N^2 = 0 \). This contradiction implies that \( A = 0 \).

Finally if \( T \) is a semi-simple ring, \( R \) reduces to a (triangular) ring of finite global dimension, since on the idempotents of \( A \) the notions of connected sequences of cycles and connected cycles coincide (e.g. C5, C6 and [II]). Therefore we always have:

**Theorem 10.** \( R \) is a ring direct product of a quasi-Frobenius subring \( T \), with a semi-primary ring \( A \) of finite global dimension. As a matter of fact l. inj. dim. \( R = s \) if and only if gl. dim. \( A = s \). (Including the possibility \( T = 0 \) or \( A = 0 \)).

This in particular yields:

**Theorem 11.** Let \( R \) be a semi-primary ring with radical of square zero. Then l. inj. dim. \( R < \infty \) if and only if r. inj. dim. \( R < \infty \). The equality l. inj. dim. \( R = r. \) inj. dim. \( R \) always holds.

**Theorem 12.** Let \( R \) be a semi-primary ring with radical of square zero. If l. inj. dim. \( R < \infty \), then \( R \) is a residue ring of a semi-primary ring \( \Sigma \) such that l. inj. dim. \( \Sigma \leq 1 \).

**Proof.** Just take \( \Sigma \) to be the direct product of \( T \) with the semi-primary hereditary overring \( \Omega \) of \( A \) (e.g. [II]).

Finally, observe that the factors \( T \) and \( A \) are independent of the choice of the complete decomposition of \( R \).

An interesting consequence is:

**Corollary 13.** For semi-primary rings with radical of square zero, the finiteness of the injective dimension is a Morita-invariant.

We recall that (i) \( R \) is a quasi-Frobenius ring with radical of square zero if every component of \( R \) contains a unique minimal ideal, and different components contain different minimal ideals [10], and (ii) \( R \) is a semi-primary ring with radical of square zero, of finite global dimension if and only if \( R \) is a residue ring of a semi-primary hereditary ring (e.g. [7], [II]).

These together with Theorem 10 yield the answer to the structure of semi-primary rings with radical of square zero, that have a finite injective dimension.
3. L. Inj. Dim. $R/N^2 < \infty$

Let $R$ be a semi-primary ring with radical $N$ such that $l.\ inj.\ dim\ R/N^2 < \infty$. For the rest we assume that $R$ is an indecomposable ring. This will not cause any restriction, since the finiteness of the left injective dimension of a direct product of rings $T$ and $T'$ is equivalent to the finiteness of left injective dimension of both $T$ and $T'$, and the same goes to the residue ring modulo the square of the corresponding radicals.

Therefore, we have either $\text{gl. dim. } R/N^2 < \infty$—in which case $R$ is a residue ring of a semi-primary hereditary ring [II]—or else $l.\ inj.\ dim. \ R/N^2 = 0$, hence $R/N^2$ is a quasi-Frobenius ring.

Let $R/N^2$ be a quasi-Frobenius ring. Let

$$R = \sum_{i=1}^{t} \sum_{j=1}^{s} R_{ij}$$

be a complete decomposition of $R$, such that $R_{ik}$ is isomorphic to $R_{pj}$ if and only if $i = p$. We may further assume that $e_{ik}N_{pq} \not\subseteq N^2$ if and only if $p = i + 1$, setting $e_{(i+1)q} = e_{iq}$. This is possible since $R/N^2$ is indecomposable as $R$ is, and we may assume that $N \neq 0$. Since $R/N^2$ is a quasi-Frobenius ring, and since $N$ is a nilpotent ideal in $R$, it follows that

$$Ne_{(p+1)q} = R_{p}\ne_{(p+1)e}$$

where $e_{p}\ne_{(p+1)e}$ is any nonzero element in $N$ that is not contained in $N^2$. This readily implies that $N^e$ is a principal left ideal for every primitive idempotent $e$. Furthermore, $R \supset N \supset N^2 \supset \cdots \supset 0$ is a Jordan–Hölder series. As a consequence one obtains the result that $R$ is an Artinian uniserial ring. We thus proved:

**Theorem 14.** Let $R$ be a semi-primary ring. The following are equivalent:

(i) $l.\ inj.\ dim.\ R/N^2 = 0$.

(ii) $R$ is an Artinian uniserial ring.

If $R$ is a local ring, then $R$ itself is a quasi-Frobenius ring. Furthermore, $R$ is a restricted quasi-Frobenius ring (see also [10]). However if $R$ is not a local ring it may happen that $l.\ inj.\ dim.\ R \neq 0$.

**Example.** Let $R$ be a subring of the $3 \times 3$ matrix algebra over a field $F$. A matrix $M$ belongs to $R$ if and only if $M$ is of the form:

$$\begin{bmatrix}
m_{11} & 0 & 0 \\
m_{21} & m_{22} & 0 \\
m_{31} & m_{32} & m_{33}
\end{bmatrix}$$
where \( m_{ij} \in F \) for every pair of integers \( 1 \leq j \leq i \leq 3 \) and \( m_{11} = m_{33} \).

One verifies by straightforward projective resolutions of the simple \( R \)-modules that \( \text{gl. dim. } R = 2 \).

By checking the annihilating ideals of \( R/N^2 \) it follows that \( R/N^2 \) is a quasi-Frobenius ring.

Remark that \( R/I \) is a quasi-Frobenius ring for every nonzero two-sided ideal \( I \). This follows by straightforward computations observing that the only possibilities for \( I \) are \( N, N^2, Re_iR, Re_iR, \) and \( Re_iR | N^2 \), where \( e_i \) denotes the matrix whose \( pq \)th entry is \( \delta_{1p}\delta_{1q} + \delta_{2p}\delta_{2q} \) and \( e_i \) denotes the matrix whose \( pq \)th entry is \( \delta_{2p}\delta_{2q} \).

4. Restricted Quasi-Frobenius Rings

This section is devoted to the study of restricted quasi-Frobenius, semi-local rings. For the local case we refer to [10]. The commutative case is discussed in full details in the sequence of papers [3], [5], [9] and [12]. Notice that if \( N \neq 0 \) and \( R \) is an Artinian local ring, then \( \text{gl. dim. } R/N^2 = \infty \). Therefore, we get using [10]:

**Theorem 15.** Let \( R \) be a Noetherian local ring with maximal ideal \( N \), such that \( 1. \text{inj. dim.}_R R/N^2 < \infty \), then 1. inj. dim. \( R/N^2 = 0 \) and (i) \( N \) is not nilpotent, then \( R \) is a principal ideal domain or, (ii) \( 0 \neq N \) is nilpotent, then \( R \) is an Artinian principal ideal ring.

In both cases 1. inj. dim.\( R/I \) \( R/I = 0 \) for every twosided ideal \( I \), whenever \( I \neq 0 \) (in case (ii) this also holds for \( I = 0 \)).

Let \( R \) be a Noetherian semi-local ring, with a finite set of maximal left ideals \( N_1, \ldots, N_k \). Set \( M = N_1 \cap \cdots \cap N_k \), and assume that \( R/I \) is a quasi-Frobenius ring for every nonzero twosided ideal \( I \). If \( M \) is nilpotent then we are reduced to the Artinian case, thus assume that \( M \) is not nilpotent.

Considering the quasi-Frobenius ring \( R/M^2 \) one easily finds elements \( n_1, \ldots, n_t \) in \( R \) such that \( N_i = Rn_i + M^2 = n_iR + M^2 \) for \( i = 1, \ldots, t \). This set of equalities imply \( N_i = Rn_i = n_iR \) for every \( i \). If this was not the case for some \( i \), say \( i = 1 \), \( R/\cap_{i=1}^t N_i \) is a quasi-Frobenius ring. There results the existence of an integer \( k \), such that \( N_1^k = N_1^{k+1} \). Hence for an appropriate element \( r \) in \( R \) one gets \( n_1^k = r n_1^{k+1} \) or \( n_1^k (1 - r n_1) = 0 \).

Therefore, \( n_1^k R(1 - n_1) = n_1^n(1 - n_1) = 0 \). But \( R/J \) being a quasi-Frobenius ring for every nonzero twosided ideal \( J \) assures the existence of an integer \( s \) such that \( M^s \subset J \). The hypothesis that \( M \) is not nilpotent therefore implies that \( R \) is a prime ring. Since \( n_1^k \neq 0 \), and of course \( 1 - n_1 \neq 0 \), this contradiction proves our claim.

As a consequence we have that \( R \) is a domain. This follows from the fact
that every left ideal is a two-sided ideal. Of course, let \( r \) be any nonzero element in \( R \), then we have \( r = r_n N_i \) with \( r_n \notin N_i \) (\( i \) may be zero). We proceed this way to get \( r = r_n N_i \cdots N_i \) with \( r_n \notin N_i \) for \( i = 1, \ldots, t \). Therefore \( r_n \) is invertible in \( R \), and \( Rr = N_i \cdots N_i \). In particular this implies that every left ideal is a two-sided ideal. Since each \( N_i \) is a (strongly) prime ideal, it follows that \( N_i \cap N_j = N_i N_j \) whenever \( i \neq j \). As a consequence every left ideal being a product of powers of the maximal ideals \( N_i \), and since \( N_i = Rn_i = n_i R \) for every integer \( i, 1 \leq i \leq t \), it follows that \( J \) is a principal ideal.

The assumption being left right symmetric so are the results. We thus obtain:

\textbf{Theorem 16.} Let \( R \) be a Noetherian semi-local ring with Jacobson radical \( M \), such that \( R/I \) is a quasi-Frobenius ring for every nonzero two-sided ideal \( I \). Then

(i) \( M \) is not nilpotent, \( R \) is a principal ideal domain, every ideal is a two-sided ideal, and every ideal is a product of powers of maximal ideals.

or, (ii) \( M \) is nilpotent, \( R \) is an Artinian ring.

Restricting ourselves to commutative rings this reduces to:

\textbf{Corollary 17.} A commutative restricted quasi-Frobenius ring, that is semi-local, is a principal ideal ring.

\textit{Proof.} Observing that a commutative Artinian ring is a uniserial ring if and only if it is a principal ideal ring, the corollary follows.

An immediate consequence is the well-known result:

\textbf{Corollary 18.} A commutative Dedekind domain with a finite number of maximal ideals is a principal ideal domain.

As far as case (ii) of Theorem 16 is concerned, if \( M^2 \neq 0 \) then Theorem 14 is applicable while if \( M^2 = 0 \) then some pathological cases may occur, as for instance the local ring with \( M \) being isomorphic to the direct sum of two-minimal left ideals that are two-sided ideals.

5. \textit{Appendix}

In the sequel we made use of the left injective dimension of a ring \( R \). The problems we dealt with turned out to be left-right symmetric, and we could use the right injective dimension as well. In the general setting there is no much relation between the left and right injective dimension. Professor Maurice Auslander suggested the following Lemma, and we owe him many
thanks for pointing out the basic idea of its proof, i.e. using weak dimension for comparison.

**Lemma A.** Let $R$ be a Noetherian ring. If $l.\text{ inj. dim.}_R R = s < \infty$ and if $r.\text{ inj. dim.}_R R = t < \infty$, then $s = t$. The same hypothesis on a semi-primary ring $R$, leads to the same conclusion.

**Proof.** Consider the functorial homomorphism:

$$\text{Hom}_R(\text{Ext}_R^i (A, R), \mathcal{Q}) \cong \text{Tor}_R^k (\text{Hom}_R (R, \mathcal{Q}), A) \cong \text{Tor}_R^k (\mathcal{Q}, A)$$

for every left finitely generated module $A$ and right injective module $\mathcal{Q}$.

If $l.\text{ inj. dim.}_R R = t$ then there exists a finitely generated left module $A$ (take $A = S$ in case $R$ is semi-primary) such that $\text{Ext}_R^i (A, R) \neq 0$. Thus there exists an injective right module $\mathcal{Q}$, namely the injective envelop of $\text{Ext}_R^i (A, R)$ such that $\text{Tor}_R^k (\mathcal{Q}, A) \neq 0$.

Define $f. l.\text{ w. dim.}_R$ to be the supremum of the flat dimension of left modules of finite flat dimension, and $l.\text{ w. dim.}_R$ to be the supremum of the flat dimension of left injective modules. Respectively, the left-right symmetrization $f. r.\text{ w. dim.}_R$ and $r.\text{ w. dim.}_R$. We just went proving the inequality $l.\text{ inj. dim.}_R R \leq r.\text{ w. inj. dim.}_R R$, and of course if $l.\text{ inj. dim.}_R R$ is finite we furthermore have the inequality $r.\text{ w. inj. dim.}_R R \leq f. r.\text{ w. dim.}_R R$.

Next observe that for any left module $B$, of finite flat dimension, say $l.\text{ w. dim.}_B = r < \infty$, there exists a right module $C$ such that $\text{Tor}_R^k (C, B) \neq 0$. Let $\mathcal{Q}$ be the injective envelop of $C$, then the exact sequence

$$0 \to C \to \mathcal{Q} \to D \to 0$$

implies the exact sequence:

$$0 = \text{Tor}_R^{r+1} (D, B) \to \text{Tor}_R^r (C, B) \to \text{Tor}_R^r (\mathcal{Q}, B)$$

whence $\text{Tor}_R^r (\mathcal{Q}, B) \neq 0$. In particular there results the existence of a finitely generated left module $B'$, such that $\text{Tor}_R^r (\mathcal{Q}, B') \neq 0$.

Therefore, $f. l.\text{ w. dim.}_R \leq r.\text{ w. inj. dim.}_R R = l.\text{ inj. dim.}_R R$.

In case $R$ is semi-primary the last inequality is a consequence of the following reasoning:

By [1], $l.\text{ w. dim.}_B = l.\text{ p. dim.}_B$ for every left module, while we always have $l.\text{ p. dim.}_B \leq l.\text{ inj. dim.}_R R$ whenever $l.\text{ p. dim.}_B = k < \infty$, since then $\text{Ext}_R^k (B, R) \neq 0$.

We therefore have the following inequalities in case $l.\text{ inj. dim.}_R R < \infty$ for $l.\text{ w. dim.}_R \leq l.\text{ inj. dim.}_R R \leq r.\text{ w. inj. dim.}_R R \leq f. r.\text{ w. dim.}_R R$.

By the left-right symmetrization of these inequalities that hold because of the hypothesis $r.\text{ inj. dim.}_R R < \infty$, the conclusion is evident.
Our next aim is to prove the existence of Artinian rings $R$ of left injective dimension 1, even so, $\text{inj. dim.}_R R/N^2 = \infty$. Thus the converse of Theorem 12 does not hold in general. To this extent we make use of:

**Lemma B.** Let $R$ be a semi-primary ring such that $\text{inj. dim.}_R R = t$, then $\text{inj. dim.}_T T = t + 1$, where $T$ denotes the ring of (lower) $n \times n$ triangular matrices over $R(n > 1)$.

Using this Lemma, take $R$ to be any quasi-Frobenius ring that is not a semi-simple ring. Then the ring of (lower) $n \times n$ triangular matrices over $R(n > 1)$ is of left injective dimension 1, and $R/N^2$ is not a quasi-Frobenius ring as can easily be checked via the minimal ideals of $R/N^2$.

**Proof of Lemma B.** Let $e_{ij}$ denote the $n \times n$ matrix whose $pq$th entry is $\delta_{ip} \cdot \delta_{jq}$. The natural embedding of $R$ in $T$ makes $T$ into a left (right) free $R$-module, and so are $T e_{ij}(e_{ij} T)$ whenever $e_{ij} \in T$.

Set $T(L) = T \otimes_R L$, for every left $R$-module $L$, and set $e_{ij} T(L) = e_{ij} T \otimes_R L$ whenever $e_{ij} \in T$.

For any $R$-module $M$, and for any projective $R$-resolution of $M$:

$$P_m \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

there result a projective $T$-resolution for $T(M)$, and $e_{ij} T(M)$

$$T(P_m) \rightarrow \cdots \rightarrow T(P_1) \rightarrow T(P_0) \rightarrow T(M) \rightarrow 0$$

$$e_{ij} T(P_m) \rightarrow \cdots \rightarrow e_{ij} T(P_1) \rightarrow e_{ij} T(P_0) \rightarrow e_{ij} T(M) \rightarrow 0$$

whenever $e_{ij} \in T$.

Finally, there is an isomorphism between $\text{Hom}_T(T(M), T)$ and $\text{Hom}_R(M, R) \otimes_R T$. Thus

$$\text{Ext}_R^m(M, R), \text{ Ext}_T^m(T(M), T)$$

and $\text{Ext}_R^m(e_{ij} T(M), T)$

all vanish if and only if one of them does, for every integer, whenever $e_{ij} \in T$.

If $e_{ij} \in T$ and $i < n$, then $e_{(i+1)j} T(M) \subset e_{ij} T(M)$, and there results an exact sequence for all positive integer $m$:

$$\cdots \text{Ext}_R^m(L, T) \rightarrow \text{Ext}_T^m(e_{ij} T(M), T) \rightarrow H$$

$$\rightarrow \text{Ext}_T^m(e_{(i+1)j} T(M), T) \rightarrow \text{Ext}_T^{m+1}(L, T) \cdots$$

where $H$ is an epimorphism only if $\text{Ext}_R^m(M, R) = 0$, and $L$ is the left module $e_{ij} T(M) e_{(i+1)j} T(M)$.

In particular, $\text{Ext}_R^m(M, R) \neq 0$ and $\text{Ext}_R^{m+1}(M, R) = 0$ imply

$$\text{Ext}_T^{m+1}(L, T) \neq 0.$$
Checking the injective dimension of $T$ via the minimal ideals, the desired result follows.

It seems that the conclusion may hold in more general cases. However we dealt only with the semi-primary case, since it perfectly suits the purposes we were concerned with in this note.

References