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Injective Dimension of Semi-Primary Rings*

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In recent works by Eilenberg Nagao and Nakayama [*4*], Jans and Nakayama [*8*], Harada [*7*], and Zaks [*11*] the problem of characterizing semi-primary rings all of whose residue rings have finite global dimension, was solved. It turns out that if N is the radical of a semi-primary ring, then the ring R/N^2 plays a crucial role. It also plays an important role in the study of restricted quasi-Frobenius rings, in works by Faith [*5*], Levy [*9*], and Zaks [*12*].

The purpose of this paper is to investigate the possibility of replacing the global dimension by injective dimension. The ideas beyond the treatment are very much influenced by the ones used in the global dimension case, as for instance starting the study from rings with radical of square zero. The most surprizing result seems to be that if $N^2 = 0$ and l. inj. dim $R < \infty$, then R is a ring direct product of a quasi-Frobenius ring and a ring of finite global dimension.

The results obtained for rings with radical of square zero are applied in Section 3 to arbitrary semi-primary rings, and in Section 4 to Noetherian semi-local rings. As a consequence of the general result we deduce the well-known theorem that a Dedekind domain with a finite number of maximal ideals is a principal ideal domain.

We add an Appendix in which we discuss the relation between l. inj. $\dim_R R$ and r. inj. $\dim_R R$ for semi-primary rings and for Noetherian rings. Finally, we introduce an example of an Artinian ring R with l. inj. $\dim_R R = 1$ and l. inj. dim. ${}_{R/N^2} R/N^2 = \infty$, that is of course interesting for the comparison of the finite global dimension case and the finite injective dimension case.

1. PRELIMINARIES

A semi-primary ring R, is a ring with nilpotent (Jacobson) radical N, and such that $S = R/N$ is a semi-simple Artinian ring. For any primitive idempotent e in R we set $eS = eR/eN$, and $Se = Re/Ne$.

* Dedicated to the memory of the late Professor Eri Jabotinsky.

All ideals are left ideals, and all modules are unitary left modules unless otherwise stated.

A Noetherian ring, will denote a left and right Noetherian ring.

A local ring is a ring that has a unique maximal left (right) ideal. This ideal is of course the unique maximal twosided ideal.

A left (right) semi-local ring R is a ring that has a finite number of maximal left (right) ideals, $N_1, \ldots, N_s$, such that each N_i is a twosided ideal. Let $M = N_1 \cap \cdots \cap N_s$ denote the (Jacobson) radical of R, then this is equivalent to saying that R/M is a direct sum of division rings. Whence R is a left semi-local ring if and only if it is a right semi-local ring. We thus say that R is a semi-local ring.

We refer to [*7*] and [*11*] for details concerning triangular rings, to [*2*] for general homological concepts, and to [*10*] and [*12*] for general information concerning quasi-Frobenius rings, and rings all of whose homomorphic images are quasi-Frobenius.

An Artinian uniserial ring is a ring in which every component has a unique composition series.

2. $N^2 = 0$

In this section R will stand for a semi-primary ring with radical N of square zero, and l. inj. dim.$_R R < \infty$, unless otherwise stated. In particular we have on N an R-module structure and an S-module structure that naturally coïncide.

LEMMA 1. *Let R be a left self-injective ring, then R is a quasi-Frobenius ring.*

Proof. For every minimal left ideal I in R, its injective envelop $E(I)$ in R is a direct summand of R. Let J be a minimal left ideal in R, then I and J are isomorphic to each other if and only if so are $E(I)$ and $E(J)$. Since $N^2 = 0$, every indecomposable left ideal in R is either a direct summand of R, or else it is a minimal left ideal. In particular, let $R = Re_1 + \cdots + Re_t$ be a complete decomposition of R, then for each i, $1 \leqslant i \leqslant t$, Re_i is either a minimal ideal, or else $0 \subset Ne_i \subset Re_i$ is a Jordan–Hölder series, Re_i being the injective envelop of the minimal left ideal Ne_i. Thus R is an Artinian ring, whence a quasi-Frobenius ring.

We proceed studying some properties of idempotents. One may verify that the finiteness of the injective dimension is not essential for Lemmas 2 and 3′.

LEMMA 2. *Let e, f be primitive idempotents in R, such that $eNf \neq 0$. If for some positive integer i, $\mathrm{Ext}_R^{i+1}(Sf, R) = 0$ then $\mathrm{Ext}_R^{i}(Se, R) = 0$.*

Proof. Since $Nf \neq 0$, and since Rf is an indecomposable ideal, it results that Sf is not a projective module. Consider the exact sequence:

$$0 \to Nf \to Rf \to Sf \to 0$$

Then $\mathrm{Ext}_R^{k+1}(Sf, R)$ is isomorphic with $\mathrm{Ext}_R^{k}(Nf, R)$ for all positive integers k.

Since $N^2 = 0$, then Nf is isomorphic to a direct sum of modules, $\sum_{u \in U} Se_u$. Given $eNf \neq 0$ there results the existence of v, $v \in U$, such that Se_v is isomorphic to Se. In particular $\mathrm{Ext}_R^{j}(Nf, R) = 0$ implies $\mathrm{Ext}_R^{j}(Se, R) = 0$ for every positive integer j.

The conclusion now follows immediately.

Under the finiteness hypothesis on l. inj. dim.$_R$ R, we notice that if $fNe \neq 0$ and $eNf \neq 0$ then Lemma 2 enables us to conclude that

$$\mathrm{Ext}_R^{i}(Sf, R) = \mathrm{Ext}_R^{i}(Se, R) = 0$$

for every positive integer i. Furthermore, let $R = Re_1 + \cdots + Re_t$ be a complete decomposition for R then:

LEMMA 3. *Let $e_1, \ldots, e_n$ be such that*

$$e_i N e_{i+1} \neq 0 \quad \textit{for all} \quad i, i = 1, \ldots, (n-1) \quad \textit{and} \quad e_n N e_1 \neq 0.$$

Set $T = (e_1 + \cdots + e_n) R (e_1 + \cdots + e_n)$. If l. inj. dim.$_R$ $R = s < \infty$, *then T is a quasi-Frobenius ring, if $e_k N e_i = 0$ whenever $k > n$ and $i \leqslant n$.*

Proof. Since T is a semi-primary ring, it follows by Lemma 1 that it suffices to prove that T is left self-injective.

Successive application of Lemma 2, yields $\mathrm{Ext}_R^{i}(Se_j, R) = 0$ for all positive integers i and for all j, $1 \leqslant j \leqslant n$.

Let $\{e_1', \ldots, e_u'\}$ be the maximal subset of $\{e_1, \ldots, e_t\}$ such that

$$\mathrm{Ext}_R^{i}(Se_j', R) = 0$$

for every positive integer i whenever $j \leqslant u$.

Let $T' = (e_1' + \cdots + e_u') R (e_1' + \cdots + e_u')$. Assume T' is proved to be a quasi-Frobenius ring, then the vanishing hypothesis on $e_k N e_i$ imply that T is a direct summand of T', whence a quasi-Frobenius ring.

Therefore we'll be done once we prove the Lemma under the assumption that $\{e_1, \ldots, e_n\}$ is a maximal subset of primitive idempotents with respect to the property:

$$\mathrm{Ext}_R^{i}(Se_j, R) = 0$$

for every positive integer i and for all j, $j \leqslant n$. In general the condition $e_i N e_{i+1} \neq 0$ is no more valid. However, the condition $e_p N e_i \neq 0$ for some $i \leqslant n$ and p, $1 \leqslant p \leqslant t$, now implies $p \leqslant n$. Whence $Te_i = Re_i$ for any i, $i \leqslant n$. As a consequence the exact sequence:

$$0 \to Ne_j \to Re_j \to Se_j \to 0$$

can be regarded as an exact sequence of R-modules as well as T-modules. For the rest of the proof take a fixed j.

If M denotes the radical of T, then $Me_j = Ne_j$.

Since $\mathrm{Ext}_R^1(Se_j, R) = 0$, then the induced map

$$\mathrm{Hom}_R(Re_j, R) \to \mathrm{Hom}_R(Ne_j, R)$$

is an epimorphism.

Let F be any T-homomorphism of Me_j into Te_j then F is an R-homomorphism of Ne_j into R, regarding T as a subset of R. Therefore, there exists an element r in R, such that $F(me_j) = me_j r$ for every m in M. Since $F(me_j) \in T$ it follows that $me_j r = me_j r\,(e_1 + \cdots + e_n)$, in particular F is induced by $e_j r(e_1 + \cdots + e_n)$ which is an element of T. Hence the induced map on $\mathrm{Hom}_T(Te_j, T) \to \mathrm{Hom}_T(Me_j, T)$ is an epimorphism, thus $\mathrm{Ext}_T^1(Se_j, T) = 0$.

Since this holds for $j = 1,\ldots, n$, it follows by [1] that T is a self-injective ring, and this completes the proof.

Observe that the same proof applies to:

LEMMA 3′. *Let $e_1,\ldots, e_n$ be mutually orthogonal primitive idempotents in R, such that $\mathrm{Ext}_R^i(Se_j, R) = 0$ for every positive integer i, and for all j $1 \leqslant j \leqslant n$. Set $T = (e_1 + \cdots + e_n) R(e_1 + \cdots + e_n)$. Then T is a quasi-Frobenius ring, if $e_k N e_i = 0$ whenever $k > n$ and $i \leqslant n$.*

Since in a quasi-Frobenius ring every component is the injective envelop of its minimal ideal we obtain:

COROLLARY 4. *Let $f_1,\ldots, f_s$ be mutually orthogonal primitive idempotents in T, such that $Q = (f_1 + \cdots + f_s)\, T(f_1 + \cdots + f_s)$ is an indecomposable factor of T. Then $f_i T f_i$ is a division ring for all i, $1 \leqslant i \leqslant s$, or else Q is isomorphic to an $s \times s$ matrix algebra over a quasi-Frobenius local ring. In case that $f_i T f_i$ is a division ring there results a unique ordering, (up to cyclic permutations), say $h_1,\ldots, h_p$, such that:* (i) *for every i, $1 \leqslant i \leqslant s$, there exists a unique j, $1 \leqslant j \leqslant p$, such that Tf_i is isomorphic to Th_j, and* (ii) *$h_m M h_q \neq 0$ if and only if $q = m + 1$ (setting $h_{p+1} = h_1$), where M denotes the radical of Q.*

For the rest let $R = Re_1 + \cdots + Re_t$ be a complete decomposition for R, e, e', e'', will denote primitive idempotents, all idempotents to appear are

presumed to be from the set $\{e_1, \ldots, e_t\}$, and l. inj. dim.$_R R = s < \infty$, unless otherwise specified. Finally, let $\mathrm{Ext}_R^i(Se_j, R) = 0$ for all positive integers i, if and only if $j \leqslant n$ (including for the moment the possibility of $n = 0$ for the vacuous case).

Set $T = (e_1 + \cdots + e_n) R(e_1 + \cdots + e_n)$. If $T \neq 0$ then by Lemma 3′ T is a quasi-Frobenius ring. Furthermore, by Lemma 2 and the assumption l. inj. dim.$_R R = s < \infty$ it results that $e_k N e_k = 0$ for $k > n$, thus:

LEMMA 5. *$e_k R e_k$ is a division ring for every k, $k > n$.*

DEFINITION. $C(e_j) = \{e_m \mid$ there exists idempotents $e_1', \ldots, e_h'$ such that (i) $e_1' = e_h' = e_j$, (ii) $e_i' = e_m$ for some i, $1 \leqslant i \leqslant h$, and (iii) $e_u' R e_{u+1}' \neq 0$ for $u = 1, \ldots, (h-1)\}$. We refer to $C(e_j)$ as the *cycle* defined by the idempotent e_j. Obviously $C(e_j)$ contains e_j and every idempotent e' such that Re' is isomorphic to Re_j.

It is important to keep in mind that the cycle defined by an idempotent e, depends in general on the prescribed decomposition of R. It is possible to define the cycles without the restiction on the idempotents to belong to a prescribed decomposition of R, however there is nothing that we know of that can be gained in this way except for complications.

We list some properties of cycles

C1 : Let $e', e'' \in C(e)$. If for some idempotent f we have $e'Rf \neq 0$ and $fRe'' \neq 0$, then $f \in C(e)$.

C2 : If $e \in C(e')$ then $e' \in C(e)$.

C3 : If $e \notin C(e')$ then $C(e) \cap C(e') = \varnothing$.

C4 : If $e \in C(e')$ and $\mathrm{Ext}_R^i(Se', R) = 0$ for every positive integer i, then $\mathrm{Ext}_R^i(Se, R) = 0$ for every positive integer i.

C5 : If $\mathrm{Ext}_R^i(Se', R) = 0$ for every positive integer i, and if $eNe' \neq 0$ then $e \in C(e')$.

C6 : If $e \in C(e')$ implies that Re is isomorphic to Re', then eRe is a division ring, or $\mathrm{Ext}_R^i(Se, R) = 0$ for every positive integer i.

C7 : If $e \in C(e')$ and Re is not isomorphic to Re', then fRf is a division ring for every f in $C(e')$. Furthermore, $\mathrm{Ext}_R^i(Sf, R) = 0$ for every positive integer i and for every idempotent f in $C(e')$. This results from Corollary 4 after observing that the hypothesis in discussion leads to a situation as described in Lemma 3.

C8 : The subring $T' = \sum fRg$—the sum taken over all pairs of idempotent f, g such that $f \in C(g)$—is a quasi-Frobenius ring.

DEFINITION. $C(e_0'), \ldots, C(e_m')$ is a *connected sequence of cycles* of length m

if there exist idempotents $f_j \in C(e_j')$ for every j, $0 \leqslant j \leqslant m$, such that $f_j N f_{j+1} \neq 0$, and $f_j \notin C(e'_{j+1})$ for every $j, j = 0,..., (m-1)$. We say that $C(e')$ and $C(e'')$ are connected if and only if there exists a connected sequence starting at $C(e')$ and ending at $C(e'')$.

Some useful properties of connected sequences of cycles are listed below. Most of them are straightforward, and for the rest we state some hints, leaving the details as an excercise for the interested reader.

S1 : If $C(e')$ and $C(e)$ are not connected then $e'Re = 0$ whenever $e \notin C(e')$.

S2 : If $C(e_0')....., C(e_m')$ is a connected sequence of cycles and $C(e_0') = C(e_m')$, then $m = 0$.

This is a result of the hypothesis l. inj. dim.$_R R = s < \infty$ together with Lemma 3 and Corollary 4. We may restate this as:

PROPOSITION 6. *If* l. inj. dim.$_R R = s < \infty$, *then connected sequences of cycles are bounded in length. We set* $r =$ *Supremum of the lengths of connected cycles.*

S3 : Any cycle $C(e)$ that may occur in a connected sequence $C(e')$, $C(e)$ consists entirely of idempotents f such that Rf is isomorphic to Re, and fRf is a division ring.

From Corollary 4 and Lemma 2 it results that there exists a positive integer j such that $\mathrm{Ext}_R^j(Se, R) \neq 0$. The result follows from C6 and C7.

S4 : Let $C(e_0'),..., C(e_m')$ be a connected sequence of cycles, if $m \neq 0$ then there exist integers i_j such that $j \leqslant i_j$, and $\mathrm{Ext}_R^{i_j}(Se_j', R) \neq 0$. One can get $i_0 < i_1 < i_2 < \cdots < i_m$.

This is an immediate consequence of Lemma 2, taking into account the previous properties of connected sequences.

We rephrase this property as:

PROPOSITION 7. $r \leqslant s$

Also it results by similar arguments that there always exists an idempotent e for which $\mathrm{Ext}_R^i(Se, R) = 0$ for every positive integer i. In particular, for the ring T defined on top we always have $n \neq 0$.

Finally if T is a semi-simple ring, then one easily verifies that we are reduced to the triangular ring case, and gl. dim. $R = s$. The connected sequences of cycles turn out to be exactly the connected sequences of idempotent as introduced by Jans and Nakayama in [*8*]. For further discussion of this setting we refer to [*7*] and [*11*].

In particular, assuming for the rest that gl. dim. $R = \infty$, we may equally well require that T be a quasi-Frobenius ring with nonzero radical.

PROPOSITION 8. $s \leqslant r$

Proof. There exists as idempotent e for which $\mathrm{Ext}_R{}^s(Se, R) \neq 0$ and $\mathrm{Ext}_R^{s+i}(Se, R) = 0$ for every positive integer i. Consider the exact sequence:

$$0 \to Ne \to Re \to Se \to 0$$

then $\mathrm{Ext}_R{}^j(Ne, R) = \mathrm{Ext}_R^{j+1}(Se, R)$ for every positive integer j. If $s = 0$ we are done, otherwise $\mathrm{Ext}_R^{s-1}(Ne, R) \neq 0$. Since $N^2 = 0$, Ne is isomorphic to a direct sum of minimal modules hence there exists an e' such that $\mathrm{Ext}_R^{s-1}(Se', R) \neq 0$, $e'Ne \neq 0$, and $\mathrm{Ext}_R^{s+k}(Se', R) = 0$ for every k, $k \geqslant 0$. In particular $C(e')$, $C(e)$ is a connected sequence of cycles, since obviously $e' \notin C(e)$ whenever $s \neq 0$.

Proceeding this going down process we end with a connected sequence of cycles $C(e_0') \cdots C(e_s')$ such that $\mathrm{Ext}_R{}^i(Se_j', R)$ vanish whenever $i > j$, and $\mathrm{Ext}_R{}^j(Se_j', R) \neq 0$ for all j, $j = 0,\ldots, s$.

Collecting Propositions 7 and 8 we get:

THEOREM 9. *Let R be a semi-primary ring with radical of square zero and let* l. inj. dim.$_R R = s < \infty$. *Then $r = s$.*

The hypothesis l. inj. dim.$_R R = s < \infty$ is essential even in case $s = 0$, since not every Artinian local ring with radical of square zero is a quasi-Frobenius ring. Still for this type of rings we obviously have $r = 0$.

We defined in R the subring T. Another subring of R consists of the sum $\Lambda = \sum e_i R e_j$—the sum taken over all pairs of idempotents e_i, e_j such that e_i does not belong to T; this necessarily implies that e_j does not belong to T. In other words, we set Λ to be $(e_{n+1} + \cdots + e_t) R(e_{n+1} + \cdots + e_t)$. Obviously if $n = t$ we take $\Lambda = 0$. We have thus defined on R a splitting, $R = T + \Lambda + A$, where

$$A = (e_{n+1} + \cdots + e_t) R(e_1 + \cdots + e_n) + (e_1 + \cdots + e_n) R(e_{n+1} + \cdots + e_t).$$

Since the first of the summands vanishes we have

$$A = (e_1 + \cdots + e_n) R(e_{n+1} + \cdots + e_t),$$

or $\Lambda A = AT = 0$, and $TA = A\Lambda = A$. In particular, on the left hand side of A, the R and T-module structure coïncide.

Let e be any idempotent in T, then $\mathrm{Ext}_R{}^1(Se, R) = 0$, which means that the induced map $\mathrm{Hom}_R(Re, R) \to \mathrm{Hom}_R(Ne, R)$ is an epimorphism.

Assumming that T is a quasi-Frobenius ring, that is non-semi-simple, one easily verifies that we may presume the existence of a splitting as above with the hypothesis that if T is a direct product of two subring T_1 and T_2, then both T_1 and T_2 are quasi-Frobenius rings that are not semi-simple rings. This further assumption enables us the following reasoning: Unless $A = 0$

there exists an element a in A and a primitive idempotent f in T such that $Ta = Ra$ is isomorphic to Nf.

Consider the exact sequence:

$$0 \to Nf \to Rf \to Sf \to 0$$

and the isomorphism map $F : Nf \to Ra$. Since $\mathrm{Hom}_R(Rf, R) \to \mathrm{Hom}_R(Nf, R)$ is an epimorphism, there exists an r in R, such that $F(nf) = nfr \subset Ra$. Therefore we may assume that $r \in r\Lambda$, but then $fr \in A \subset N$, whence $nfr \in N^2 = 0$. This contradiction implies that $A = 0$.

Finally if T is a semi-simple ring, R reduces to a (triangular) ring of finite global dimension, since on the idempotents of Λ the notions of connected sequences of cycles and connected cycles coincide (e.g. C5, C6 and [*11*]). Therefore we always have:

THEOREM 10. *R is a ring direct product of a quasi-Frobenius subring T, with a semi-primary ring Λ of finite global dimension. As a matter of fact* l. inj. dim. $R = s$ *if and only if* gl. dim. $\Lambda = s$. (*Including the possibility* $T = 0$ *or* $\Lambda = 0$).

This in particular yields:

THEOREM 11. *Let R be a semi-primary ring with radical of square zero. Then* l. inj. $\dim._R R < \infty$ *if and only if* r. inj. $\dim._R R < \infty$. *The equality* l. inj. $\dim._R R =$ r. inj. $\dim._R R$ *always holds.*

THEOREM 12. *Let R be a semi-primary ring with radical of square zero. If* l. inj. $\dim._R R < \infty$, *then R is a residue ring of a semi-primary ring Σ such that* l. inj. dim. $\Sigma \leqslant 1$.

Proof. Just take Σ to be the direct product of T with the semi-primary hereditary overring Ω of Λ (e.g. [*11*]).

Finally, observe that the factors T and Λ are independent of the choice of the complete decomposition of R.

An interesting consequence is:

COROLLARY 13. *For semi-primary rings with radical of square zero, the finiteness of the injective dimension is a Morita-invariant.*

We recall that (i) R is a quasi-Frobenius ring with radical of square zero if every component of R contains a unique minimal ideal, and different components contain different minimal ideals [*10*], and (ii) R is a semi-primary ring with radical of square zero, of finite global dimension if and only if R is a residue ring of a semi-primary hereditary ring (e.g. [*7*], [*11*]).

These together with Theorem 10 yield the answer to the structure of semi-primary rings with radical of square zero, that have a finite injective dimension.

3. L. Inj. Dim. $R/N^2 < \infty$

Let R be a semi-primary ring with radical N such that l. inj. dim $R/N^2 < \infty$. For the rest we assume that R is an indecomposable ring. This will not cause any restriction, since the finiteness of the left injective dimension of a direct product of rings T and T' is equivalent to the finiteness of left injective dimension of both T and T', and the same goes to the residue ring modulo the square of the corresponding radicals.

Therefore, we have either gl. dim. $R/N^2 < \infty$—in which case R is a residue ring of a semi-primary hereditary ring [*11*]—or else l. inj. dim. $R/N^2 = 0$, hence R/N^2 is a quasi-Frobenius ring.

Let R/N^2 be a quasi-Frobenius ring. Let

$$R = \sum_{\substack{i=1,\ldots,t \\ j=1,\ldots,j_i}} Re_{ij}$$

be a complete decomposition of R, such that Re_{ik} is isomorphic to Re_{pq} if and only if $i = p$. We may further assume that $e_{ik}Ne_{pq} \not\subset N^2$ if and only if $p = i + 1$, setting $e_{(t+1)q} = e_{1q}$. This is possible since R/N^2 is indecomposable as R is, and we may assume that $N \neq 0$. Since R/N^2 is a quasi-Frobenius ring, and since N is a nilpotent ideal in R, it follows that

$$Ne_{(p+1)q} = Re_{p1}ne_{(p+1)q}$$

where $e_{p1}ne_{(p+1)q}$ is any nonzero element in N that is not contained in N^2. This readily implies that $N^m e$ is a principal left ideal for every primitive idempotent e. Furthermore, $Re \supset Ne \supset N^2e \supset \cdots 0$ is a Jordan–Hölder series. As a consequence one obtains the result that R is an Artinian uniserial ring. We thus proved:

Theorem 14. *Let R be a semi-primary ring. The following are equivalent*:

(i) l. inj. dim. $R/N^2 = 0$.

(ii) *R is an Artinian uniserial ring.*

If R is a local ring, then R itself is a quasi-Frobenius ring. Furthermore, R is a restricted quasi-Frobenius ring (see also [*10*]). However if R is not a local ring it may happen that l. inj. dim.$_R$ $R \neq 0$.

Example. Let R be a subring of the 3×3 matrix algebra over a field F. A matrix M belongs to R if and only if M is of the form:

$$\begin{Vmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & m_{33} \end{Vmatrix}$$

where $m_{ij} \in F$ for every pair of integers $1 \leqslant j \leqslant i \leqslant 3$ and $m_{11} = m_{33}$. One verifies by straightforward projective resolutions of the simple R-modules that gl. dim. $R = 2$.

By checking the annihilating ideals of R/N^2 it follows that R/N^2 is a quasi-Frobenius ring.

Remark that R/I is a quasi-Frobenius ring for every nonzero two-sided ideal I. This follows by straightforward computations observing that the only possibilities for I are N, N^2, Re_1R, Re_2R, and $Re_2R + N^2$, where e_1 denotes the matrix whose pqth entry is $\delta_{1p}\delta_{1q} + \delta_{3p}\delta_{3q}$ and e_2 denotes the matrix whose pqth entry is $\delta_{2p}\delta_{2q}$.

4. Restricted Quasi-Frobenius Rings

This section is devoted to the study of restricted quasi-Frobenius, semi-local rings. For the local case we refer to [*10*]. The commutative case is discussed in full details in the sequence of papers [*3*], [*5*], [*9*] and [*12*]. Notice that if $N \neq 0$ and R is an Artinian local ring, then gl. dim. $R/N^2 = \infty$. Therefore, we get using [*10*]:

Theorem 15. *Let R be a Noetherian local ring with maximal ideal N, such that* l. inj. dim.$_{R/N^2}$ $R/N^2 < \infty$, *then* l. inj. dim. $R/N^2 = 0$ *and* (i) *N is not nilpotent, then R is a principal ideal domain or,* (ii) *$0 \neq N$ is nilpotent, then R is an Artinian principal ideal ring.*

In both cases l. inj. dim.$_{R/I}$ $R/I = 0$ for every twosided ideal I, whenever $I \neq 0$ (in case (ii) this also holds for $I = 0$).

Let R be a Noetherian semi-local ring, with a finite set of maximal left ideals $N_1, \ldots, N_k$. Set $M = N_1 \cap \cdots \cap N_k$, and assume that R/I is a quasi-Frobenius ring for every nonzero twosided ideal I. If M is nilpotent then we are reduced to the Artinian case, thus assume that M is not nilpotent.

Considering the quasi-Frobenius ring R/M^2 one easily finds elements $n_1, \ldots, n_t$ in R such that $N_i = Rn_i + M^2 = n_iR + M^2$ for $i = 1, \ldots, t$. This set of equalities imply $N_i = Rn_i = n_iR$ for every i, $1 \leqslant i \leqslant t$. We claim $\bigcap_{j=1}^{\infty} N_i^j = 0$ for every i. If this was not the case for some i, say $i = 1$, $R/\bigcap_{j=1}^{\infty} N_1^i$ is a quasi-Frobenius ring. There results the existence of an integer k, such that $N_1^k = N_1^{k+1}$. Hence for an appropriate element r in R one gets $n_1^k = r_i n_1^{k+1}$, or $n_1^k(1 - r_i n_1) = 0$.

Therefore, $n_1^k R(1 - rn_1) = Rn_1^k(1 - rn_1) = 0$. But R/J being a quasi-Frobenius ring for every nonzero twosided ideal J assures the existence of an integer ℓ such that $M^\ell \subset J$. The hypothesis that M is not nilpotent therefore implies that R is a prime ring. Since $n_1^k \neq 0$, and of course $1 - rn_1 \neq 0$, this contradiction proves our claim.

As a consequence we have that R is a domain. This follows from the fact

that every left ideal is a two sided ideal. Of course, let r be any nonzero element in R, then we have $r = r_1 n_t^{i_t}$ with $r_1 \notin N_t$ (i_t may be zero). We proceed this way to get $r = r_t n_1^{i_1} \cdots n_t^{i_t}$ with $r_t \notin N_i$ for $i = 1,\ldots, t$. Therefore r_t is invertible in R, and $Rr = N_1^{i} \cdots N_t^{i_t}$. In particular this implies that every left ideal is a two-sided ideal. Since each N_i is a (strongly) prime ideal, it follows that $N_i \cap N_j = N_i N_j$ whenever $i \neq j$. As a consequence every left ideal being a product of powers of the maximal ideals N_i, and since $N_i = Rn_i = n_i R$ for every integer i, $1 \leqslant i \leqslant t$, it follows that J is a principal ideal.

The assumption being left right symmetric so are the results. We thus obtain:

THEOREM 16. *Let R be a Noetherian semi-local ring with Jacobson radical M, such that R/I is a quasi-Frobenius ring for every nonzero twosided ideal I. Then*

(i) *M is not nilpotent, R is a principal ideal domain, every ideal is a two-sided ideal, and every ideal is a product of powers of maximal ideals.*

or, (ii) *M is nilpotent, R is an Artinian ring.*

Restricting ourselves to commutative rings this reduces to:

COROLLARY 17. *A commutative restricted quasi-Frobenius ring, that is semi-local, is a principal ideal ring.*

Proof. Observing that a commutative Artinian ring is a uniserial ring if and only if it is a principal ideal ring, the corollary follows.

An immediate consequence is the well-known result:

COROLLARY 18. *A commutative Dedekind domain with a finite number of maximal ideals is a principal ideal domain.*

As far as case (ii) of Theorem 16 is concerned, if $M^2 \neq 0$ then Theorem 14 is applicable while if $M^2 = 0$ then some pathological cases may occur, as for instance the local ring with M being isomorphic to the direct sum of two-minimal left ideals that are twosided ideals.

5. APPENDIX

In the sequel we made use of the left injective dimension of a ring R. The problems we dealt with turned out to be left-right symmetric, and we could use the right injective dimension as well. In the general setting there is no much relation between the left and right injective dimension. Professor Maurice Auslander suggested the following Lemma, and we owe him many

thanks for pointing out the basic idea of its proof, i.e. using weak dimension for comparison.

LEMMA A. *Let R be a Noetherian ring. If* l. inj. dim.$_R$ $R = s < \infty$ *and if* r. inj. dim.$_R$ $R = t < \infty$, *then $s = t$. The same hypothesis on a semi-primary ring R, leads to the same conclusion.*

Proof. Consider the functorial homomorphism:

$$\mathrm{Hom}_R(\mathrm{Ext}_R{}^i(A, R), Q) \cong \mathrm{Tor}_i{}^R(\mathrm{Hom}_R(R, Q), A) \cong \mathrm{Tor}_i{}^R(Q, A)$$

for every left finitely generated module A and right injective module Q.

If l. inj. dim.$_R$ $R = t$ then there exists a finitely generated left module A (take $A = S$ in case R is semi-primary) such that $\mathrm{Ext}_R{}^t(A, R) \neq 0$. Thus there exists an injective right module Q, namely the injective envelop of $\mathrm{Ext}_R{}^t(A, R)$ such that $\mathrm{Tor}_t{}^R(Q, A) \neq 0$.

Define f. l. w. dim. R to be the supremum of the flat dimension of left modules of finite flat dimension, and l. w. inj. dim. R to be the supremum of the flat dimension of left injective modules. Respectively, the left-right symmetrization f. r. w. dim. R and r. w. inj. dim. R. We just went proving the inequality l. inj. dim.$_R$ $R \leqslant$ r. w. inj. dim. R, and of course if l. inj. dim.$_R$ R is finite we furthermore have the inequality r. w. inj. dim. $R \leqslant$ f. r. w. dim. R.

Next observe that for any left module B, of finite flat dimension, say l. w. dim. $B = r < \infty$, there exists a right module C such that $\mathrm{Tor}_r{}^R(C, B) \neq 0$. Let Q be the injective envelop of C, then the exact sequence

$$0 \to C \to Q \to D \to 0$$

implies the exact sequence:

$$0 = \mathrm{Tor}_{r+1}^R(D, B) \to \mathrm{Tor}_r{}^R(C, B) \to \mathrm{Tor}_r{}^R(Q, B)$$

whence $\mathrm{Tor}_r{}^R(Q, B) \neq 0$. In particular there results the existence of a finitely generated left module B', such that $\mathrm{Tor}_r{}^R(Q, B') \neq 0$.

Therefore, f. l. w. dim. $R \leqslant$ r. w. inj. dim. $R =$ l. inj. dim.$_R$ R.

In case R is semi-primary the last inequality is a consequence of the following reasoning:

By [*1*], w. l. dim. $B =$ l. p. dim. B for every left module, while we always have l. p. dim. $B \leqslant$ l. inj. dim.$_R$ R whenever l. p. dim. $B = k < \infty$, since then $\mathrm{Ext}_R{}^k(B, R) \neq 0$.

We therefore have the following inequalities in case l. inj. dim.$_R$ $R < \infty$ f. l. w. dim. $R \leqslant$ l. inj. dim.$_R$ $R \leqslant$ r. w. inj. dim. $R \leqslant$ f. r. w. dim. R.

By the left-right symmetrization of these inequalities that hold because of the hypothesis r. inj. dim.$_R$ $R < \infty$, the conclusion is evident.

Our next aim is to prove the existence of Artinian rings R of left injective dimension 1, evenso l. inj. dim.$_R$ $R/N^2 = \infty$. Thus the converse of Theorem 12 does not hold in general. To this extent we make use of:

LEMMA B. *Let R be a semi-primary ring such that* l. inj. dim.$_R$ $R = t$, *then* l. inj. dim.$_T$ $T = t + 1$, *where T denotes the ring of (lower) $n \times n$ triangular matrices over $R(n > 1)$.*

Using this Lemma, take R to be any quasi-Frobenius ring that is not a semi-simple ring. Then the ring of (lower) $n \times n$ triangular matrices over $R(n > 1)$ is of left injective dimension 1, and R/N^2 is not a quasi-Frobenius ring as can easily be checked via the minimal ideals of R/N^2.

Proof of Lemma B. Let e_{ij} denote the $n \times n$ matrix whose pqth entry is $\delta_{ip} \cdot \delta_{jq}$. The natural embedding of R in T makes T into a left (right) free R-module, and so are $Te_{ij}(e_{ij}T)$ whenever $e_{ij} \in T$.

Set $T(L) = T \otimes_R L$, for every left R-module L, and set $e_{ij}T(L) = e_{ij}T \otimes_R L$ whenever $e_{ij} \in T$.

For any R-module M, and for any projective R-resolution of M:

$$P_m \to \cdots \to P_0 \to M \to 0$$

there result a projective T-resolution for $T(M)$, and $e_{ij}T(M)$

$$T(P_m) \to \cdots \to T(P_0) \to T(M) \to 0$$

$$e_{ij}T(P_m) \to \cdots \to e_{ij}T(P_0) \to e_{ij}T(M) \to 0$$

whenever $e_{ij} \in T$.

Finally, there is an isomorphism between $\mathrm{Hom}_T(T(M), T)$ and $\mathrm{Hom}_R(M, R) \otimes_R T$. Thus

$$\mathrm{Ext}_R^m(M, R), \qquad \mathrm{Ext}_T^m(T(M), T) \qquad \text{and} \qquad \mathrm{Ext}_T^m(e_{ij}T(M), T)$$

all vanish if and only if one of them does, for every integer, whenever $e_{ij} \in T$.

If $e_{ij} \in T$ and $i < n$, then $e_{(i+1)j}T(M) \subset e_{ij}T(M)$, and there results an exact sequence for all positive integer m:

$$\cdots \mathrm{Ext}_T^m(L, T) \to \mathrm{Ext}_T^m(e_{ij}T(M), T) \xrightarrow{H}$$
$$\to \mathrm{Ext}_T^m(e_{(i+1)j}T(M), T) \to \mathrm{Ext}_T^{m+1}(L, T) \cdots$$

where H is an epimorphism only if $\mathrm{Ext}_R^m(M, R) = 0$, and L is the left module $e_{ij}T(M)/e_{(i+1)j}T(M)$.

In particular, $\mathrm{Ext}_R^m(M, R) \neq 0$ and $\mathrm{Ext}_R^{m+1}(M, R) = 0$ imply

$$\mathrm{Ext}_T^{m+1}(L, T) \neq 0.$$

Checking the injective dimension of T via the minimal ideals, the desired result follows.

It seems that the conclusion may hold in more general cases. However we dealt only with the semi-primary case, since it perfectly suits the purposes we were concerned with in this note.

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