SPECTRAL DENSITY ESTIMATION FOR
STATIONARY STABLE PROCESSES

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Received 10 January 1983
Revised 16 September 1983

Weakly and strongly consistent nonparametric estimates, along with rates of convergence, are
established for the spectral density of certain stationary stable processes. This spectral density
plays a role, in linear inference problems, analogous to that played by the usual power spectral
density of second order stationary processes.

AMS 1970 Subject Classification: Primary 60G10, 62M15
stationary stable processes * nonparametric spectral density estimation

1. Introduction and summary

Consistent estimates of the spectral density function \( \phi(\lambda) \) of fourth order, zero
mean, mean square continuous, stationary processes \( X(t), -\infty < t < \infty \), have been
studied extensively in the literature [2, 11, 16, 17]. Given an observation of \( X \) over
the interval \([0, T]\), a nonparametric naive estimate of \( \phi(\lambda) \) is the periodogram

\[
I_T(\lambda) = \frac{1}{2\pi T} \left| \int_0^T e^{-i\lambda t} X(t) \, dt \right|^2
\]

which is not a consistent estimate of \( \phi(\lambda) \), as \( T \to \infty \). However, smoothing of the
periodogram by a spectral window leads to a consistent estimate of \( \phi(\lambda) \), whose
precise asymptotic bias and covariance are known [2, 11, 16, 17]. These processes
\( X \) have in the complex case a spectral representation

\[
X(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \, d\xi(\lambda), \quad -\infty < t < \infty,
\]  

* Research supported by the Office of Naval Research under Contract No. N00014-75-C-0652.
II Research supported by the Air Force Office of Scientific Research under Contract No. AFOSR
F-49620-82-C-0009.
where the process $\xi$ has orthogonal increments and
\[
E[|d\xi(\lambda)|^2] = \phi(\lambda) \, d\lambda,
\] (1.2)
and covariance function given by
\[
\text{Covariance}(X(t), X(s)) = \int_{-\infty}^{\infty} e^{i\pi \gamma} \phi(\lambda) \, d\lambda.
\] (1.3)

In this paper we consider an important class of stationary processes which have
the spectral representation (1.1), but whose second moments are infinite. Namely
we consider strictly stationary complex symmetric $\alpha$-stable (SaS) processes $X,$
$0 < \alpha < 2,$ having the spectral representation (1.1) where $\xi$ is an SaS process with
independent increments and
\[
\{E[|d\xi(\lambda)|^p]\}^{\alpha/p} = \text{Const}(p, \alpha) \phi(\lambda) \, d\lambda \quad \text{for all } 0 < p < \alpha,
\] (1.4)
where the constant depends only on $p$ and $\alpha$ (and not on $\xi$) and $\phi(\lambda)$ is a nonnegative
integrable function called the spectral density of $X.$ (Some properties of $\xi$ are
collected in Section 7B.) The integral in (1.1) is defined by means of convergence
in probability or (equivalently) in $p$th mean, $0 < p < \alpha,$ and the finite dimensional
characteristic functions of $X$ are given by
\[
E\exp\left\{i \Re \sum_{n=1}^{N} z_n^* X(t_n) \right\} = \exp\left\{-c_n \int \left| \sum_{n=1}^{N} z_n^* e^{i\gamma \lambda} \right|^\alpha \phi(\lambda) \, d\lambda \right\}
\] (1.5)
where $z_n = r_n + is_n,$ $c_n = (\alpha \pi)^{1/2} \int_0^\infty \cos \theta |\gamma|^\alpha \, d\theta,$ $c_2 = \sqrt{3/10},$ so that the spectral
density $\phi$ describes fully the distribution of the process $X.$ When the index \(\alpha = 2,\)
the process $X$ is Gaussian, the function $\phi$ appearing in (1.4) and (1.5) is the usual
power spectral density, and the standard spectral analysis of fourth-order processes
described earlier is then applicable. When $0 < \alpha < 2,$ as we assume henceforth, the
process $X$ has finite moments of order only less than $\alpha,$ so that $X$ is not even of
second order. Its spectral density $\phi$ does not represent a power spectral density in
the usual sense, but it has been shown in [5] and [6] that in problems of linear
prediction and filtering, it plays a role analogous to that played by the power spectral
density function of a second order process. Moreover, when $1 < \alpha < 2,$ the
covariation of $X(t)$ with $X(s),$ introduced in [3, 13], plays in regression problems [13, 5]
and in sample function properties [4] a role analogous to that of the covariation
of a second order process. The covariation function of the SaS stationary process $X$
under consideration here (with $1 < \alpha < 2$) is given by
\[
\text{Covariation}(X(t), X(s)) = \int_{-\infty}^{\infty} e^{i\pi \gamma \lambda} \phi(\lambda) \, d\lambda
\] (1.6)
\[\text{[3]},\] and thus has an identical representation in terms of the spectral density $\phi$ as
the covariation function (1.3) of a second order stationary process. (Here the
covariation as introduced in [3, 13] is multiplied by $\alpha$ so as to equal the covariation
when $\alpha = 2.$) Hence the need to obtain consistent estimates of the spectral density
$\phi$ of such stationary stable processes $X,$ from finite length observations, is quite clear.
Our goal is to establish nonparametric estimates for the spectral density $\phi(\lambda)$ of the stationary stable process $X$ and study their asymptotic statistical properties. Given an observation of $X$ over the finite interval $[-T, T]$, we form in Section 2 the (real part of the) finite tapered Fourier transform

$$d_T(\lambda) = A_T \text{Re} \int_{-T}^{T} e^{-i\lambda h(t/T)}X(t) \, dt.$$  \hfill (1.7)

This is a $S_{\alpha S}$ process and asymptotically, as $T \to \infty$, it has independent values at distinct frequencies (Theorem 2.2) and characteristic function $\exp(-c_\alpha |r|^{\alpha} \phi(\lambda))$. This generalizes a result of Hosoya (Theorem 4.3 in [9] and Theorem 2.3 in [10]) where the discrete-time case with no tapering is considered. Next it is natural to consider the periodogram

$$J_T(\lambda) = \text{Const}(d_T(\lambda))^\alpha$$  \hfill (1.8)

which unfortunately has infinite mean and whose smoothed version by means of a spectral window seems difficult to study. To circumvent these difficulties we introduce in Section 3 the modified periodogram

$$I_T(\lambda) = C_{\rho,\alpha}(1/d_T(\lambda))^{\rho}$$

with $0 < \rho < \alpha/2$. This is an asymptotically unbiased estimate of $(\phi(\lambda))^{\rho/\alpha}$ (Theorem 3.1) and its asymptotic variance is obtained (Theorem 3.2). The modified periodogram is not a consistent estimate of $(\phi(\lambda))^{\rho/\alpha}$. However by smoothing it via a spectral window we obtain in Section 4 expressions for the mean (Theorem 4.1) and variance (Theorem 4.2) of the smoothed estimate

$$f_T(\lambda) = \int_{-\infty}^{\infty} W_T(\lambda-u) I_T(u) \, du$$

from which follow its mean-square consistency as an estimate of $(\phi(\lambda))^{\rho/\alpha}$, along with rates of convergence (Theorem 4.3). Finally, a consistent estimate of $\phi(\lambda)$ is obtained in Section 5 by

$$\hat{\phi}_T(\lambda) = \{f_T(\lambda)\}^{\rho/\alpha}$$

and its convergence in probability to $\phi(\lambda)$, along with rates of convergence, is established (Theorem 5.1). A strongly consistent estimate, along with a rate of convergence, can be obtained for an appropriate subsequence (Theorem 5.2). All these results are obtained under appropriate conditions on the spectral density $\phi(\lambda)$, listed at the end of this section, and on the data and spectral windows. Due to the method of proof, the rates of weak and strong convergence in Theorems 5.1 and 5.2 do not reduce to those available in the Gaussian case $\alpha = 2$. It would therefore be desirable to obtain sharper rates and, more significantly, to investigate whether the standard periodogram (1.8), when smoothed via a spectral window, provides a consistent estimate of the spectral density. It would also be more natural to use the finite Fourier transform itself rather than its real part (1.7); we have adopted the latter because our analysis employs the identity (3.3), valid for real numbers.
In Section 6 it is pointed out that these results described above for complex processes are also valid for real processes under somewhat stronger conditions, and that in both cases observation of $X$ over the one-sided interval $[0, T]$, rather than the two-sided $[-T, T]$, may be used in estimating the spectral density.

To the best of our knowledge the available literature deals with discrete-time stationary (finite order) autoregressive SαS processes and is concerned with the estimation of the autoregressive coefficients $[1, 7, 8, 12, 18]$. These stationary SαS processes are (infinite) moving averages of independent and identically distributed SαS random variables and thus, as shown in $[6$, Theorem 3.3], they form a class of discrete-time processes distinct from the class of discrete-time analogs of the processes considered here, i.e. $\{X_n\}_{n=-\infty}^{\infty}$ with

$$X_n = \int_{-\infty}^{\infty} e^{i \lambda n} d\xi(\lambda), \quad n = 0, \pm 1, \ldots$$

(for which, of course, the analogs of our result hold).

Conditions on the spectral density $\phi(\lambda), -\infty < \lambda < \infty$

$$(\phi 1) \phi$$ is bounded.

$$(\phi 2) \phi$$ is bounded and uniformly continuous.

$$(\phi 3) \phi$$ is continuously differentiable with bounded derivative.

$$(\phi 4) \phi$$ is twice continuously differentiable with bounded second derivative.

2. Finite tapered transform

In this section we develop the statistical properties of the real part of the finite tapered Fourier transform $d_T(\lambda)$ defined in (1.7)

$$d_T(\lambda) = A_T \Re \int_{-T}^{T} e^{i \lambda t} h(t/T)X(t) dt. \quad (2.1)$$

We first state the assumptions on the data window or taper: $h(t)$ is a bounded even function vanishing for $|t| > 1$. Assuming that its Fourier transform

$$H(\lambda) = \int_{-1}^{1} h(t) e^{-i \lambda t} dt$$

satisfies

$$B_\alpha \equiv \int_{-1}^{1} |H(\lambda)|^\alpha d\lambda < \infty,$$

we define the kernels

$$H_T(\lambda) = (T / B_\alpha)^{1/\alpha} H(T\lambda) = A_T H(T\lambda) \quad (2.2)$$
so that
\[ \int_{-\infty}^{\infty} |H_T(\lambda)|^{\alpha} \, d\lambda = 1. \]

Some standard examples are
\[ h(t) = 1_{[-1,1]}(t), \quad H(\lambda) = 2 \frac{\sin \lambda}{\lambda}, \]
\[ h(t) = (1-|t|)1_{[-1,1]}(t), \quad H(\lambda) = \left\{ \frac{\sin \lambda/2}{\lambda/2} \right\}^2, \]

which, when \( \omega = 2 \), give rise to the usual Dirichlet and Fejer kernels. Throughout
the paper we shall use various conditions on the rate of decay of \( H(\lambda) \) as \( |\lambda| \to \infty \).
These are stated below.

Conditions on the data window

(H) \( |H(\lambda)| \leq \text{Const}/(1 + \lambda^2)^{\beta/2} \) for all \( \lambda \) and some \( \beta \geq 1 \).

(H') (H) with \( \beta > 2/\alpha \).

(H'') (H) with \( \beta > 3/\alpha \).

Note that when no tapering is used, \( h(t) = 1_{[-1,1]}(t) \), condition (H) is satisfied
with \( \beta = 1 \). Substituting the integral representation (1.1) for \( X(t) \) in (2.1) and
interchanging the order of integration (cf. Theorem 4.6 in [5]) we obtain

\[
d_T(\lambda) = A_T \text{Re} \int_{-\infty}^{\infty} \left\{ \int_{-T}^{T} h\left( \frac{t}{T} \right) e^{-i(t-u)} \, dt \right\} \, d\xi(u)
\]
\[ = A_T \text{Re} \int_{-\infty}^{\infty} TH[ T(\lambda-u) ] \, d\xi(u) \]
\[ - \text{Re} \int_{-\infty}^{\infty} H_T(\lambda-u) \, d\xi(u) = \int_{-\infty}^{\infty} H_T(\lambda-u) \, d\xi_1(u), \quad (2.3) \]

where \( \xi_1 = \text{Re}[\xi] \) is an \( S\alpha S \) independent increments process with \( E \exp\{ir\xi(B)\} = \exp\{-c_\alpha |r|^\alpha \int_B \phi \} \) [3]. Thus for each fixed \( T \), the finite tapered transform \( d_T(\lambda) \),
\( -\infty < \lambda < \infty \), is an \( S\alpha S \) process and we now find its asymptotic finite dimensional
distributions as \( T \to \infty \), which, once they exist, are by necessity \( S\alpha S \). The description
of the asymptotic \( S\alpha S \) univariate distribution of \( d_T(\lambda) \) is given in Theorem 2.1, and
Theorem 2.2 establishes asymptotic independence; both results include rates of
convergence for the corresponding characteristic functions.

**Theorem 2.1.** Let \( \phi \) be continuous at \( \lambda \).

(i) If either \( \phi \) is bounded or \( u|H(u)|^{\alpha} \to 0 \) as \( |u| \to \infty \), then

\[
E \exp\{i\rho d_T(\lambda)\} = \exp\{-c_\alpha |r|^{\alpha} \phi(\lambda)\} + o(1).
\]
(ii) If Conditions (φ3) and (H') are satisfied then
\[ E \exp\{ird_T(\lambda)\} = \exp\{-c_\alpha |r|^\alpha \phi(\lambda)\} + o\left(\frac{1}{T}\right). \]

The following proposition is needed for the proof of Theorem 2.1.

**Proposition 2.1.** Let
\[ \psi_T(\lambda) = \int_{-\infty}^{\infty} |H_T(\lambda - u)|^\alpha \phi(u) \, du. \]  

(i) If φ is continuous at λ and either φ is bounded or \( u|H(u)|^\alpha \to 0 \) as \( |u| \to \infty \), then
\[ \psi_T(\lambda) = \phi(\lambda) + o(1). \]

(ii) If Conditions (φ3) and (H') are satisfied, then
\[ \psi_T(\lambda) = \phi(\lambda) + o\left(\frac{1}{T}\right). \]

**Proof.** (i) When φ is bounded the result follows by dominated convergence and when \( u|H(u)|^\alpha \to 0 \) as \( |u| \to \infty \) by the argument of Bochner's theorem (see Theorem 1A in [15]).

(ii) We have
\[ B_n T[\psi_T(\lambda) - \phi(\lambda)] - \int_{-\infty}^{\infty} v|H(v)|^\alpha \, dv \]
and as the integrand tends to \( v|H(v)|^\alpha \phi'(\lambda) \) as \( T \to \infty \) and is bounded absolutely by \( \|\phi\|_{\infty} |v| |H(v)|^\alpha \), which is integrable over the indicated range, it follows by dominated convergence that
\[ B_n T[\psi_T(\lambda) - \phi(\lambda)] - \phi'(\lambda) \int_{-\infty}^{\infty} v|H(v)|^\alpha \, dv = 0. \]

**Proof of Theorem 2.1.** From (7.1) we have
\[ E \exp\left\{ i \int f_1 \, d\xi_1 \right\} = \exp\left\{ -c_\alpha \int |f_1|^\alpha \phi \right\} \]  
and thus we have from (2.3)
\[ E \exp\{ird_T(\lambda)\} = \exp\left\{ -c_\alpha |r|^\alpha \int |H_T(\lambda - u)|^\alpha \phi(u) \, du \right\}. \]

Thus putting
\[ c_T(\lambda) = \int |H_T(\lambda - u)|^\alpha \phi(u) \, du - \phi(\lambda) \]  

(2.7)
and using $|e^{-x} - e^{-y}| \leq |x - y|$ for $x, y \geq 0$, we obtain
\[ |E e^{itd_T}\lambda) - e^{-c_1|\Lambda|^\alpha} | \leq c_\alpha |\Lambda|^\alpha |e_T(\lambda)|.\]
The results then follow from Proposition 2.1. \(\square\)

We next consider the multidimensional case and we first study the convergence to zero of
\[
\epsilon_T(r, \lambda) = \int_{-\infty}^{\infty} \left| \sum_{k=1}^{n} r_k H_T(\lambda_k - u) \right|^\alpha \phi(u) \, du 
- \sum_{k=1}^{n} |r_k|^\alpha \int_{-\infty}^{\infty} |H_T(\lambda_k - u)|^\alpha \phi(u) \, du
\]
where $r = (r_1, \ldots, r_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$.

**Proposition 2.2.** If $\lambda_1, \ldots, \lambda_n$ are distinct points and Condition (H) is satisfied, then
\[
\epsilon_T(r, \lambda) = \begin{cases} 
O(\ln T / T^{\alpha^2 + 1}) & \text{for } 0 < \alpha \leq 1, \\
o(1 / T^\beta) & \text{for } 1 < \alpha < 2,
\end{cases}
\]
where it is assumed $\alpha \beta > 1$ when $0 < \alpha \leq 1$. If in addition $\phi$ is bounded (Condition (ϕ1)), then, for $1 < \alpha < 2$,
\[
\epsilon_T(r, \lambda) = O(1 / T^{\beta - 1/\alpha}).
\]

Propositions 2.1 and 2.2 together imply that under the combined conditions
\[
\int_{-\infty}^{\infty} \left| \sum_{k=1}^{n} r_k H_T(\lambda_k - u) \right|^\alpha \phi(u) \, du \to \sum_{k=1}^{n} |r_k|^\alpha \phi(\lambda_k)
\]
and convergence rates are also provided. This generalizes Theorem 4.1 in [9] where the case of Dirichlet kernel is considered (and no rates are provided).

**Proof of Proposition 2.2.** Let $E_k = (\lambda_k - \gamma_T, \lambda_k + \gamma_T)$, $k = 1, \ldots, n$, where $\gamma_T$ is such that $\gamma_T \to 0$ as $T \to \infty$. Then in (2.8) we split the integral of the first term into the integral over $U_k E_k$ and $(U_k E_k)^c$, and each of the integrals of the second term into the integrals over $E_k$ and $E_k^c$ to obtain
\[
\epsilon_T(r, \lambda) = \sum_k \int_{E_k} \left\{ \left| \sum_j r_j H_T(\lambda_j - u) \right|^\alpha - \left| r_k H_T(\lambda_k - u) \right|^\alpha \right\} \phi(u) \, du 
+ \int_{(U_k E_k)^c} \left| \sum_j r_j H_T(\lambda_j - u) \right|^\alpha \phi(u) \, du
- \sum_k |r_k|^\alpha \int_{E_k^c} |H_T(\lambda_k - u)|^\alpha \phi(u) \, du.
\]
Denote the first term by \( A \) and by \( B \) the remaining of (2.9). It is easily seen (using the \( c_1 \) inequality) that

\[
|B| \leq \text{Const} \sum_k |r_k|^\alpha \int_{E_k} |H_T(\lambda_k - u)|^\alpha \phi(u) \, du.
\]

Each integral is bounded by \( \sup_{|v| > y_T} |H_T(v)|^\alpha \cdot \int \phi \), and using (2.2) and Condition (H) we obtain

\[
|B| \leq \text{Const} \frac{T}{(Ty_T)^{\alpha\beta}}, \quad (2.10a)
\]

Under (\( \phi 1 \)), each integral is bounded by \( \|\phi\|_x \int_{|v| > y_T} |H_T(v)|^\alpha \, dv \) which gives similarly

\[
|B| \leq \text{Const} \frac{1}{(Ty_T)^{\alpha\beta - 1}}, \quad (2.10b)
\]

We now consider the term \( A \). Since \( H_T(u) \) is continuous we have from (2.9) by the mean value theorem

\[
|A| \leq \sum_k \left\{ \left| \sum_{j \neq k} r_j H_T(\lambda_j - u_k) \right|^\alpha + \alpha |r_k H_T(\lambda_k - u_k)|^\alpha \left| \sum_{j \neq k} r_j H_T(\lambda_j - u_k) \right| \right\} \int_{E_k} \phi,
\]

and by Condition (H) and the \( c_1 \) inequality

\[
|A| \leq \text{Const} \sum_k \sum_{j \neq k} \left\{ \frac{|r_j|^\alpha T}{[1 + T^2(\lambda_j - u_k)^2]^{\alpha\beta/2}} \frac{\alpha |r_k|^\alpha - |r_j| T}{[1 + T^2(\lambda_j - u_k)^2]^{\beta/2}} \right\} \int_{E_k} \phi.
\]

Since \( \phi \) is integrable, \( \int_{E_k} \phi = o(1) \) as \( T \to \infty \) and when \( \phi \) is also bounded (or just locally bounded on neighborhoods of \( \lambda_k \)'s) the integral is \( O(\gamma_T) \). Hence

\[
|A| \leq \text{Const} \left( \frac{1}{T^{\alpha\beta - 1}} + \frac{1}{T^{\beta - 1}} \right) o(1) = o(1/T^{\beta - 1}) \quad \text{under (H).} \quad (2.12)
\]

From (2.10) and (2.12) we have under (H),

\[
\mathcal{E}_T(t, \lambda) = \frac{\text{Const}}{T^{\alpha\beta - 1}(\gamma_T)^{\alpha\beta}} + o(1/T^{\beta - 1})
\]
and the choice $(\gamma_T)^{-\alpha\beta} = \ln T$ gives the result for $1 < \alpha < 2$. Similarly, under $(H)$ and $(\phi 1)$ we have

$$e_T(r, \lambda) \leq \text{Const} \left( \frac{\gamma_T}{T^{\beta-1}} + \frac{1}{(T\gamma_T)^{\alpha\beta-1}} \right)$$

and the optimal choice of $\gamma_T, \gamma_T = T^{-(1-1/\alpha)}$, gives the result.

When $0 < \alpha \leq 1$, we have from the $c_1$ inequality $|x + y|^\alpha - |x|^\alpha \leq |y|^\alpha$, and thus from (2.11) we obtain

$$|A| \leq \text{Const} \sum_k \left| \sum_{r \neq k} r_j H_T(\lambda_j - u_k) \right|^{\alpha} \int_{E_k} \phi$$

$$\leq \text{Const} \sum_k \sum_{j \neq k} \frac{|r_j|^\alpha T}{1 + T^2(\lambda_j - u_k)^2} \int_{E_k} \phi$$

and as $\int_{E_k} \phi$ is $o(1)$ when $\phi$ is integrable and $O(\gamma_T)$ when $\phi$ is bounded, we have

$$|A| \leq \frac{\text{Const}}{T^{\alpha\beta-1}} o(1) \quad \text{under (H),}$$

$$|A| \leq \frac{\gamma_T}{T^{\alpha\beta-1}} \quad \text{under (II) and (\phi 1)}$$

(2.13)

It is now seen from (2.10) and (2.13) that the term $B$ dominates and the result for $0 < \alpha \leq 1$ follows by choosing $(\gamma_T)^{-\alpha\beta} = \ln T$. \qed

We now establish the asymptotic independence of the tapered transform $d_T(\lambda)$.

**Theorem 2.2.** Let $\phi$ be continuous at the distinct points $\lambda_1, \ldots, \lambda_n$. Under Condition $(H)$ with $\alpha\beta > 1$, $d_T(\lambda_1), \ldots, d_T(\lambda_n)$ are asymptotically independent $S\alpha S$ variables and

$$E_T \triangleq E \exp \left\{ i \sum_{k=1}^n r_k d_T(\lambda_k) \right\} - \exp \left\{ -c_\alpha \sum_{k=1}^n |r_k|^\alpha \phi(\lambda_k) \right\} = o(1).$$

Under Conditions $(\phi 1), (\phi 3)$ and $(H')$, the difference $E_T$ satisfies

$$E_T = \begin{cases} o\left( \frac{1}{T} \right) & \text{for } 0 < \alpha \leq 1, \\ o\left( \frac{1}{T} \right) + O\left( \frac{1}{T^{\beta-1/\alpha}} \right) & \text{for } 1 < \alpha < 2. \end{cases}$$

**Proof.** From (2.3) and (2.5) we obtain

$$E \left\{ \exp \left[ i \sum_{k=1}^n r_k d_T(\lambda_k) \right] \right\} = \exp \left\{ -c_\alpha \int_{-\infty}^\infty \left| \sum_{k=1}^n r_k H_T(\lambda_k - u) \right|^\alpha \phi(u) \, du \right\}$$
so that

\[ |E_T| \leq \exp \left\{ -c_\alpha \left| \sum_{k=1}^n r_k H_T(\lambda_k - u) \phi(u) \right|^\alpha du \right\} \]

\[ - \exp \left\{ -c_\alpha \sum_{k=1}^n |r_k|^\alpha \int H_T(\lambda_k - u) \phi(u) du \right\} \]

\[ + \left| \exp \left\{ -c_\alpha \sum_{k=1}^n |r_k|^\alpha \int H_T(\lambda_k - u) \phi(u) du \right\} \right| \]

\[ - \exp \left\{ -c_\alpha \sum_{k=1}^n |r_k|^\alpha \phi(\lambda_k) \right\} \].

Thus with \( E_T(r, \lambda) \) defined in (2.8) and \( e_T(\lambda) \) in (2.7) we obtain, using \(|e^{-x} - e^{-y}| \leq |x - y|\) for \( x, y \geq 0\),

\[ |E_T| \leq c_\alpha \left\{ |E_T(r, \lambda)| + \sum_{k=1}^n |r_k|^\alpha |e_T(\lambda_k)| \right\}.

The results then follow from Propositions 2.1 and 2.2.

The asymptotic independence of discrete-time finite untapered transforms (sine and cosine) has been established in Theorem 4.3 in [9].

### 3. A fractional-power periodogram

In this section we study the statistical properties of the modified periodogram

\[ I_T(\lambda) = C_{p,\alpha}|d_T(\lambda)|^p, \quad 0 < p < \alpha. \tag{3.1} \]

as a naive estimate for the fractional-power \( p/\alpha \) of the spectral density \( \phi(\lambda) \), i.e. for

\[ f(\lambda) = [\phi(\lambda)]^{p/\alpha}. \tag{3.2} \]

The normalization constant \( C_{p,\alpha} \) is given by

\[ C_{p,\alpha} = \frac{D_p}{F_{p,\alpha}(r, \alpha)^{p/\alpha}} \]

where

\[ D_p = \int_{-\infty}^{\infty} \frac{1 - \cos u}{|u|^{1+p}} \, du, \quad 0 < p < 2, \]

\[ F_{p,\alpha} = \int_{-\epsilon}^{\epsilon} \frac{1 - e^{-|u|^\alpha}}{|u|^{1+p}} \, du, \quad 0 < p < \alpha. \]

We first show that \( I_T(\lambda) \) is an asymptotically unbiased estimate of \( f(\lambda) \).
Theorem 3.1. Let $0 < p < \alpha$. Then

$$EI_T(\lambda) = \left\{ \int_{-\infty}^{\infty} |H_T(\lambda - u)|^{-\alpha} \phi(u) \, du \right\}^{p/\alpha} = \{\psi_T(\lambda)\}^{p/\alpha}.$$

Define

$$\text{Bias}[I_T(\lambda)] = EI_T(\lambda) - f(\lambda).$$

(i) If $\phi$ is continuous at $\lambda$ and Condition (\phi 1) or (H) is satisfied, then

$$\text{Bias}[I_T(\lambda)] = o(1).$$

(ii) Under Conditions (\phi 3) and (H'),

$$\text{Bias}[I_T(\lambda)] = o\left(\frac{1}{T^{p/\alpha}}\right)$$

and the $o(\cdot)$ term is uniform in $\lambda$.

(iii) Under Conditions (\phi 3), (H') and $\phi(\lambda) \neq 0$,

$$\text{Bias}[I_T(\lambda)] = \frac{1}{\{\phi(\lambda)\}^{1-p/\alpha}} o\left(\frac{1}{T}\right).$$

Proof. Using the identity

$$|x|^p = D_p^{-1} \int_{-\infty}^{\infty} \frac{1 - \cos(xu)}{|u|^{1+p}} \, du = D_p^{-1} \Re \int_{-\infty}^{\infty} \frac{1 - e^{iu\lambda}}{|u|^{1+p}} \, du,$$  \hspace{1cm} (3.3)

valid for all real $x$ and $0 < p < 2$, we obtain

$$I_T(\lambda) = \frac{1}{F_{p,\alpha} e^{p/\alpha}} \Re \int_{-\infty}^{\infty} \frac{1 - e^{u\lambda}}{|u|^{1+p}} \, du = \psi_T(\lambda)^{p/\alpha}$$  \hspace{1cm} (3.4)

and thus, by (2.5),

$$EI_T(\lambda) = \frac{1}{F_{p,\alpha} e^{p/\alpha}} \int_{-\infty}^{\infty} \frac{1 - e^{-c_{\alpha} |u|^{\alpha} \phi_T(\lambda)}}{|u|^{1+p}} \, du = \{\psi_T(\lambda)\}^{p/\alpha}$$  \hspace{1cm} (3.5)

since $\psi_T(\lambda) > 0$. Hence

$$\text{Bias}[I_T(\lambda)] = \{\psi_T(\lambda)\}^{p/\alpha} - \{\phi(\lambda)\}^{p/\alpha}.$$

Part (i) now follows from (i) of Proposition 2.1. Parts (ii) and (iii) follow from (ii) of Proposition 2.1 and, respectively, the following inequalities where $x, y \geq 0$ and $r = p/\alpha \in (0, 1)$: $|x^r - y^r| \leq |x - y|^r$ and

$$|x^r - y^r| \leq \frac{r}{2} |x - y|(x^{r-1} + y^{r-1}), \quad x, y > 0,$$  \hspace{1cm} (3.6)

of which the latter follows from $x^r - y^r = r \int_y^x u^{r-1} \, du$.  \hspace{1cm} □
The asymptotic variance of $I_T(\lambda)$ is now shown to be proportional to $f^2(\lambda)$ when $0 < p < \alpha/2$.

**Theorem 3.2.** Let $0 < p < \alpha/2$. Then

$$\text{Var}[I_T(\lambda)] = V_{p,\alpha}\{\psi_T(\lambda)\}^{2p/\alpha}$$

where $V_{p,\alpha} = C_{p,\alpha}^2/C_{2p,\alpha} - 1$, and under the same conditions as in Theorem 3.1, $\text{Var}[I_T(\lambda)] - V_{p,\alpha}f^2(\lambda)$ is $o(1)$ under (i), $O(1/T^{2p/\alpha})$ under (ii), and $(\phi(\lambda))^{-1+2p/\alpha}o(1/T)$ under (iii).

**Proof.** From (3.1) and (3.3) we obtain

$$EI_T^2(\lambda) = C_{p,\alpha}^2D_{2p}^{-1}\int_{-\infty}^{\infty} \frac{1 - e^{-\zeta_1|u|^\alpha}\phi_T(\lambda)}{|u|^{1+2p/\alpha}} du$$

$$= C_{p,\alpha}^2D_{2p}^{-1}C_{2p,\alpha}\{c_{p,\alpha}\psi_T(\lambda)\}^{2p/\alpha} = C_{p,\alpha}^2C_{2p,\alpha}\{c_{p,\alpha}\psi_T(\lambda)\}^{2p/\alpha}.$$

Hence $\text{Var}[I_T(\lambda)] = V_{p,\alpha}\{\psi_T(\lambda)\}^{2p/\alpha}$, and the results now follow as in the proof of Theorem 3.1. \(\square\)

Theorem 2.2 implies the asymptotic independence of the values of $I_T(\lambda)$, $-\infty < \lambda < \infty$, and we now derive the rate at which its covariance tends to zero, when $0 < p < \alpha/2$. Even though this result per se is not used further on, its proof is included here because it is heavily used later on.

**Theorem 3.3.** Let $0 < p < \alpha/2$ and $\phi$ be continuous at the distinct points $\lambda_1$ and $\lambda_2$. If $\phi(\lambda_1) \neq 0 \neq \phi(\lambda_2)$ and Conditions (\(\phi_1\)) and (H) are satisfied then

$$\text{Cov}[I_T(\lambda_1), I_T(\lambda_2)] = O(1/T^s)$$

where

$$s = \begin{cases} \alpha\beta - 11 & \text{if } 1 < \alpha\beta < 2, \\
1 & \text{if } \alpha\beta = 2, \\
\alpha\beta/2 & \text{if } 2 < \alpha\beta, \end{cases}$$

(3.7)

and the notation $x = (a)$ indicates that $x < a$ but can take a value arbitrarily close to $a$.

**Remark 3.1.** When $\phi$ vanishes at $\lambda_n$, the method of the proof of Theorem 3.3 is not applicable and a crude bound on the covariance of $I_T(\lambda_1)$ and $I_T(\lambda_2)$ can be obtained from Theorem 3.2 using the Cauchy–Schwarz inequality. For example, if $\phi(\lambda_1) = 0$ or $\phi(\lambda_2) = 0$ then $\text{Cov}[I_T(\lambda_1), I_T(\lambda_2)] = o(1)$ under (i) of Theorem 3.1, and $= O(1/T^{2p/\alpha})$ under (ii) of Theorem 3.1. If $\phi(\lambda_1) = \phi(\lambda_2) = 0$, then under (ii) of Theorem 3.1, $\text{Cov}[I_T(\lambda_1), I_T(\lambda_2)] = O(1/T^{2p/\alpha})$.

The following proposition is essential to the proof of Theorem 3.3.
Proposition 3.1. Define

$$\Delta(\lambda) = \int_{-\infty}^{\infty} \left| H(u) \right|^{\alpha/2} \left| H(\lambda - u) \right|^{\alpha/2} \, du. \quad (3.8)$$

Then under Condition (H),

$$\Delta(\lambda) \leq \frac{\text{Const}}{(1 + \lambda^2)^{s/2}}$$

where $s$ is given in (3.7).

Proof. As $\Delta(\lambda)$ is the convolution of $L_2$ functions, it is bounded and uniformly continuous and we establish its rate of decay as $|\lambda| \to \infty$. Since $\Delta(\lambda)$ is even, assume $\lambda \to \infty$ and split the range of integration in (3.8) as follows:

$$\Delta(\lambda) = \int_{-\lambda/2}^{\lambda/2} + \int_{-\lambda/2}^{-\lambda/2} + \left\{ \int_{-\infty}^{-\lambda/2} + \int_{\lambda/2}^{\lambda/2} \right\} \Delta J_1 + J_2 + J_3.$$

Using the Cauchy–Schwarz inequality and (H) we have

$$|J_3| \leq \int_{|u| > \lambda/2} |H(u)|^\alpha \, du \leq \frac{\text{Const}}{\lambda^{\alpha \beta - 1}}.$$

When $\alpha \beta \neq 2$, using (H) we obtain

$$|J_1 + J_2| \leq \text{Const} \int_{|u| < \lambda/2} |H(u)|^{\alpha/2} \, du$$

and for large $\lambda$,

$$\int_{|u| < \lambda/2} |H(u)|^{\alpha/2} \, du \leq \text{Const} + \text{Const} \int_{1}^{\lambda/2} \frac{du}{u^{\alpha \beta / 2}} \leq \text{Const} \{1 + \lambda^{-\alpha \beta / 2}\}, \quad (3.9)$$

we finally have

$$|J_1 + J_2| \leq \text{Const} \frac{1 + \lambda^{-\alpha \beta / 2}}{\lambda^{\alpha \beta / 2}} = \text{Const} \left\{ \frac{1}{\lambda^{\alpha \beta / 2}} + \frac{1}{\lambda^{\alpha \beta - 1}} \right\}.$$

We thus have

$$|\Delta(\lambda)| \leq \text{Const} \left\{ \frac{1}{\lambda^{\alpha \beta / 2}} + \frac{1}{\lambda^{\alpha \beta - 1}} \right\}$$

and the result follows. When $\alpha \beta = 2$, in evaluating the integral in (3.9) we obtain $\ln(\lambda/2)$ so that the bound on $J_1 + J_2$, as well as on $\Delta(\lambda)$, becomes $\text{Const} \ln \lambda / \lambda$ which is smaller than $\text{Const} \lambda^{-r}$ for all $0 < r < 1$. \[\square\]

Proof of Theorem 3.3. By (3.4) and (3.5) we have

$$I_\gamma(\lambda) - E I_\gamma(\lambda) = F_{\rho,\alpha}^{-1} c_{\rho/\alpha} \int_{-\infty}^{\infty} \{ \text{Re} \left[ e^{|u|d(\lambda)} - e^{-c_{\rho/\alpha} |u| d(\lambda)} \right] \} \frac{du}{|u|^{1+p}}$$
and thus
\[
\text{Cov}[I_T(\lambda_1), I_T(\lambda_2)] = E \prod_{k=1}^{2} \cos u_k d_T(\lambda_k) \\
= \frac{1}{2} \exp \left[ -c_\alpha \int \left( \sum_{k=1}^{2} u_k H_T(\lambda_k - v) \right)^{\alpha} \phi(v) \, dv \right] \\
+ \frac{1}{2} \exp \left[ -c_\alpha \int \left( \sum_{k=1}^{2} (-1)^k u_k H_T(\lambda_k - v) \right)^{\alpha} \phi(v) \, dv \right]
\]

For simplicity we denote this covariance by \( C(\lambda_1, \lambda_2) \). From \( 2 \cos x \cos y = \cos(x + y) + \cos(x - y) \) we obtain
\[
E \prod_{k=1}^{2} \cos u_k d_T(\lambda_k) \\
= \frac{1}{2} \exp \left[ -c_\alpha \int \left( \sum_{k=1}^{2} u_k H_T(\lambda_k - v) \right)^{\alpha} \phi(v) \, dv \right] \\
+ \frac{1}{2} \exp \left[ -c_\alpha \int \left( \sum_{k=1}^{2} (-1)^k u_k H_T(\lambda_k - v) \right)^{\alpha} \phi(v) \, dv \right]
\]

and substituting in the expression for \( C(\lambda_1, \lambda_2) \), and changing the variable \( u_2 \) to \(-u_2\) in the second term, we have
\[
C(\lambda_1, \lambda_2) = F_{p, \alpha} c_{\alpha}^{-2p/\alpha} \int_{\chi} \left\{ e^{-a} - e^{-b} \right\} \frac{du_1 \, du_2}{|u_1 u_2|^{1+\rho}}
\]

where
\[
a = c_\alpha \int_{-\chi}^{\chi} \left( \sum_{k=1}^{2} u_k H_T(\lambda_k - v) \right)^{\alpha} \phi(v) \, dv,
\]
\[
b = c_\alpha \sum_{k=1}^{2} |u_k|^{\alpha} \psi_T(\lambda_k) = c_\alpha \sum_{k=1}^{2} |u_k|^{\alpha} \int_{-\chi}^{\chi} |H_T(\lambda_k - v)|^{\alpha} \phi(v) \, dv.
\]

Since \( a, b > 0 \) we have
\[
|e^{-a} - e^{-b}| = e^{-b} |e^{-a} - 1| = e^{-b} |a - b| e^{-a} - 1
\]

so that
\[
|C(\lambda_1, \lambda_2)| \leq \text{Const} \int_{\chi} \int |a - b| e^{-|k_1 - k_2|} \frac{du_1 \, du_2}{|u_1 u_2|^{1+\rho}}
\]

Now
\[
a - b = c_\alpha \int_{-\chi}^{\chi} \left\{ \left( \sum_{k=1}^{2} u_k H_T(\lambda_k - v) \right)^{\alpha} - \sum_{k=1}^{2} |u_k H_T(\lambda_k - v)|^{\alpha} \right\} \phi(v) \, dv
\]
and using the inequality in Theorem 7.1(a), it follows that

\[ |a - b| \leq 2c_a |u_1 u_2|^{\alpha/2} \int_{-\infty}^{\infty} |H_T(\lambda_1 - \nu)H_T(\lambda_2 - \nu)|^{\alpha/2} \phi(\nu) \, d\nu \]
\[ \leq 2c_a (\sup \phi) |u_1 u_2|^{\alpha/2} \Delta[T(\lambda_1 - \lambda_2)]. \]  

(3.14)

Also from (3.11) and (3.14),

\[ |a - b| - b \leq 2c_a \|\phi\|_{\infty} |u_1 u_2|^{\alpha/2} \Delta[T(\lambda_1 - \lambda_2)] - c_a \sum_{k=1}^{2} |u_k|^{\alpha} \psi_T(\lambda_k) \]
\[ = -c_a \sum_{k=1}^{2} |u_k|^{\alpha} \{ \psi_T(\lambda_k) - \|\phi\|_{\infty} \Delta[T(\lambda_1 - \lambda_2)] \} \]
\[ = -c_a \sum_{k=1}^{2} |u_k|^{\alpha} \{ \phi(\lambda_k) + o(1) \}, \]  

(3.15)

since \( \psi_T(\lambda) = \phi(\lambda) + o(1) \) (cf. Theorem 3.1) and \( \Delta[T(\lambda_1 - \lambda_2)] = o(1) \) (cf. Proposition 3.1). Using (3.14) and (3.15) in (3.13) we have

\[ |C(\lambda_1, \lambda_2)| \leq \text{Const} \Delta[T(\lambda_1 - \lambda_2)] \prod_{k=1}^{2} \int_{-\infty}^{\infty} e^{-c_a |u_k|^{\alpha} \{ \phi(\lambda_k) + o(1) \}} \left| u_k \right|^{1+p-\alpha/2} \, du_k. \]

If \( \phi(\lambda_1) \neq 0 \neq \phi(\lambda_2) \) then for large enough \( T \) the terms \( \phi(\lambda_k) + o(1) \) are positive and thus

\[ |C(\lambda_1, \lambda_2)| \leq \frac{\text{Const}}{\left\{ \phi(\lambda_1)\phi(\lambda_2) \right\}^{1/2-p/\alpha}} \Delta[T(\lambda_1 - \lambda_2)] \]  

(3.16)

and the result follows from Proposition 3.1. \( \Box \)

### 4. Consistent estimate of \( \{ \phi(\lambda) \}^{p/\alpha} \)

While the modified periodogram \( I_T(\lambda) \) is an asymptotically unbiased estimate of \( \{ \phi(\lambda) \}^{p/\alpha} \), it is not a mean-square consistent estimate as seen from its variance expression (Theorem 3.2). By smoothing \( I_T(\lambda) \) via appropriate spectral windows we obtain in this section consistent estimates of \( f(\lambda) = \{ \phi(\lambda) \}^{p/\alpha} \) with \( p < \alpha/2 \). We set

\[ f_T(\lambda) = \int_{-\infty}^{\infty} W_T(\lambda - u) I_T(u) \, du \]  

(4.1)

where the spectral window is generated as follows: Let \( W(\lambda) \) be an even, nonnegative, continuous function vanishing for \( |\lambda| > i \), such that

\[ \int_{-i}^{i} W(\lambda) \, d\lambda = i. \]

The spectral window \( W_T(\lambda) \) is defined by

\[ W_T(\lambda) = M_T W(M_T \lambda) \]  

(4.2)
where \( M_T \) satisfies

\[
M_T \to \infty \quad \text{and} \quad M_T / T \to 0 \quad \text{as} \quad T \to \infty.
\] (4.3)

The bandwidth of the spectral window is then proportional to \( 1 / M_T \).

We first show that the smoothed modified periodogram \( f_T(\lambda) \) is an asymptotically unbiased estimator of \( f(\lambda) \), where \( 0 < p < \alpha \).

**Theorem 4.1.** Let \( 0 < p < \alpha \) and \( \text{Bias}[f_T(\lambda)] = Ef_T(\lambda) - f(\lambda) \).

(i) **Under Condition (\( \phi_2 \)),**

\[
\text{Bias}[f_T(\lambda)] = o(1)
\]

and the term \( o(\cdot) \) is uniform in \( \lambda \).

(ii) **Under Conditions (\( \phi_3 \) and (\( H' \)),**

\[
\text{Bias}[f_T(\lambda)] = O\left( \frac{1}{M_T^{p/\alpha}} \right)
\]

and the term \( O(\cdot) \) is uniform in \( \lambda \). If furthermore \( \phi(\lambda) \neq 0 \), then

\[
\text{Bias}[f_T(\lambda)] = \frac{1}{\{\phi(\lambda)\}^{1-p/\alpha}} O\left( \frac{1}{M_T} \right)
\]

and the term \( O(\cdot) \) is uniform in \( \lambda \).

(iii) **Under Conditions (\( \phi_4 \), (\( H'' \)) and \( \phi(\lambda) \neq 0 \),**

\[
\text{Bias}[f_T(\lambda)] = \frac{1}{\{\phi(\lambda)\}^{1-p/\alpha}} O\left( \frac{1}{M_T^2} \right)
\]

We remark that without dropping the assumption of positivity on the spectral window, the rate \( M_T^{-2} \) cannot be improved by assuming additional smoothness for the spectral density \( \phi \) (beyond twice differentiability).

**Proof.** From (4.1) and Theorem 3.1 we have

\[
Ef_T(\lambda) = \int_{-1}^{1} W(v) \left\{ \psi_T\left( \lambda - \frac{v}{M_T} \right) \right\}^{p-\alpha} dv
\]

and thus

\[
\text{Bias}[f_T(\lambda)] = \int_{-1}^{1} W(u) \left[ \left\{ \psi_T\left( \lambda - \frac{u}{M_T} \right) \right\}^{p-\alpha} - \{\phi(\lambda)\}^{p-\alpha} \right] du.
\]

and since \( p/\alpha < 1 \),

\[
||\text{Bias}[f_T(\lambda)]|| = \int_{-1}^{1} W(u) \left\| \psi_T\left( \lambda - \frac{u}{M_T} \right) - \phi(\lambda) \right\|^{p-\alpha} du \tag{4.4}
\]
Using (2.4) we obtain
\[
\psi_T \left( \lambda - \frac{u}{M_T} \right) - \phi(\lambda) = B_a^{-1} \int_{-\infty}^{\infty} |H(v)|^\alpha \left[ \phi \left( \lambda - \frac{u}{M_T} - \frac{v}{T} \right) - \phi(\lambda) \right] \, dv.
\] (4.5)

It follows that under (\phi 2),
\[
\psi_T \left( \lambda - \frac{u}{M_T} \right) - \phi(\lambda) \to 0 \quad \text{uniformly in } |u| \leq 1
\] (4.6)

establishing the result in (i). Under (\phi 3),
\[
\left| \phi \left( \lambda - \frac{u}{M_T} - \frac{v}{T} \right) - \phi(\lambda) \right| \leq (\sup |\phi'|) \left| \frac{u}{M_T} + \frac{v}{T} \right|
\]
and using (H'), we have from (4.5),
\[
\left| \psi_T \left( \lambda - \frac{u}{M_T} \right) - \phi(\lambda) \right| \leq \text{Const} \left( \frac{|u|}{M_T} + \frac{1}{T} \right) \leq \frac{\text{Const}}{M_T}
\] (4.7)

and the first part of (ii) follows from (4.4). Using the inequality (3.6) we find
\[
\left| \left\{ \psi_T \left( \lambda - \frac{u}{M_T} \right) \right\}^{p/\alpha} - \{\phi(\lambda)\}^{p/\alpha} \right| \leq \frac{p}{2\alpha} \psi_T \left( \lambda - \frac{u}{M_T} \right) - \phi(\lambda) \left[ \frac{1+O(1)}{[\phi(\lambda)]^{1-p/\alpha}} \right]
\]
\[
= \frac{1}{[\phi(\lambda)]^{1-p/\alpha}} O \left( \frac{1}{M_T} \right)
\]

by (4.7) where the O(·) is uniform in \lambda and |u| \leq 1. The second part of (ii) now follows from (4.4). Using \( y' - x' \sim r(y-x)/x^{1-r} \) as \( y \to x \neq 0, 0 < r < 1 \), we obtain
\[
\left\{ \psi_T \left( \lambda - \frac{u}{M_T} \right) \right\}^{p/\alpha} - \{\phi(\lambda)\}^{p/\alpha} \sim \frac{p/\alpha}{\{\phi(\lambda)\}^{1-p/\alpha}} \left[ \psi_T \left( \lambda - \frac{u}{M_T} \right) - \phi(\lambda) \right]
\]
uniformly in \( u \), when \( u \) belongs to a compact interval. It follows that
\[
\text{Bias}[\hat{f}_T(\lambda)] = \frac{F/\alpha}{\{\phi(\lambda)\}^{1-p/\alpha}} \int_{-1}^{1} W(u) \left[ \psi_T \left( \lambda - \frac{u}{M_T} \right) - \phi(\lambda) \right] \, du.
\] (4.8)

From (4.5) we have
\[
\psi_T \left( \lambda - \frac{u}{M_T} \right) - \phi(\lambda) = B_a^{-1} \int_{-\infty}^{\infty} |H(v)|^\alpha \left\{ \phi' \left( \lambda - \frac{u}{M_T} - \frac{v}{T} \right) \phi''(\lambda) \right\} \, dv
\]
\[
+ \frac{1}{2} \left( \frac{u}{M_T} + \frac{v}{T} \right)^2 \phi''(w) \right\} \, dv
\]
\[
= - \phi'(\lambda) \frac{u}{M_T} + O \left( \frac{1}{M_T} \right)
\] (4.9)
since $H$ is symmetric and the second term in (4.9) is bounded by
\[
\frac{1}{2} \sup |\phi''|B_{\alpha^{-1}} \left[ \frac{1}{M_T^2} + \frac{B_{\alpha^{-1}}}{T^2} \int_{-\infty}^{\infty} v^2 |H(v)|^\alpha \, dv \right] = O\left( \frac{1}{M_T^2} \right)
\]
uniformly in $\lambda$ and $|u| \leq 1$ under $(H'')$. By (4.8) and (4.10), since $W$ is symmetric, we obtain
\[
\text{Bias}[f_T(\lambda)] \approx \frac{1}{\{\phi(\lambda)\}^{1-p/\alpha}} O\left( \frac{1}{M_T^2} \right)
\]
which gives Part (iii). □

We now show that the variance of the smoothed modified periodogram tends to zero.

**Theorem 4.2.** Let $0 < p < \alpha/2$. If $\phi(\lambda) \neq 0$ and Conditions (φ2) and (H) are satisfied, then
\[
\text{Var}[f_T(\lambda)] \leq \text{Const} f^2(\lambda)(M_T/T)^r
\]
where
\[
r = \begin{cases} 
\{(\alpha\beta - 1) \quad \text{if } 1 < \alpha\beta \leq 2, \\
1 \quad \text{if } 2 < \alpha\beta.
\end{cases}
\]

**Remark 4.1.** When $\alpha\beta > 2$ the convergence rate $(M_T/T)^r$, $r < 1$, can be strengthened to $(M_T/T) \ln(T/M_T)$ as seen from the proof of the theorem.

**Remark 4.2.** When $\phi(\lambda) = 0$, the method of proof of Theorem 4.2 is not applicable and a crude bound on $\text{Var}[f_T(\lambda)]$ can be obtained from (4.14):
\[
\text{Var}[f_T(\lambda)] \leq \left\{ \int_{-1}^1 W(x) \, \text{Var}^{1/2} \left[ I_T\left( \lambda - \frac{x}{M_T} \right) \right] \, dx \right\}^2
\]
\[
= \text{Var}_{\alpha,p} \left\{ \int_{-1}^1 W(x) \left[ \psi_T\left( \lambda - \frac{x}{M_T} \right) \right] \frac{\varphi''(\lambda)}{\varphi(\lambda)^{1+p/\alpha}} \, dx \right\}^2
\]
by the Cauchy–Schwarz inequality and Theorem 3.2. Now using (4.6) and (4.7) in the bias analysis of Theorem 4.1 (where we can set $M_T = T$) we find that under Conditions (φ2) and (H) we have $\text{Var}[f_T(\lambda)] = o(1)$ and under Conditions (φ3) and (H'), $\text{Var}[f_T(\lambda)] = O(1/T^{2p/\alpha})$.

The proof of Theorem 4.2 makes use of the following proposition.
Proposition 4.1. If $\phi$ is continuous at $\lambda$, Conditions $(\phi 1)$ and $(H)$ are satisfied, and

$$G_T(v) = T^{1/2} \int_{-\infty}^{\infty} W_T(v-u)|H_T(u)|^{\alpha/2} \, du,$$

then

$$\frac{1}{T} \int_{-\infty}^{\infty} \phi(\lambda - v)G_T^2(v) \, dv \leq \text{Const} \lambda(M_T/T)^\nu$$

where $\nu = (\alpha - 1)_+$ for $1 < \alpha \leq 2$ and $\nu = 1$ for $\alpha > 2$.

Proof. Dropping $T$ in $M_T$ throughout, we have by (2.2) and (4.2),

$$\tilde{G}_T(v) \triangleq \frac{1}{M} G_T\left(\frac{v}{M}\right) = B_\alpha^{-1/2} \int_{-\infty}^{\infty} W\left(v - \frac{M}{T} u\right)|H(u)|^{\alpha/2} \, du \quad (4.11)$$

and

$$S_T(\lambda) \triangleq \frac{1}{T} \int_{-\infty}^{\infty} \phi(\lambda - v)G_T^2(v) \, dv = \frac{M}{T} \int_{-\infty}^{\infty} \phi\left(\lambda - \frac{u}{M}\right)\tilde{G}_T^2(u) \, du. \quad (4.12)$$

We first consider the case $\alpha > 2$, i.e. $(H')$ is satisfied. Since $|H(v)|^{\alpha/2} \in L_1$ and $W$ is bounded and continuous we have from (4.11) by dominated convergence

$$\tilde{G}_T(v) \to B_\alpha^{-1/2} W(v) \int |H|^{\alpha/2} \text{ as } T \to \infty.$$

Hence the integrand in (4.12) tends to $B_\alpha^{-1} \left[\int |H|^{\alpha/2}\right]^2 W^2(u)\phi(\lambda)$ and if we show that it is bounded by an integrable function, independent of $T$, then by dominated convergence we will have from (4.12),

$$\lim_{T \to \infty} \frac{T}{M} S_T(\lambda) = B_\alpha^{-1} \left\{ \int |H|^{\alpha/2}\right\}^2 \left\{ \int W^2 \right\} \phi(\lambda)$$

establishing the result. Since $\phi$ is bounded it suffices to show that $\tilde{G}_T^2(u)$ is bounded by an integrable function. It is clear from (4.11) that it is uniformly bounded in $T$ since for all $T$ and $u$,

$$\tilde{G}_T(u) \leq B_\alpha^{-1/2} (\sup W) \int |H(v)|^{\alpha/2} \, dv < \infty. \quad (4.13)$$

Next, using the support of $W$ we have from (4.11) and $(H)$

$$\tilde{G}_T(v) \leq B_\alpha^{-1/2} (\sup W) \frac{T}{M} \int_{-1}^{1} \left| H\left[ \frac{T}{M} (v-u) \right] \right|^{\alpha/2} \, du$$

$$\leq \text{Const} \frac{T}{M} \int_{-1}^{1} \frac{du}{\{1 + \left[ T(v-u)/M \right]^2\}^{\alpha\beta/4}}.$$
so that for large $|v|$, 
\[
\tilde{G}_T(v) \leq \frac{\text{Const} \, T/M}{\{1 + [T(\lfloor x \rfloor - 1)/M]^2\}^{\alpha \beta /4}} \leq \text{Const} \left(\frac{M}{T}\right)^{\alpha \beta /2 - 1} \leq \frac{1}{|v|^{\alpha \beta /2}} \leq \text{Const} \left|v\right|^{\alpha \beta /2}.
\] (4.14)

Thus by (4.13) and (4.14), $\tilde{G}_T^2(v) \leq \text{Const}(1 + v^2)^{-\alpha \beta /2}$ which is integrable since $\alpha \beta > 2$.

The case $1 < \alpha \beta \leq 2$ is handled similarly, except that instead of using (4.8) directly, we first apply Hölder’s inequality, with $q > 1$,

\[
\tilde{G}_T(v) = B_\alpha^{-1/2} \int_{(T/M)(v + 1)}^{(T/M)(v - 1)} W\left(v - \frac{M}{T} u\right) |H(u)|^{\alpha /2} \, du 
\leq B_\alpha^{-1/2} \left(2 \frac{T}{M}\right)^{1/q'} \left\{ \int_{-\infty}^{\infty} W^q\left(v - \frac{M}{T} u\right) |H(u)|^{\alpha q/2} \right\}^{1/q} 
= B_\alpha^{-1/2} \left(2 \frac{T}{M}\right)^{1/q'} \left\{ K_T(v)\right\}^{1/2}
\]

where

\[
K_T(v) = \left\{ \int_{-\infty}^{\infty} W^q\left(v - \frac{M}{T} u\right) |H(u)|^{\alpha q/2} \, du \right\}^{2/q}.
\] (4.15)

Then by (4.12),

\[
\left(\frac{T}{M}\right)^{2/q'} S_T(\lambda) \leq 2^2/q' B_\alpha^{-1} \int_{-\infty}^{\infty} \phi\left(\frac{\lambda - v}{M}\right) K_T(v) \, dv.
\] (4.16)

Since $1 < \alpha \beta \leq 2$, $|H(v)|^{\alpha q/2} \in L_1$ so that by dominated convergence and (4.15),

\[
K_T(v) \to W^2(v) \left\{ \int |H|^{\alpha q/2} \right\}^{2/q} \text{ as } T \to \infty.
\]

Also, $K_T(v)$ can be shown to be bounded for all $T$ by an integrable function, by an argument similar to that used for $\tilde{G}_T(v)$,

\[
K_T(v) \leq \text{Const} \left(\frac{M}{T}\right)^{\alpha \beta /2} \frac{1}{\left(1 + v^2\right)^{\alpha \beta /2}} \leq \text{Const} \frac{1}{\left(1 + v^2\right)^{\alpha \beta /2}}
\]

provided $\alpha \beta > 2/q$. Thus the integral on the right side of (4.16) tends to $\phi(\lambda)\left\{ \int W^2\right\} \left\{ \int |H|^{\alpha q/2} \right\}^{2/q}$ and

\[
\lim_{T \to \infty} \left(\frac{T}{M}\right)^{2/q'} S_T(\lambda) \leq 2^2/q' B_\alpha^{-1} \left\{ \int W^2\right\} \left\{ \int |H|^{\alpha q/2} \right\}^{2/q} \phi(\lambda)
\]

provided $q > 1$ and $\alpha \beta > 2/q$. These constraints are satisfied for all $1 < \alpha \beta < 2$ by choosing any $q > 2/(\alpha \beta)$. 
\[\square\]
Proof of Theorem 4.2. By (4.1) we have, putting $C(u, v) = \text{Cov}[I_T(u), I_T(v)]$,

$$\text{Var}[f_T(\lambda)] = \int_{-\infty}^{\infty} W_T(\lambda - u_1) W_T(\lambda - u_2) C(u_1, u_2) \, du_1 \, du_2$$

$$= \int_{-1}^{1} W(x_1) W(x_2) C\left(\lambda - \frac{x_1}{M_T}, \lambda - \frac{x_2}{M_T}\right) \, dx_1 \, dx_2$$

$$= \int_{|x_1 - x_2| < \varepsilon_T} + \int_{|x_1 - x_2| > \varepsilon_T} \triangleq J_1 + J_2$$

(4.17)

where $\varepsilon_T \to 0$ as $T \to \infty$. For the term $J_1$, using the Cauchy–Schwarz inequality and

$$\text{Var}\left[I_T\left(\lambda - \frac{x}{M_T}\right)\right] = V_{\alpha, \beta}\left[\psi_T\left(\lambda - \frac{x}{M_T}\right)\right]^{2p/\alpha} \to V_{\alpha, \beta}(\phi(\lambda))^{2p/\alpha}$$

uniformly in $|x| \leq 1$ (as in the proof of Theorem 4.1(i)). we obtain

$$|J_1| \leq \text{Const}\{\phi(\lambda)\}^{2p/\alpha} \int_{|x_1 - x_2| < \varepsilon_T} W(x_1) W(x_2) \, dx_1 \, dx_2 \leq \text{Const} f^2(\lambda) \varepsilon_T.$$  (4.18)

For the term $J_2$ we bound the covariance $C(\lambda - x_1/M_T, \lambda - x_2/M_T) \triangleq C(\lambda_1, \lambda_2)$ as in the proof of Theorem 3.3 by (3.13) with $a, b$ given in (3.11). We now use, just as in (3.14),

$$|a - b| \leq 2c_a |u_1 u_2|^{\alpha/2} \int_{-\infty}^{\infty} \left|H_T\left(\lambda - \frac{x_1}{M_T} - v\right)H_T\left(\lambda - \frac{x_2}{M_T} - v\right)\right|^{\alpha/2} \phi(v) \, dv$$

$$\triangleq 2c_a |u_1 u_2|^{\alpha/2} Q_T(\lambda, x_1, x_2)$$

(4.19)

to write

$$|J_2| \leq \text{Const} \int_{|x_1 - x_2| < \varepsilon_T} dx_1 \, dx_2 W(x_1) W(x_2) Q_T(\lambda, x_1, x_2)$$

$$\times \int_{\varepsilon_T} e^{-b - (a - b)} \frac{du_1 \, du_2}{|u_1 u_2|^{1 + \rho \alpha/2}}.$$

Now the inner double integral is shown to be bounded for large $T$ uniformly in $|x_k| \leq 1$, provided $T \varepsilon_T / M_T \to \infty$, and $\phi(\lambda) \neq 0$, just as in the proof of Theorem 3.3, using the observations that

(i) $\sup_{|x_1 - x_2| > \varepsilon_T} Q_T(\lambda, x_1, x_2) = o(1)$ as $T \to \infty$
which follows from (4.19) and Proposition 3.1 since

\[ Q_T(\lambda, x_1, x_2) \leq (\sup \phi) \Delta \left[ \frac{T}{M} (x_1 - x_2) \right] \leq \frac{\text{Const}}{1 + \left[ T(x_1 - x_2)/M \right]^{23/2}}. \]

(ii) \( \psi_T(\lambda - x_k / M_T) \to \phi(\lambda) \) uniformly in \( |x_k| \leq 1 \) (as in the proof of Theorem 4.1(i)). Thus

\[ |J_2| \leq \frac{\text{Const}}{\{ \phi(\lambda) \}^{1 - 2p/\alpha}} \int_1^1 dx_1 \, dx_2 \, W(x_1) W(x_2) Q_T(\lambda, x_1, x_2) \]

and as the double integral equals, by (4.19),

\[ \int_{-1}^1 dx, \, d \phi(v) \left\{ \int_{-1}^1 dx \, W(x) \left| H_T \left( \lambda - \frac{x}{M_T} - v \right) \right|^{\alpha/2} \right\}^2 = \frac{1}{T} \int_{-v}^v \phi(\lambda - v) G_T^2(v) \, dv \]

it follows by Proposition 4.1 that

\[ |J_3| \leq \text{Const} \{ \phi(\lambda) \}^{2p/\alpha} (M_T / T)^v = \text{Const} f^2(\lambda)(M_T / T)^v. \quad (4.20) \]

From (4.17), (4.18) and (4.20) we have

\[ \text{Var}[f_T(\lambda)] \leq \text{Const} f^2(\lambda) (\epsilon_T + (M_T / T)^v). \]

When \( \alpha \beta > 2 \) the rate is \( \epsilon_T + M_T / T \) and as \( T \epsilon_T / M_T \to \infty \) we can choose \( \epsilon_T = (M_T / T) \ln(T/M_T) \) so that

\[ \text{Var}[f_T(\lambda)] \leq \text{Const} f^2(\lambda)(M_T / T) \ln(T/M_T). \]

When \( 1 < \alpha \beta \leq 2 \) the rate is \( \epsilon_T + (M_T / T)^v \) for every \( v < \alpha \beta - 1 \) so we can choose \( \epsilon_T = (M_T / T)^v \). \( \square \)

The mean square consistency of \( f_T(\lambda) \) as an estimator for \( f(\lambda) \) follows from Theorems 4.1 and 4.2, via the relationship

\[ \text{MSE}[f_T(\lambda)] = E \{ f_T(\lambda) - f(\lambda) \}^2 = \text{Bias}^2[f_T(\lambda)] + \text{Var}[f_T(\lambda)]. \]

We state below the mean square convergence rate of \( f_T(\lambda) \) when an optimal choice of the bandwidth parameter \( M_T \) is selected.

**Theorem 4.3.** Let \( 0 < p < \alpha/2 \). Under Conditions (\phi 2) and (II), and any \( M_T \) satisfying (4.3),

\[ \text{MSE}[f_T(\lambda)] = O(1). \]
Under (\( \phi 3 \)) and (\( H' \)),

\[
\text{MSE}[f_T(\lambda)] = \begin{cases} 
\mathcal{O}\left(\frac{\ln T}{T^{2/3}}\right) & \text{if } \phi(\lambda) \neq 0 \text{ and } M_T = T^{1/3}, \\
\mathcal{O}\left(\frac{1}{T^{2p/\alpha}}\right) & \text{if } \phi(\lambda) \neq 0 \text{ and } M_T = T. 
\end{cases}
\]

Under (\( \phi 4 \)) and (\( H'' \)),

\[
\text{MSE}[f_T(\lambda)] = \mathcal{O}\left(\frac{\ln T}{T^{4/5}}\right) \text{ if } \phi(\lambda) \neq 0 \text{ and } M_T = T^{1/5}.
\]

These results follow immediately from Theorems 4.1, 4.2 and Remarks 4.1, 4.2 (in case \( \phi \) vanishes at \( \lambda \), the choice \( M_T = T \) is legitimate as can be seen from Theorem 4.1 and Remark 4.2).

5. Consistent estimate of the spectral density \( \phi(\lambda) \)

We now show that

\[
\phi_T(\lambda) = \left\{ f_T(\lambda) \right\}^{p/\alpha}
\]

is a consistent estimator of the spectral density \( \phi(\lambda) \) when \( 0 < p < \alpha/2 \), \( f_T(\lambda) \) is defined in (4.1), and the bandwidth parameter \( M_T \) is chosen as in Theorem 4.3.

**Theorem 5.1.** Let \( 0 < p < \alpha/2 \). Then

\[
a_T\left\{ \phi_T(\lambda) - \phi(\lambda) \right\} \to 0 \quad \text{in probability}
\]

under the conditions and with the values of \( a_T \) specified in Table 1.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>( \phi(\lambda) \neq 0 )</th>
<th>( \phi(\lambda) = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under (( \phi 2 )) and (( H ))</td>
<td>( T^{1/3}/\ln T )</td>
<td>( T^n )</td>
</tr>
<tr>
<td>Under (( \phi 3 )) and (( H' ))</td>
<td>( T^{1/3}/\ln T )</td>
<td>( T^n )</td>
</tr>
<tr>
<td>Under (( \phi 4 )) and (( H'' ))</td>
<td>( T^{2/p/\alpha}/\ln T )</td>
<td>( T^n )</td>
</tr>
</tbody>
</table>

Note that the smoother \( \phi(\lambda) \) is the faster the rate of convergence in probability; however, when no data tapering is used (Condition (\( H \)) with \( \beta = 1 \)) no convergence rate is available yet. This is in contrast to the standard spectral estimation of fourth order processes where specific convergence rates are available even when no tapering is used: For example, when \( \phi(\lambda) > 0 \) is twice differentiable (\( \phi 4 \)), the bias is \( \mathcal{O}(1/M_T^2) \) and the variance is \( \mathcal{O}(M_T/T) \); so that the mean square error is \( \mathcal{O}(1/T^{4/5}) \) which
gives convergence in probability rate of \( a_T = T^{2/5}/\ln T \). While this rate is identical to the one given in Theorem 5.1 for stationary stable processes under Condition \((\phi 4)\), we had to introduce tapering (Condition \((H'')\)) to achieve such a rate.

**Proof.** Using the inequality

\[
y^a - x^a = q \int_x^y u^{q-1} du \leq \frac{q}{2} |y - x|(x^{q-1} + y^{q-1})
\]  

with \( q = \alpha/p > 2 \) and \( x, y \geq 0 \) we obtain from (5.1), omitting \( \lambda \),

\[
|\phi_T - \phi| \leq \frac{q}{2} |f_T - f|(f_T^{q-1} + f^{q-1}).
\]

(5.3)

By Theorem 4.3, \( f_T^{q-1} + f^{q-1} \to 2f^{q-1} \) in probability as \( T \to \infty \). Also, \( E(f_T - f)^2 \leq \text{Const}/\theta(T) \), for \( \theta(T) \) given in Theorem 4.3, so that for every \( \varepsilon > 0 \),

\[
P( a_T | f_T - f| > \varepsilon ) \leq \frac{a_T^2 E(f_T - f)^2}{\varepsilon^2} \leq \text{Const} \frac{a_T^2}{\varepsilon^2 \theta(T)}
\]

which tends to zero as \( T \to \infty \) provided \( a_T^2/\theta(T) \to 0 \). With the choice of \( a(T) \) indicated in the statement of the theorem, the result follows from (5.3). \( \square \)

We finally remark that by speeding up the stretch of the sample function used, i.e. by considering subsequences of \( \phi_T(\lambda) \), we can obtain strongly consistent estimators of \( \phi(\lambda) \). In Theorem 5.2 we give the rate of a.s. convergence for the simplest estimator of this type (where the bandwidth parameter \( M_T \) is as in Theorem 4.3).

**Theorem 5.2.** Assume that the taper \( h(t) \) is continuous on \([-1, 1]\), \( 0 < p < \alpha/2 \), and let \( \tilde{T} = T^{\gamma} \) with \( \gamma > 1 \). Then

\[
\tilde{T}^\delta \ln \tilde{T} |\phi_T(\lambda) - \phi(\lambda)| \to 0 \text{ with probability one}
\]

under the conditions and the values of \( \delta \) specified in Table 2. the permissible values of the speed up parameter \( \gamma \) in each case are those for which the indicated values of \( \delta \) are positive.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>( \phi(\lambda) \neq 0 )</th>
<th>( \phi(\lambda) = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under ((\phi 3)) and ((H'))</td>
<td>( \begin{pmatrix} 1 &amp; 1 \ \frac{\gamma}{3} &amp; 2\gamma \end{pmatrix} )</td>
<td>( \begin{pmatrix} p &amp; 1 \ \frac{\alpha}{\gamma} &amp; 2\gamma \end{pmatrix} )</td>
</tr>
<tr>
<td>Under ((\phi 4)) and ((H'))</td>
<td>( \begin{pmatrix} 2 &amp; 1 \ \frac{\gamma}{5} &amp; 2\gamma \end{pmatrix} )</td>
<td>( \begin{pmatrix} p &amp; 1 \ \frac{\alpha}{\gamma} &amp; 2\gamma \end{pmatrix} )</td>
</tr>
</tbody>
</table>
Proof. It follows from inequality (5.3) that \( \tilde{T} \delta |f - f| \to 0 \) a.s. implies \( \tilde{T} \delta |\phi - \phi| \to 0 \) a.s.; it suffices therefore to establish the former. It is seen from (2.1), (3.1), (4.1) and the assumptions of the theorem that for fixed \( \lambda \), \( f(\lambda) \) is a.s. continuous in \( T \). Thus the process \( Y_\tau \), \( 0 \leq \tau \leq 1 \), defined for fixed \( \lambda \) by

\[
Y_\tau = f_{1/\tau}(\lambda), \quad 0 < \tau \leq 1, \quad Y_0 = f(\lambda),
\]

is a.s. continuous on \((0, 1]\), hence separable. Using \( E(f_T - f)^2 \leq \operatorname{Const}/\theta(T) \), with \( \theta(T) = T^r/\ln T \) (cf. Theorem 4.3), we have for \( \tau \in (0, 1] \) and \( \epsilon(\tau) > 0 \),

\[
P(|Y_\tau - Y_0| \geq \epsilon(\tau)) - P(|f_{1/\tau} - f| \geq \epsilon(\tau)) \leq \frac{\operatorname{Const}}{\epsilon^2(\tau) \theta(1/\tau^r)} \eta(\tau).
\]

If \( \int_0^\infty \epsilon(\tau) \tau^{-1} d\tau < \infty \) and \( \int_0^\infty \eta(\tau) \tau^{-2} d\tau < \infty \), it then follows by Kolmogorov’s theorem (see [14], p. 97) that \( Y \) has a.s. continuous paths on \([0, 1]\) and that in fact

\[
\sup_{0 \leq s \leq \tau} |Y_s - Y_0| \leq \left( \frac{4}{\ln 2} \right) \int_0^\tau \frac{\epsilon(u)}{u} \, du, \quad 0 < \tau < Z,
\]

where \( Z \) is a strictly positive random variable. Taking

\[
\epsilon(\tau) = \tau^b \{ b \ln(1/\tau) + 2 \} / \ln^3(1/\tau), \quad b > 0,
\]

we have

\[
\int_{0^+} \eta(\tau) \tau^{-2} \, d\tau \leq \operatorname{Const} \int_{0^+} \frac{\{ \ln(1/\tau) \}^{4+m}}{\tau^{2(1+b) - \gamma}} \, d\tau < \infty
\]

for all \( m = 0, 1, \ldots \) provided \( 0 < b < (\gamma r - 1)/2 \). Then \( |Y_\tau - Y_0| \leq \operatorname{Const} \tau^b / \ln^2(1/\tau) \) for \( 0 < \tau < Z \) and \( \tau^\gamma \ln(1/\tau) |Y_\tau - Y_0| \to 0 \) a.s. as \( \tau \to 0 \). Thus \( T^b \ln T |f_T - f| \to 0 \) a.s. as \( T \to \infty \) and the result follows with \( \delta = b/\gamma < (\gamma r - 1)/2\gamma \), i.e. \( \delta = [(\gamma r - 1)/2] \). □

6. Real processes and one-sided observations

The case where the process \( X \) is real requires a few modifications. \( X \) has the spectral representation

\[
X(t) = \text{Re} \int_{-\infty}^{\infty} e^{i\lambda t} \, d\xi(\lambda)
\]

(where as before \( \xi \) has complex \( \text{SdS} \) independent increments) finite dimensional characteristic functions

\[
E \exp \left\{ i \sum_{n=1}^N r_n X(t_n) \right\} = \exp \left\{ -\kappa \int_{-\infty}^{\infty} \left| \sum_{n=1}^N r_n e^{i\lambda t} \right|^\alpha \phi(\lambda) \, d\lambda \right\}, \quad (6.1)
\]
Covariation \( (X(t), X(s)) = \alpha c_\alpha \Re \int_{-\infty}^{\infty} e^{i(t-s)\lambda} \phi(\lambda) \, d\lambda \) (6.2)

when \( 1 < \alpha < 2 \) (see Theorem 7.2(v)). The unpleasant constant \( \alpha c_\alpha \) appears only because we chose in (6.1) the same scaling for \( \phi \) as in the complex case (1.5), and would readily disappear had we rescaled \( \phi \) by choosing \( c = \alpha^{-1} \) in (6.1). It is clear from (6.1) that \( \phi \) is an even function (its odd part, if any, does not contribute to the integral in (6.1)). In this case we use the finite tapered cosine transform

\[
d_T(\lambda) = Q(\lambda) A_T \int_{-\infty}^{\infty} h(t/T) X(t) \cos(\lambda t) \, dt
\]

where \( Q(\lambda) = 2^{1-1/\alpha} \) for \( \lambda \neq 0 \) and = 1 for \( \lambda = 0 \). By a calculation similar to that leading to (2.3) we find

\[
d_{T}(\lambda) = \frac{1}{2} Q(\lambda) \int_{-\infty}^{\infty} \{ H_T(\lambda - u) + H_T(-\lambda - u) \} \, d\xi_1(u),
\]

and by Propositions 2.1 and 2.2 (see statement following Proposition 2.2) we have for \( \lambda \neq 0 \),

\[
E \exp[i L d_{T}(\lambda)] = \exp\left\{ -c_\alpha \left[ \frac{r}{2} Q(\lambda)^{\alpha} \left| H_T(\lambda - u) + H_T(-\lambda - u) \right| \right. \right.
\]

\[
\left. \left. - \exp\left( -c_\alpha \left[ \frac{r}{2} Q(\lambda)^{\alpha} \right] \left[ \phi(\lambda) + \phi(-\lambda) \right] \right) \right) \right\}
\]

\[
= \exp(-c_\alpha |\lambda|^\alpha \phi(\lambda))
\]

with convergence rates provided there. The same result for \( \lambda = 0 \) follows from Proposition 2.1. All subsequent results remain valid with either identical rates of convergence under somewhat modified conditions, or sometimes slower rates of convergence under the same conditions. Because of the modified form (6.3) of the characteristic function of \( d_{T}(\lambda) \) (instead of (2.6)), Proposition 2.2 is used along with Proposition 2.1 resulting in changes in Theorems 3.1, 3.2 and 4.1 whereas Theorems 3.3 and 4.2 remain unchanged. The resulting convergence rate in Theorem 4.3 under Conditions (\( \phi, 3 \)), (\( H_T \)) and \( \phi(\lambda) \neq 0 \), becomes with \( M_T = T^{1-3} \),

\[
\operatorname{MSE}[f_{T}(\lambda)] =
\begin{cases}
O\left( \frac{\ln T}{T^{2-\alpha}} \right) & \text{if } \lambda = 0 \text{ and if } \lambda \neq 0, \, 0 < \alpha \leq 1.5, \\
O\left( \frac{\ln T}{T^{2-\alpha} + \frac{1}{T^{2(\beta_1 - 1)}}} \right) & \text{if } \lambda \neq 0, \, 1.5 < \alpha < 2.
\end{cases}
\]

In the second case, when \( \beta = \frac{4}{3} \) (rather than merely \( \beta \geq 2/\alpha \)). In \( T/T^{2-3} \) becomes the dominant term and thus the rate of convergence remains equal to that of the
complex case. If in addition Condition $(\phi 1)$ is satisfied, then the term $T^{-2(\beta - 1)}$ in the second case becomes $T^{-2(\beta - 1)/\alpha}$ and $\ln T / T^{-2/3}$ becomes the dominant term, so that with this additional condition the mean square convergence rate for real processes is identical to that for complex processes. As a consequence the weak and strong convergence rates for $q_T(\lambda)$ remain the same as in the complex case under the additional Condition $(\phi 1)$, whereas without $(\phi 1)$ the rates change only when $1.5 < \alpha < 2$ and $2/\alpha < \beta < 3/4$.

In practice it is more common to have observations of $X$ over the one-sided interval $[0, T]$, in which case the estimate $q_T(\lambda)$ is modified as follows. When the process $X$ is complex we form

$$d_T(\lambda) = A_T \text{Re} \int_0^T e^{-iu} h(t/T) X(t) \, dt = \text{Re} \int_{-\infty}^\infty K_T(\lambda - u) \, d\xi(u)$$

and when $X$ is real

$$d_T(\lambda) = Q(\lambda) A_T \int_0^T h(t/T) X(t) \cos(\lambda t) \, dt$$

$$= \frac{1}{2} Q(\lambda) \Re \int_{-\infty}^\infty \{K_T(\lambda - u) + K_T(-\lambda - u)\} \, d\xi(u)$$

where

$$K(\lambda) = \int_0^1 h(t) e^{-\lambda t} \, dt$$

and $K_T$ is defined from $K$ just as $H_T$ from $H$ in (2.2). While $H_T$ is real and even, $K_T$ is complex and $|K_T|$ is even. The analysis goes through similarly in both cases, and the resulting rates are likewise sometimes different.

7. Appendix

A. Two useful inequalities

The following inequalities proved useful in earlier sections.

**Theorem 7.1.** (a) For all real $x$, $y$ and $0 < \alpha \leq 2$,

$$| |x + y|^\alpha - |x|^\alpha - |y|^\alpha| \leq 2|xy|^ {\alpha - 1}$$

and thus

$$| |x + y|^\alpha - |x|^\alpha| \leq |y|^\alpha + 2|xy|^ {\alpha - 1/2}.$$

(b) When $1 \leq \alpha \leq 2$, for all $x$, $y \geq 0$,

$$(x + y)^\alpha \leq x^\alpha + y^\alpha + \alpha x^{\alpha - 1} y$$
and thus for all real x, y,

\[ | |x + y|^\alpha - |x|^\alpha| \leq |y|^\alpha + \alpha |x|^{\alpha-1} |y|. \]

**Proof.** (a) It suffices to show

\[ | |1 + t|^\alpha - 1 - |t|^\alpha| \leq 2|t|^{\alpha/2} \]

and as this inequality remains unchanged by replacing t by \( t^{-1} \), it suffices to show it for \( |t| \leq 1 \). For \( 0 \leq t \leq 1 \), it is written, when \( 1 \leq \alpha \leq 2 \) as \( (1 + t)^\alpha - 1 - t^\alpha \leq 2t^{\alpha/2} \) or

\( (1 + t)^\alpha \leq (1 + t^{\alpha/2})^2 \)

and follows from \( (1 + t)^{\alpha/2} \leq 1 + t \leq 1 + t^{\alpha/2} \); and when \( 0 < \alpha \leq 1 \) as \( 1 + t^\alpha - (1 + t)^\alpha \leq 2t^{\alpha/2} \) or \( (1 - t^\alpha)^2 \leq (1 + t)^\alpha \) which is clearly valid. For \( -1 < t < 0 \), putting \( s = -t \in [0, 1] \) the inequality is written as \( 1 + s^\alpha - (1 - s)^\alpha \leq 2s^{\alpha/2} \) or

\( (1 - s^{\alpha/2})^2 \leq (1 - s)^\alpha \)

which follows from \( 1 - s^{\alpha/2} \leq 1 - s \leq (1 - s)^{\alpha/2} \).

(b) It is easily seen that it suffices to show

\[ (1 + t)^\alpha \leq 1 + t^\alpha + \alpha t \quad \text{for } 0 \leq t \leq 1. \]

For \( 0 \leq t \leq 1 \) we have the Taylor series

\[ (1 + t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha - 1)}{2} t^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} t^3 + \ldots \]

Since \( 1 \leq \alpha \leq 2 \) the terms beyond the third one have alternating signs and decreasing magnitude so that

\[ (1 + t)^\alpha \leq 1 + \alpha t + \frac{\alpha(\alpha - 1)}{2} t^2 \leq 1 + \alpha t + t^\alpha. \]

**B. Certain properties of the \( \text{SaS} \) process \( \xi \)**

Here we collect certain properties of the complex \( \text{SaS} \) independent increments process \( \xi = \xi_1 + i\xi_2 \) appearing in the spectral representation (1.1) of \( X \). \( X \) defined by (1.1) is strictly stationary (as was assumed) if and only if \( \xi \) has isotropic or rotationally invariant increments, i.e. the distribution of the process of increments \( e^{i\omega} \xi(e) \) for \( -\infty < \lambda < \infty \), does not depend on the rotation \( \theta \) \([3, 10]\). Thus, throughout this section \( \xi \) is a complex \( \text{SaS} \) process with independent and isotropic increments. In the Gaussian case \( \alpha = 2 \) the real and imaginary parts \( \xi_1 \) and \( \xi_2 \) are independent (uncorrelated in the more general second order case) satisfying \( E(d\xi_1(\lambda))d\lambda = \frac{1}{2} \phi(\lambda) d\lambda \) and (1.2). When \( 0 < \alpha < 2 \) we have the following properties.

**Theorem 7.2.** (i) When \( 1 < \alpha < 2 \), \( \xi_1(A) \), and \( \xi_2(B) \) have zero covariation for all Borel sets \( A \) and \( B \).

(ii) For disjoint Borel sets \( A \) and \( B \), \( \xi_1(A) \) and \( \xi_2(B) \) are independent.

(iii) For every Borel set \( A \),

\[ (\xi_1(A), \xi_2(A)) \leq \sqrt{2} \left( c_\alpha \int_A \phi \right)^{1/\alpha} R^{1/2}(Z_1, Z_2) \]
where \( = \) indicates equality in distribution, \( R \) is positive \( \alpha/2 \) stable with \( E \exp(-uR) = \exp(-u^{-\alpha/2}) \), \( u > 0 \), \( Z_1 \) and \( Z_2 \) are standard normal and \( R, Z_1, Z_2 \) are independent.

(iv) For \( 0 < p < \alpha \) and \( k = 1, 2 \), we have

\[
\{E|d\xi_k(\lambda)|^p\}^{\alpha/p} = c_\alpha \left\{ \frac{2^{p+1}}{\alpha \sqrt{\pi}} \frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma(-p/2)} \right\}^{\alpha/p} \phi(\lambda) \, d\lambda,
\]

\[
\{E|d\xi(\lambda)|^p\}^{\alpha/p} = c_\alpha \left\{ \frac{2^{p}}{\alpha} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma(-p/2)} \right\}^{\alpha/p} \phi(\lambda) \, d\lambda.
\]

(v) (6.2) is satisfied.

**Proof.** In accordance with (1.5), the characteristic function of the integral \( \int f \, d\xi \) with \( f = f_1 + if_2 \in L_\alpha(\phi) \) is given by

\[
E \exp\left\{ i \text{Re} \left[ \int f \, d\xi \right] \right\} = E \exp\left\{ i \left[ \int f_1 \, d\xi_1 - \int f_2 \, d\xi_2 \right] \right\}
\]

\[
= \exp\left\{ -c_\alpha \int |f|^{\alpha/2} \phi \right\} \quad (7.1)
\]

(see [3, 10]). We thus have

\[
E \exp[i(s\xi_1(A) + t\xi_2(B))] = \exp\left\{ -c_\alpha \int_{-\infty}^{\infty} (s^2 1_A + t^2 1_B)^{\alpha/2} \phi \right\}.
\]

(i) We have from (7.2) and [5], (with the adjusted definition),

\[
\text{Covariation}(\xi_1(A), \xi_2(B))
\]

\[
= -\frac{d}{dr} \ln E \exp[i[r\xi_1(A) + \xi_2(B)]]|_{r=0}
\]

\[
= c_\alpha \frac{d}{dr} \left\{ |r|^{\alpha} \int_{A \cap B} \phi + (r^2 + 1)^{\alpha/2} \int_{A \cap B} \phi + \int_{B \cap A} \phi \right\}|_{r=0}
\]

\[
= \alpha c_\alpha \left\{ |r|^{\alpha-1} \text{sgn}(r) \int_{A \cap B} \phi + r(r^2 + 1)^{1-\alpha/2} \int_{A \cap B} \phi \right\}|_{r=0}
\]

\[
= 0.
\]

(ii) When \( A \cap B \neq \emptyset \), (7.1) is written as

\[
E \exp[i(s\xi_1(A) + t\xi_2(B))] = \exp\left\{ -c_\alpha \left[ |s|^{\alpha} \int_A \phi + |t|^{\alpha} \int_B \phi \right] \right\}
\]

and thus \( \xi_1(A), \xi_2(B) \) are independent.
(iii) Since for all $s$ and $t$ we have
\[ E \exp \left\{ \frac{i}{\sqrt{2}} \left( c_\alpha \int_A \phi \right)^{1/\alpha} R^{1/2}(sZ_1 + tZ_2) \right\} \]
\[ = E \exp \left\{ - \left( c_\alpha \int_A \phi \right)^{2/\alpha} R(s^2 + t^2) \right\} = \exp \left\{ - c_\alpha \int_A \phi (s^2 + t^2)^{\alpha/2} \right\} \]
\[ = E \exp \{ i[s\xi_1(A) + t\xi_2(A)] \} \]
the result follows.

(iv) is established using (iii). For instance,
\[ E' \xi(A)^p = E[\xi_1^2(A) + \xi_2^2(A)]^{p/2} \]
\[ = 2^{p/2} \left( c_\alpha \int_A \phi \right)^{p/\alpha} E(R^{p/2}) E(Z_1^2 + Z_2^2)^{p/2} \]
\[ = 2^{p/2} \left( c_\alpha \int_A \phi \right)^{p/\alpha} \frac{2\Gamma(-p/\alpha) \Gamma(p/2)}{\alpha \Gamma(-p/2) 2^{1-p/2}} \]
\[ = \frac{2^p \Gamma(p/2) \Gamma(-p/\alpha)}{\alpha \Gamma(-p/2)} \left( c_\alpha \int_A \phi \right)^{p/\alpha} . \]

(v) We have from (7.2) and [5],
\[ \text{Covariance}(X(t), X(s)) \]
\[ = - \frac{d}{dr} \ln E \exp \{ i[rX(t) + X(s)] \}_{r=0} \]
\[ = c_\alpha \frac{d}{dr} \int_{-\infty}^{\infty} |r e^{i\lambda} + e^{i\lambda} |^\alpha \phi(\lambda) \ d\lambda |_{r=0} \]
\[ = c_\alpha \frac{d}{dr} \int_{-\infty}^{\infty} \left[ r^2 + 1 + 2r \cos(t - s) \lambda \right]^{\alpha/2} \phi(\lambda) \ d\lambda |_{r=0} \]
\[ = \alpha c_\alpha \int_{-\infty}^{\infty} \cos((t - s) \lambda) \phi(\lambda) \ d\lambda \quad \Box \]

References