STRUCTURAL STABILITY OF ASYMPTOTIC LINES ON SURFACES IMMERSED IN $\mathbb{R}^3$

BY

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ABSTRACT. – In this paper are studied immersions of surfaces into to $\mathbb{R}^3$ whose nets of asymptotic lines are topologically undisturbed under small perturbations of the immersion. These immersions are called structurally asymptotic stable. Sufficient conditions to belong to this class are established here. These conditions focus on the stable patterns around parabolic points, parabolic separatrix connections, periodic asymptotic lines (including those that intercept the parabolic lines) as well the exclusion of recurrent asymptotic lines. The class of immersions that are structurally stable in this sense is open in the $C^5$-topology.

Keywords: Structural stability, asymptotic foliations

RÉSUMÉ. – Dans ce travail sont étudiées les plongements des surfaces dans l’espace $\mathbb{R}^3$ pour lesquelles ces réseaux des lignes asymptotiques sont préservées topologiquement pour les pétites déformations du plongement. Ces plongements sont appelés asymptotique structurellement stables. Ces conditions focalisent sur le comportement des lignes asymptotiques dans une voisinage des lignes paraboliques, sur l’absence des connections des séparatrices paraboliques, sur les lignes asymptotiques fermées et aussi sur l’absence des récurrences non triviales des lignes asymptotiques. La classe des plongements structurellement stable est ouverte dans la topologie $C^5$. © Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

Consider a $C^r$, $r \geq 5$, immersion $\alpha$ of a smooth, compact and oriented, two-dimensional manifold $\mathbb{M}$ into Euclidean space $\mathbb{R}^3$.

The Fundamental Forms of $\alpha$ at a point $p$ of $\mathbb{M}$ are the symmetric bilinear forms on $T_p\mathbb{M}$ defined as follows [26,25]:

The First Fundamental Form:

$$I_\alpha(p; u, v) = \langle D\alpha(p); u, D\alpha(p); v \rangle.$$

The Second Fundamental Form:

$$II_\alpha(p; u, v) = -\langle DN_\alpha(p); u, D\alpha(p); v \rangle.$$

Here, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^3$ and $N_\alpha$ is the positive normal of the immersion:

$$N_\alpha = \frac{\alpha_u \wedge \alpha_v}{|\alpha_u \wedge \alpha_v|},$$

where $(u, v)$ is a positive chart on $\mathbb{M}$ and $\wedge$ is the vector (wedge) product associated to a once for all fixed orientation on $\mathbb{R}^3$, $\alpha_u = \frac{\partial}{\partial u}$ and $\alpha_v = \frac{\partial}{\partial v}$.

A line $\ell = v$, tangent at a point $p$ of $\mathbb{M}$ (i.e., $v \in T_p\mathbb{M}\setminus 0$), along which the normal curvature $k_n(p; \ell)$ vanishes, is called an asymptotic direction of $\alpha$ at $p$.

A maximal, regular curve $c: (a, b) \to \mathbb{M}$, parametrized by arc length $s$, whose tangent line is an asymptotic direction is called an asymptotic line of $\alpha$. That is, for every $s$ in $(a, b)$, it holds that $II_\alpha(c(s); c'(s), c'(s)) = 0$.

Through every point $p$ of the hyperbolic region $\mathbb{H}_\alpha$ of the immersion $\alpha$, characterized by the condition that the Gaussian Curvature $K_\alpha = \det(DN_\alpha)$ is negative, pass two transverse asymptotic lines of $\alpha$, tangent to the two asymptotic directions through $p$. This follows from the usual existence and uniqueness theorems on Ordinary Differential Equations. In fact, on $\mathbb{H}_\alpha$ the local line fields are defined by the kernels $\mathcal{L}_{\alpha,1}$, $\mathcal{L}_{\alpha,2}$ of the smooth one-forms $\omega_{\alpha,1}$, $\omega_{\alpha,2}$ which locally split $II_\alpha = \omega_{\alpha,1} \otimes \omega_{\alpha,2}$. 
The forms $\omega_{i,j}$ are locally defined up to a non-vanishing factor and a permutation of their indices. Therefore, their kernels and integral foliations are locally well defined only up to a permutation of their indices.

Under the orientability hypothesis imposed on $\mathbb{M}$, it is possible to globalize, to the whole $\mathbb{H}_a$, the definition of the line fields $L_{a,1}$, $L_{a,2}$ and of the choice of an ordering between them, as follows:

Consider the field $C$ of tangent cones on $\mathbb{H}_a$, defined by the non-negative part of the second fundamental form, i.e., $I_a(p; v, v) = 1$; $II_a(p; v, v) \geq 0$, oriented compatibly with $\mathbb{M}$. Call $\{e_1(p), e_2(p)\}$ a positive basis for $T_p\mathbb{M}$ consisting of unit asymptotic vectors, positive also for $C(p)$.

This choice of a basis can also be defined as follows:

$$D\alpha(p, e_1(p)) \wedge D\alpha(p, e_2(p)) = N_a(p) \text{ and } II_a(p; v, v) > 0, \text{ for } v = e_1(p) + e_2(p).$$

There is only one other different choice, $\{e'(p), e'_2(p)\}$, for such a basis; both choices define the same asymptotic line fields of $\alpha$:

$$L_{a,1}(p) = \mathbb{R}.e_1(p) = \mathbb{R}.e'_1(p) \text{ and } L_{a,2}(p) = \mathbb{R}.e_2(p) = \mathbb{R}.e'_2(p).$$

These two line fields, called the asymptotic line fields of $\alpha$, are smooth on $\mathbb{H}_a$; they are distinctly defined together with the ordering between them given by the subindices $\{1, 2\}$ which define their orientation ordering: “1” for the first asymptotic line field $L_{a,1}$, “2” for the second asymptotic line field $L_{a,2}$. They will be presented as an ordered pair $L_a = \{L_{a,1}, L_{a,2}\}$.

The asymptotic foliations of $\alpha$ are the integral foliations $A_{a,1}$ of $L_{a,1}$ and $A_{a,2}$ of $L_{a,2}$; they fill out the hyperbolic region $\mathbb{H}_a$. The ordered asymptotic net of the immersion $\alpha$ is the ordered pair $A_a = \{A_{a,1}, A_{a,2}\}$, the index $i = \{1, 2\}$ will be called the orientation ordering of the asymptotic foliation.

Clearly, an exchange in the orientations either of $\mathbb{M}$ or of $\mathbb{R}^3$ produces an inversion in the orientation ordering of the asymptotic line fields.

When non-empty, the region $\mathbb{H}_a$ is bounded by the set (generically, i.e., for most $\alpha$’s, a regular curve [11,20,6,8]) $\mathbb{P}_a$ of parabolic points of $\alpha$, on which $K_a$ vanishes. On $\mathbb{P}_a$, the pair of asymptotic directions degenerate into a single one or into the whole tangent plane at points where $II_a = 0$, called flat umbilic points.

The parabolic points will be regarded here as the singularities of the asymptotic net. In fact, in the context of Singularity Theory, $\mathbb{P}_a$ is the
singular set of the Normal Map $N_\alpha$ from $\mathbb{M}$ to the unit sphere $\mathbb{S}^2$. On the Elliptic Region $\mathbb{E}_\alpha$, defined by $\mathcal{K}_\alpha > 0$, the asymptotic directions are imaginary and will not be studied here. Thus the domain for real asymptotic directions and their integral curves in the present work will be the set $\{\mathcal{K}_\alpha \leq 0\}$ of non-elliptic points, which generically is either the empty set or a manifold with boundary coincident with $\text{Clos}(\mathbb{H}_\alpha)$.

An immersion $\alpha$ is said to be $C^s$-local asymptotic structurally stable at a compact set $S$ in $\text{Clos}(\mathbb{H}_\alpha)$ if for any sequence $\alpha_n$ converging to $\alpha$ together with its first $s$ derivatives in a compact neighborhood $V_S$ of $S$ there is a sequence of compact subsets $S_n$ and a sequence of homeomorphisms $h_n$ mapping $S$ to $S_n$, converging to the identity of $\mathbb{M}$, such that on $V_S$ it maps arcs of the asymptotic foliations $A_{\alpha,n,i}$ to arcs of that of $A_{\alpha',i}$ for $i = 1, 2$.

An immersion $\alpha$ is said $C^s$-global structurally asymptotic stable if the compact set $S$ above is the closure of the the hyperbolic region $\mathbb{H}_\alpha$.

This implies that the parabolic set must be preserved by the homeomorphism defining the topological equivalence in the case of global structural stability.

Asymptotic lines, together with geodesics and principal curvature lines are studied in Classical Differential Geometry [19,10,12,9,21,7,5,22,24–26].

For geodesics and principal lines, global structural stability and genericity properties have been developed in [1,2,14–16]. Meanwhile, for asymptotic lines the attention has been focused on their description in a small neighborhood of the curve $\mathbb{P}_\alpha$ of parabolic points [3,20,5].

This paper is devoted to the study of the simplest qualitative aspects of asymptotic lines on surfaces immersed into Euclidean space, focusing on their local and global structural stability. The results establish sufficient conditions for an immersion $\alpha$ to be $C^s$-global structurally asymptotic stable, $s \geq 5$. This extends the local results for parabolic points and periodic asymptotic lines established in [13] and reviewed below.

2. Preliminaries and formulation of the main results

On the projective bundle $\mathbb{P}\mathbb{M} = (TM\setminus0)/\{v = rw, r \neq 0\}$ of $\mathbb{M}$, consider the submanifold $\mathcal{H}_\alpha$ defined by all the asymptotic directions. That is by the zeros of the second fundamental form of $\alpha$. The first
condition to be imposed on \( \alpha \) is precisely that 0 is a regular value of the projectivization of \( H_\alpha \), that is \( DK_\alpha \neq 0 \) at parabolic points.

The restriction of the projection \( \Pi \) of \( \mathbb{P}\mathbb{M} \) to \( \mathcal{H}_\alpha \) covers \( \text{Clos}(\mathbb{H}_\alpha) \). Over \( \mathbb{H}_\alpha \) it is a double regular covering. Over \( \mathbb{P}_\alpha \) it has a Whitney fold [27,8]. Therefore the Euler–Poincaré characteristic are related by \( \chi(\mathcal{H}_\alpha) = 2\chi(\mathbb{H}_\alpha) \).

Lifting to this manifold the line fields \( \mathcal{L}_{\alpha,1} \) and \( \mathcal{L}_{\alpha,2} \) defines a single line field \( \mathcal{L}_{\alpha,1} \) on \( \Pi^{-1}(\mathbb{H}_\alpha) \), which under the conditions of regularity uniquely extends to a smooth line field \( \mathcal{L}_\alpha \) defined on the whole \( \mathcal{H}_\alpha \). Its singularities, when present, are contained in \( \mathcal{P}_\alpha = \Pi^{-1}(\mathbb{P}_\alpha) \). In a local chart \((u, v)\) the surface \( \mathcal{H}_\alpha \) is defined implicitly by the equation,

\[
F(u, v, p) = e + 2fp + gp^2 = 0, \quad p = \frac{dv}{du}
\]

and the line field \( \mathcal{L}_{\alpha,1} \) is locally given by:

\[
\begin{align*}
u' &= F_p, \\
X: \quad v' &= pF_p, \\
p' &= -(F_u + pF_v).
\end{align*}
\]

The submanifold \( \mathcal{H}_\alpha \) is a compact and oriented surface and the line field \( \mathcal{L}_{\alpha,1} \) is locally defined by a vector field, but in general is not globally orientable.

The restriction of the projection \( \Pi \) of \( \mathbb{P}\mathbb{M} \) to \( \mathcal{H}_\alpha \) covers \( \text{Clos}(\mathbb{H}_\alpha) \). Over \( \mathbb{H}_\alpha \) it is a double regular covering and over \( \mathbb{P}_\alpha \) it has a Whitney fold [27,8].

Lifting to this manifold the line fields \( \mathcal{L}_{\alpha,1} \) and \( \mathcal{L}_{\alpha,2} \) defines a single line field \( \mathcal{L}_{\alpha,1} \) on \( \Pi^{-1}(\mathbb{H}_\alpha) \), which under the conditions of regularity uniquely extends to a smooth line field \( \mathcal{L}_{\alpha,1} \) defined on the whole \( \mathcal{H}_\alpha \). Its singularities, when present, are contained in \( \mathcal{P}_\alpha = \Pi^{-1}(\mathbb{P}_\alpha) \).

When the immersion is of class \( C' \) the line field \( \mathcal{L}_{\alpha,1} \) is of class \( C'^{-3} \) on the surface \( \mathcal{H}_\alpha \).

The integral foliation of this line field is denoted by \( \mathcal{F}_{\alpha,1} \). The leaves of \( \mathcal{F}_{\alpha,1} \) contains the pullback of the leaves of the pair of asymptotic foliations \( \mathcal{A}_\alpha \). The projection of the leaves of \( \mathcal{F}_{\alpha,1} \) into \( \text{Clos}(\mathbb{H}_\alpha) \) are called the folded asymptotic lines of \( \alpha \).

On the surface \( \mathcal{H}_\alpha \) there is a canonical involution \( \varphi : \mathcal{H}_\alpha \rightarrow \mathcal{H}_\alpha \) such that \( \varphi|_{\mathcal{P}_\alpha} = \text{id} \).
If \((u, v, [p : q]) \in \mathcal{H}_a\) then \(\varphi_a(u, v, [p_1 : q_1]) = (u, v, [p_2 : q_2])\), i.e., \(\varphi_a\) sends an asymptotic direction \([p_1 : q_1]\) into an asymptotic direction \([p_2 : q_2]\).

Notice that the involution \(\varphi_a \circ \varphi_a = id\) is a diffeomorphism of \(\mathcal{H}_a\) under the regularity hypothesis of the parabolic points.

Now consider the line field on \(\mathcal{H}_a\) induced by \(\varphi_a\). That is, \(L_{a,II} = (\varphi_a)_* L_{a,1}\).

Denote by \(F_{a,II}\) the integral foliation of \(L_{a,II}\).

These two foliations are transversal in \(\mathcal{H}_a\) except at the parabolic points \(P\) where they are tangent.

In order to make a distinction between these foliations, we will say that \(F_{a,1}\) is the first asymptotic foliation and \(F_{a,II}\) is the second asymptotic foliation.

Also, as the singularities of \(L_{a,1}\) are contained in \(P\) and the involution have the fixed points formed by \(P\), it follows that \(L_{a,1}\) and \(L_{1a,1}\) have the same singular set.

Also, it is clear that the image of \(F_{a,II}\) by the projection \(\Pi : \mathcal{H}_a \to \mathbb{M}\) gives the asymptotic foliations \(A_{a,1}\) and \(A_{a,2}\).

The following conditions (inspired in \([14,16]\)) are essential for the formulation of the main stability result of this paper.

(a) Condition on parabolic points: Denote by \(\Sigma_a\) the class of immersions \(\alpha\) for which the singularities of the line field \(L_{a,1}\), which occur when \(L_{a,1}\) is tangent to \(P\), are hyperbolic (non-vanishing real part of eigenvalues). Calculations shows that when the eigenspaces are one-dimensional they are transverse to \(\Pi^{-1}(\mathbb{P}_a)\).

There are three cases to consider: the saddle (eigenvalues of opposite sign), the (proper) node (i.e., with distinct eigenvalues of the same sign) and the focus (pair of complex conjugate eigenvalues).

These conditions are expressed in terms of the curvature functions of \(\alpha\) and will be reviewed in Section 2.

(b) Condition on hyperbolic closed asymptotic lines: Denote by \(\Sigma_b\) the class of immersions for which all the regular and folded asymptotic closed lines, i.e., the periodic integral curves of \(L_{a,1}\) are hyperbolic (i.e., the derivative of the return map is different from one).

This condition can be expressed in terms of integral formulas involving the curvature functions of \(\alpha\) along the periodic asymptotic line, see Section 3.
(c) **Condition on separatrices:** Denote by $\Sigma_c$ the class of immersions such that there are no connection between separatrices of singular points of the foliation $\mathcal{F}_{a,t}$ and consequently of the $A_{a,1}$ and $A_{a,2}$.

(d) **Condition on limit sets:** Denote by $\Sigma_d$ the class of immersions such that for every leave of $\mathcal{F}_{a,t}$ the limit set is a singular point or a closed asymptotic line.

Define $\Sigma' = \Sigma'_{(a,b,c,d)} = \Sigma_a \cap \Sigma_b \cap \Sigma_c \cap \Sigma_d$.

Asymptotic lines which violate (c) for being separatrices of two parabolic points or double separatrices of the same parabolic point are called *parabolic connections*; in the second case they are also called *parabolic loops*.

An asymptotic line which violates (d) for being contained in its own limit set, without being an closed asymptotic line, is called *non-trivial recurrent asymptotic line*. An example of this type of lines is given in Section 6.

The main result of this paper is the following,

**Main Theorem.** – Let $\alpha : M \to \mathbb{R}^3$ be an immersion of class $C^r$, $r \geq 5$, of a compact and oriented surface $M$ of class $C^r$. Then:

(i) The set $\Sigma'_{(a,b,c,d)}$ is open in $\text{Imm}^{r,s}(M, \mathbb{R}^3)$, $s \geq 5$.

(ii) If $\alpha \in \Sigma'_{(a,b,c,d)}$ then $\alpha$ is $C^s$, $s \geq 5$, global structurally asymptotic stable.

Remark. – In a forthcoming paper, [17], we will prove that the class $\Sigma'_{(a,b,c,d)}$ is $C^2$-dense in the space of immersions of compact surfaces. This step will complete the analogy with lines of curvature for which the $C^2$-density have been proved in [14–16].

3. **Asymptotic lines near parabolic points**

In this section will be reviewed the local behavior of the asymptotic foliations near parabolic points, in terms of geometric invariants of the immersion $\alpha$.

Let $c : [0, L] \to \mathbb{M}^2$ be a regular arc of parabolic points, parametrized by arc length $u$. To fix the notation, suppose that $k_2|c = 0$ and $k_1|c < 0$, where $k_1$ and $k_2$ are the principal curvatures of the immersion $\alpha$. Let $\psi(u)$ the angle between $c'(u) = t(u)$ and the principal direction $L_2(\alpha)$, corresponding to $k_2$, at the point $c(u)$. Denote by $k_g(u)$ the geodesic curvature of $c$ at the point $c(u)$. 

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Theorem 3.1. – Let \( c : [0, L] \to \mathcal{M} \) be a regular curve of parabolic points as above. Then the following holds:

1° If \( \varphi(u) \neq 0 \), the asymptotic foliation, near \( c(u) \), is as shown in Fig. 1(a) (cuspidal type).

2° If \( \varphi(u) = 0 \) and \( \varphi'(u) \neq 0 \) there are three cases:
   
   (a) \( k_\varphi(u)/\varphi'(u) < 1 \),
   
   (b) \( 1 < k_\varphi(u)/\varphi'(u) < 9 \),
   
   (c) \( 9 < k_\varphi(u)/\varphi'(u) \).

In cases (a), (b) and (c) above the asymptotic foliation is as shown in the Figs. 1(b)–(d) respectively; and correspond, respectively, to the folded saddle, focus and node types parabolic points.

3° The set of immersions whose parabolic points satisfy conditions 1° and 2° is open and dense in \( C^5 \)-topology.

4° The points described in 1° and 2° are the only stable locally asymptotic structurally stable parabolic points.

Remark. – The formulation above, in terms of the geometric invariants of the immersion, is taken from [13]. See also [5,6].

4. Periodic asymptotic lines and their first return maps

In this section will be established an integral expression for the derivative of the first return map of a folded periodic asymptotic line. This derivative will be given in terms of curvature functions of the immersion \( \alpha \).

The study of closed asymptotic lines disjoint from parabolic points was carried out in [13].

4.1. Folded periodic asymptotic lines

Here will be established an integral expression for the derivative of the first return map of a folded periodic asymptotic line in terms of the curvature functions of the immersion \( \alpha \).

A folded periodic asymptotic line is a closed asymptotic curve \( c : [0, L] \to \mathcal{M} \) regular by parts, that is, there exist a finite sequence of numbers \( a_i, 0 = a_0 < a_1 < \cdots < a_l = L \), such that

\[
c_i = c|_{(a_i, a_{i+1})} : (a_i, a_{i+1}) \to Int \mathbb{H}
\]
Fig. 1. Asymptotic foliations near parabolic points.
is an asymptotic line of $\alpha$ and $p_i = c(a_i) \in \mathbb{P}_\alpha$ for $i = 1, \ldots, l - 1$. In other words, a folded periodic asymptotic line is the projection of a closed integral curve of the single line field $\mathcal{L}_\alpha$ defined on $\mathcal{H}_\alpha$, which intersects $\mathbb{P}_\alpha$.

Let $c$ be a folded periodic asymptotic line. Near each point $p_i$, consider two transversal sections to $c$, $\Sigma_{1,i}$ and $\Sigma_{2,i}$, and the Poincaré map $\sigma_i : \Sigma_{1,i} \to \Sigma_{2,i}$. Denote by $u_i' = c_i(a_i, a_i+1) \cap \Sigma_{j,i}$, $j = 1, 2$. Denote by $\pi_{i+1,i} : \Sigma_{2,i} \to \Sigma_{1,i+1}$ the Poincaré map associated to $c_i$. It follows that the Poincaré map associated to $c$, $\Pi : \Sigma_{1,1} \to \Sigma_{1,1}$ is given by:

$$\Pi = \pi_{l-1,1} \circ \sigma_{l-1} \circ \cdots \circ \pi_{i+1,i} \circ \cdots \circ \pi_{2,1} \circ \sigma_1.$$ 

The next lemma established in [13] it will be useful in what follows.

**Lemma 4.1.** Let $c : [0, L] \to \mathbb{M}^2$ be an arc of an asymptotic line parametrized by arc length $u$. Then the expression:

$$(1) \alpha(u, v) = (\alpha \circ c)(u) + v(N \wedge t)(u) + [H(a)(u)v^2 + A(u,v)v^2]N(c(u)),$$

where $A(u,0) = 0$ and $H_\alpha$ is the Mean Curvature of $\alpha$, defines a local chart of class $C^{r-2}$ around $c$.

**Proposition 4.2.** Let $c : [0, L] \to \mathbb{M}^2$ be an arc of an asymptotic line parametrized by arc length $u$ as in the lemma above and two
transversal sections \( \{u = u_0\} \) and \( \{u = u_1\} \). Then the derivative of the holonomy map \( \Pi \), associated to it is given by:

\[
\Pi'(0) = \exp \left[ \int_{u_0}^{u_1} \frac{\tau_g' - 2k_g(u)H_g(u)}{2\tau_g(u)} \, du \right],
\]

where \( k_g \) is the geodesic curvature of \( c \) and \( \tau_g = (-K_g)^{1/2} \) is the geodesic torsion of \( c \).

**Proof.** – The Darboux equations for the positive frame \( \{t, N^\wedge t, N\} \) are:

\[
t'(u) = k_g(u)(N \wedge t)(u), \\
(N \wedge t)'(u) = -k_g(u)t(u) + \tau_g(u)N(u), \\
N'(u) = -\tau_g(u)(N \wedge t)(u). 
\]

Direct calculation gives that:

\[
e(u, 0) = 0, \quad e_v(u, 0) = \tau'_g - 2H_g(u)k_g(u), \\
f(u, 0) = \tau_g(u) \quad g(u, 0) = 2H_g(u). 
\]

The differential equation of the asymptotic lines in the neighborhood of the line \( \{v = 0\} \) is given by:

\[
e + 2f\frac{dv}{du} + g(dv/du)^2 = 0. 
\]

Denote by \( v(u, r) \) the solution of the (4) with initial condition \( v(0, r) = r \). Therefore the return map \( \Pi \) is clearly given by \( \Pi(r) = v(L, r) \).

Differentiating (4) with respect to \( r \), it results that:

\[
g_vv_r(dv/du)^2 + (2gv_v + 2f_vv_r)(dv/du) + e_vv_r = 0. 
\]

Evaluating at \( v = 0 \), it follows that:

\[
2f(u, 0)v_{ur}(u, 0) + e_v(u, 0)v_r(u, 0) = 0. 
\]
Therefore, using the expressions for $f$ an $e$, found in (3), integration of (5) it is obtained:

$$\ln \Pi'(0) = \int_0^L \frac{-\tau'_g + 2H_\alpha k_\alpha}{2\tau_g} \, du.$$ 

This ends the proof. 

**Proposition 4.3.** Consider the asymptotic lines near a cuspidal parabolic point and the return map defined in the sections $\sigma_i : \Sigma_{1,i} \to \Sigma_{2,i}$. Then the function $\sigma_i$ is differentiable.

**Proof.** Near the point $p_i$ take a local chart $(U, V)$ such that the asymptotic lines are given by the differential equation $(dU/dV)^2 = U$, [3,4]. In this system of coordinates $\sigma_i : \{V = \varepsilon\} \to \{V = \varepsilon\}$ is clearly a translation $\sigma_i(u, \varepsilon) = (u + c, \varepsilon)$. Therefore $\sigma_i$ is differentiable. 

**Theorem 4.4.** Let $c : [0, L] \to \mathbb{M}$ be a folded closed asymptotic line, parametrized by arc length $u$, of an immersion $\alpha$.

If $c$ is hyperbolic, then $\alpha$ is $C^s$, $s \geq 4$, local asymptotic structurally stable at $c$.

**Remark.** When $c$ is disjoint from the parabolic set the condition of hyperbolicity of $c$ is expressed by

$$\int_0^L \frac{k_\alpha H_\alpha}{\sqrt{-k_\alpha}} \, du \neq 0.$$ 

See [13].

**5. Proof of the Main Theorem**

This section will be devoted to the proof of the main stability result of this paper.

**5.1. Openness of $\Sigma^r$ and canonical regions**

Let $\alpha \in \Sigma^r_{(a,b,c,d)}$, $r \geq 5$. Recall that $\mathcal{F}_{\alpha,1}$ or $\mathcal{F}_{\alpha,II}$ share the same set of singularities each of which is either a node, a saddle point or else a focus. The leaves of $\mathcal{F}_{\alpha,1}$ (respectively $\mathcal{F}_{\alpha,II}$) will be called as first (respectively
second) asymptotic lines. Let $\varphi_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ be the diffeomorphism taking $F_{a,II}$ (respectively $F_{a,I}$) to $F_{a,I}$ (respectively to $F_{a,II}$) and such that $\varphi_\alpha(p) = q$ if, and only if, $\pi(p) = \pi(q)$. In particular, $\varphi_\alpha$ restricted to $P_\alpha$ is the identity.

The openness of $\Sigma'$ in $\text{Imm}^{r-4}(\mathcal{M}, \mathbb{R}^3)$ follows from the local stability of the singularities together with the local stability of hyperbolic asymptotic closed lines and the continuity (on $\alpha$) of compact arcs of parabolic separatrices.

A first (respectively second) canonical region of $\alpha$, with $\alpha \in \Sigma'$, is a connected component of the complement of the union of the singularities, first (respectively second) closed asymptotic lines, the first (respectively second) strong stable separatrices of the nodes (with the orientation of attracting nodes) and first (respectively second) parabolic asymptotic separatrices of the saddle points.

The canonical regions can be parallel or cylindrical. In the first case the line field $L_{a,I}$ (respectively $L_{a,II}$) restricted to the region is topologically equivalent to $\frac{1}{uv}$ in $\mathbb{R}^2$, in the second one it is topologically equivalent to $u\frac{dv}{du} + v\frac{du}{dv}$ in $\mathbb{R}^2 \setminus \{0\}$.

Fig. 3 shows some typical examples of canonical regions. Dotted lines in the pictures of the canonical regions represent cross sections of the foliations in the region.

Let $A$ be a second (respectively first) parallel region of $\alpha$, then $\varphi_\alpha(A)$ is a first (respectively second) parallel region of $\alpha$. In either case, if $S$ is a non-empty connected component of $A \cap P_\alpha$ then $S$ is a cross section for both $A$ and $\varphi_\alpha(A)$; also these are the only canonical regions that meet $S$. 
For a cylindrical region of an asymptotic foliation, it can be that either (a) all the lines of the other foliation cross the region, as in Fig. 4(a), or that (b) the region contains at least one closed asymptotic line of the other foliation, as in Fig. 4(b).

The first (respectively second) cylindrical regions of case (a) are called transversally irreducible first (respectively second) canonical regions; those of case (b) are decomposed into the union of a finite number of transversally irreducible second (respectively first) canonical regions and two semi-transversally irreducible first regions. The boundary of a semi-transversally irreducible first (respectively second) region is the union of a first closed asymptotic line, to which the first asymptotic lines tend, and a second closed asymptotic line, to which the foliation is transversal.

In Fig. 4(b) appears a first (respectively second) canonical cylindrical region decomposed into one transversally irreducible second (respectively first) canonical region and two semi-transversally irreducible first (respectively second) regions.

It can be found a neighborhood $\mathcal{V}(\alpha)$ of $\alpha$ in the open set $\Sigma'$, $r \geq 5$, such that, along a continuous arc $\alpha_t$, $t \in [0, 1]$, in $\mathcal{V}(\alpha)$ joining $\alpha = \alpha_0$ to $\beta = \alpha_1$, there is a unique way to continue the singularities, closed asymptotic lines and asymptotic separatrices of the saddle points and nodal (strong separatrices) of both $\mathcal{F}_{a,1}$ and $\mathcal{F}_{a,II}$ in such a way that there is a natural unique continuation of the canonical regions of $\alpha_0$ into those of $\alpha_t$, which defines a one-to-one correspondence between the canonical...
regions of $\alpha$ and those of $\beta \in V(\alpha)$. Such correspondence preserves the type of canonical regions. Let $\varphi_{\alpha_t} : H_t \to H_t$ be the diffeomorphism taking $F_{\alpha_t,1}$ (respectively $F_{\alpha_t,2}$) to $F_{\beta_t,1}$ (respectively to $F_{\beta_t,2}$).

The continuation procedure defines uniquely a partial topological equivalence $h_t$ between the singular points of $F_{\alpha_t,1}$ and $F_{\beta_t,1}$ and the set of points which are simultaneously on a first and second asymptotic separatrix or closed asymptotic line of $\alpha$ with the similar set of $\alpha_t$, $t \in [0, 1]$.

At this point, by using the method of canonical regions as in [23,14,16], the continuation procedure may be used to define different topological equivalences $(H_{\alpha_t}, F_{\alpha_t,1}) \to (H_{\beta_t}, F_{\beta_t,1})$ and $(H_{\alpha_t}, F_{\alpha_t,2}) \to (H_{\beta_t}, F_{\beta_t,2})$ which extend $h_t$. Below it is indicated how to proceed in order to extend $h_t$ to a topological equivalence between $(H_{\alpha_t}, F_{\alpha_t,1}, F_{\alpha_t,2})$ and $(H_{\beta_t}, F_{\beta_t,1}, F_{\beta_t,2})$. This extension is obtained by means of a sequence of partial extensions. Once $h_t$ is defined in (part of) a canonical region $A$, it will necessarily be defined in $\varphi_{\alpha_t}(A)$ as the composition $\varphi_{\alpha_t} \circ h_t \circ \varphi_{\alpha_t}^{-1}$. This will be mentioned explicitly in most of the steps below.

As a consequence, $h_t$ will induce a topological asymptotic equivalence $H_t : \mathbb{H}_{\alpha_t} \to \mathbb{H}_{\beta_t}$ between $(\mathbb{H}_{\alpha_t}, A_{\alpha_t,1}, A_{\alpha_t,2})$ and $(\mathbb{H}_{\beta_t}, A_{\beta_t,1}, A_{\beta_t,2})$.

5.2. Construction of the asymptotic equivalence

Step 1. Given a parallel canonical region, select a specific cross section to it.

On each second parallel canonical region $R_2$ of $\alpha$, choose—once for all—a cross section $S$ according to the following directions: If $R_2$ meets $P_\alpha$, a connected component of $R_2 \cap P_\alpha$ will be taken as a cross section. If $R_2$ is disjoint of $P_\alpha$, a cross section will be taken to be an arc of a first asymptotic separatrix (which is always possible). The cross section associated to $\varphi_{\alpha}(R_2)$ will be $\varphi_{\alpha}(S)$. In this way, to each parallel region (either first or second) a cross section has been associated.

Step 2. Definition of $h_t$ on the cross sections (and so on the orbit space) of first and second parallel canonical regions.

Let $R_2$ be a second parallel canonical region of $\alpha$. Let $S$ be its associated cross section. If $S$ is an arc of a first asymptotic separatrix $\sigma$, the extremes, $a$ and $b$ of $S$ have natural continuations $h_t(a)$ and $h_t(b)$; these points define the extremes of the natural continuation $S_t$ of the arc on $\sigma_t$, the separatrix on $S_t$ which is the natural continuation of $\sigma$. Define
$h_t : S \rightarrow S_t$ to be any homeomorphism which extends the correspondence already given on the extremes.

If $S \subset P_a$, $S_t$ is an arc contained in $P_a$, the definition of $h_t$ is similar to the previous one. Recall that if an endpoint of $S$ is a node or saddle point (respectively belongs to a second asymptotic separatrix), its continuation in $S_t$ will also be of the same type.

Let $R_2$ and $S$ be as above and let $R_1 = \varphi_\alpha(R_2)$; in this case, $h_t : \varphi_\alpha(S) \rightarrow \varphi_\alpha(S_t)$ is the composition $\varphi_\alpha \circ h_1 \circ \varphi_\alpha^{-1}$. Recall that $\varphi_\alpha(S_t)$ is the cross section associated to the first parallel region $R_1_t$ of $\alpha_t$.

**Step 3. Definition of $h_t$ on the intersection of first and second parallel canonical regions.**

Let $R_2$ and $R_1$ be arbitrary second and first parallel regions, respectively. Let $S$ and $T$ be their corresponding cross sections. Denote by $s(p)$ (respectively $t(p)$) the point of intersection of the second (respectively first) asymptotic line through $p$ with $S$ (respectively $T$).

On each connected component $C$ of $R_2 \cap R_1$, define $h_t$ of $C$ onto its natural continuation $C_t$. Notice that it is already defined on the corners of $C$, which are either singular points or intersections of first with second separatrices. At a point $p$ in $C$, define $h_t(p)$ as the point in $C_t$ which is on the intersection of the second asymptotic line which passes through $h_t(s(p)) \in S_t$ with the first asymptotic line which passes through $h_t(t(p)) \in T_t$. See Fig. 5.

![Fig. 5.](image)
Notice that, wherever defined, \( \varphi_{a_t} \circ h_t = h_t \circ \varphi_a \) and therefore this procedure already defines an asymptotic topological equivalence between \( \alpha \) and \( a_t \), in the case in which there are no closed asymptotic lines.

**Step 4. Selection of a specific cross section associated to a transversally irreducible second (respectively first) cylindrical region which is not contained in a first (respectively second) region of type b.**

Let \( R_2 \) be a second transversally irreducible region of \( \alpha \). The assumptions imply that \( R_2 \) meets at least one of the following: \( \mathcal{P}_a \) or a first asymptotic separatrix or else a first closed asymptotic line. Choose -once for all- a cross section \( S \) to \( R_2 \) according to the following instructions:

If \( R_2 \) meets \( \mathcal{P}_a \), take (as a cross section) a connected component \( S \) of \( \mathcal{P}_a \setminus R_2 \).

If \( R_2 \) is disjoint of \( \mathcal{P}_a \) but it meets a first asymptotic separatrix, say \( \gamma \), select a connected component \( S \) of \( \gamma \cap R \).

If \( R \) is disjoint of \( \mathcal{P}_a \) and of every first asymptotic separatrix but it meets a first closed asymptotic line, select a connected component \( S \) of the intersection of \( R \) with this closed asymptotic line.

Now, the associated cross section to \( \varphi_a(R) \) will be \( \varphi_a(S) \). In this way, it can be associated to each transversally irreducible region (either first of second) a cross section to it.

**Step 5. Definition of \( h_t \) on the cross section associated to a transversally irreducible second (respectively first) canonical region.**

Let \( R_2 \) be a second region as in the assumption and let \( S \) be its associated cross section.

Consider the continuation \( S_t \) of \( S \) which is a cross section to the continuation \( R_{2, t} \) of \( R_2 \). The foliation \( \mathcal{F}_{\alpha_t, \Pi_t | R_{2, t}} \) defines a Poincaré map \( \pi_t : S_t \to S_t \) with only two fixed points, one attractor and one repeller, in the extremes of \( S_t \). Take a topological conjugation \( \theta_t : S \to S_t \), between \( \pi_0 \) and \( \pi_t \), that is, \( \pi_0 = \theta_t^{-1} \circ \pi_t \circ \theta_t \). Define \( h_t | S = \theta_t \). In this way, \( h_t \) is a conjugacy between the return map \( \pi_0 \) induced on \( S \) by \( \mathcal{F}_{\alpha_t, \Pi_t | R_{2, t}} \) and \( \pi_t \), induced on \( S_t \) by \( \mathcal{F}_{\alpha_t, \Pi_t | R_{2, t}} \).

Let \( R_2 \) and \( S \) be as above and let \( R_1 = \varphi_a(R_2) \); in this case, \( h_t : \varphi_a(S) \to \varphi_a(S_t) \) is the composition \( \varphi_{a_t} \circ h_t \circ \varphi_a^{-1} \). Recall that \( \varphi_a(S_t) \) is the cross section associated to the transversally irreducible first canonical region \( R_{1, t} = \varphi_a(R_{2, t}) \) of \( \alpha_t \).

**Step 6. Definition of \( h_t \) on the intersection of a cylindrical transversally irreducible canonical region with a parallel canonical region.**
Let \( R_2 \) (respectively \( R_1 \)) be a transversally irreducible second (respectively parallel first) canonical region. Let \( C = R_2 \cap R_1 \). The definition of \( h_t \) considered on the cross section associated to \( R_1 \) determines a one-to-one correspondence \( \sigma_0 \to \sigma_t \) between the leaves of \( \mathcal{F}_{a,1}|_C \) and those of \( \mathcal{F}_{a,1}|_{C_t} \), where \( C_t \) is the natural continuation of \( C \).

Extend \( h_t : C \to C_t \), so that—keeping the notations of Step 5—the intersection of \( C \) with the arc \([p, \pi_0(p)]\) of asymptotic line in \( \mathcal{F}_{a,1} \) (with \( p \in S \)) is mapped onto the intersection of \( C_t \) with the arc \([h_t(p), \pi_t(h_t(p)]\) of asymptotic line in \( \mathcal{F}_{a,1} \), preserving the correspondence \( \sigma_0 \to \sigma_t \) indicated right above; see Fig. 6. Now, extend \( h_t \) to \( \varphi_{a_t}(C) \) so that, restricted to \( C \), \( h_t \circ \varphi_a = \varphi_{a_t} \circ h_t \).

Step 7. Definition of \( h_t \) on the intersection of two cylindrical transversally irreducible canonical regions.

Let \( R_1 \) and \( R_2 \) be such first and second canonical region, respectively, which intersect each other. Let \( \lambda_0 \) (respectively \( \sigma_0 \)) be the cross section associated to \( R_1 \) (respectively \( R_2 \)). Recall that \( h_t : \lambda_0 \to \lambda_t \) is a conjugacy between the return maps induced on \( \lambda_0 \) by the first asymptotic foliation of \( \alpha \) and that induced on \( \lambda_t \) by the first asymptotic foliation \( \mathcal{F}_{a_t,1} \) of \( \alpha_t \). Here \( R_{1t} \) and \( \lambda_t \) are the corresponding natural continuations of \( R_1 \) and \( \lambda_0 \). The analogous statement is true for \( h_t : \sigma_0 \to \sigma_t \).

Define \( h_t \) on a connected component of \( R_1 \cap R_2 \) by the same procedure of Step 6: Given \( p \) in \( R_1 \cap R_2 \), it is on an asymptotic line \( \gamma_1(p) \) of \( \mathcal{F}_{a,1} \) and \( \gamma_2(p) \) of \( \mathcal{F}_{a,1} \), which intersects, respectively, \( \lambda_0 \) and \( \sigma_0 \) on orbits of the respective return maps. The asymptotic lines of \( \gamma_{1t}(p) \) of \( \mathcal{F}_{a,1} \) and \( \gamma_{2t}(p) \) of \( \mathcal{F}_{a,1} \) determined by the \( h_t \)-images of these orbits, on \( \lambda_t \) and \( \sigma_t \),
intersect on continuous curves (depending on the parameter \( t \)) only one of which, \( p_\ast \), passes through \( p \) at \( t = 0 \). Define \( h_\gamma(p) = p_\ast \). See Fig. 7.

Notice that the first cylindrical canonical region \( R_1 \) can only intersect second cylindrical canonical regions \( R_2 \) of the kind being considered in the present step as well as second parallel canonical regions.

Step 8. Definition of \( h_t \) on the intersection of a first (respectively second) cylindrical canonical region with a closed asymptotic line of the second (respectively first) asymptotic foliation.

Similar to that of Step 7.

Step 9. Definition of \( h_t \) on a closed asymptotic line of the second (respectively first) asymptotic foliation and on the first (respectively second) cylindrical canonical regions of type (a) which are contained in second (respectively first) cylindrical canonical regions of type (b).

Take a second cylindrical canonical region \( R_2 \) of type b) for \( \alpha \). Define the homeomorphism \( h_t \) on a closed asymptotic line \( \gamma_0 \) of \( \mathcal{F}_{\alpha, I} \) contained in \( R_2 \) and its natural continuation \( \gamma_t \) in \( R_{2I} \). This defines a one to one correspondence between the lines of \( \mathcal{F}_{\alpha, II} \) in \( R_2 \) and those of \( \mathcal{F}_{\alpha, II} \) in \( R_{2I} \). Now, if \( R_2 \) contains first cylindrical regions of type (a), define \( h_t \) on them following the procedure is Step 6, conjugating their return maps. See Fig. 8.

Let \( R_2, R_{2I} \) be as above and let \( A \) (respectively \( A_I \)) be the subset of \( R_2 \) (respectively of \( R_{2I} \)) where \( h_t \) has already been defined. In this case, extend \( h_t : \varphi_\alpha(A) \to \varphi_\alpha(A_I) \) as the composition \( \varphi_\alpha \circ h_t \circ \varphi_\alpha^{-1} \). In this way, \( h_t \) has been defined on the closed asymptotic lines and on the cylindrical canonical regions of type (a) satisfying the assumptions for this step.
Step 10. Definition of \( h_t \) on the semi-transversally irreducible second (respectively first) regions which are contained in second (respectively first) cylindrical canonical regions of type (b).

Let \( R_2 \) be such a second semi-transversally irreducible region contained in a cylindrical region of type (b). Notice that a second closed asymptotic line of \( \partial R_2 \) is contained either in the union of first parallel canonical regions or in a first cylindrical canonical region of type (b). This implies that \( h_t \) must have already been defined in \( \partial R_2 \), by previous steps.

Let \( p_1 \) and \( p_2 \) be points, respectively, on the closed asymptotic lines in \( \partial R_2 \) of the first and second asymptotic foliations. It may certainly be assumed that \( p_1 = h_t(p_1) \) as well as \( p_2 = h_t(p_2) \) depend continuously on \( (p_1, t) \) and \( (p_2, t) \).

Let \( \gamma_{1t}(p_1) \) and \( \gamma_{2t}(p_2) \) denote respectively the curves of \( \mathcal{F}_{\alpha, 1} \) and \( \mathcal{F}_{\alpha, r} \) passing through \( p_1 \) and \( p_2 \), then for any \( p \) in \( \gamma_{10}(p_1) \cap \gamma_{20}(p_2) \) there is a unique \( p_t \) in \( \gamma_{1t}(p_1) \cap \gamma_{2t}(p_2) \) which is its natural continuation. Define \( h_t(p) = p_t \).

In this way, all the possibilities for canonical regions have been considered for the definition of the extension of the asymptotic equivalence between \( \alpha \in \Sigma^r \), \( r \geq 4 \), and \( \beta \in \mathcal{V}(\alpha) \), when \( \mathcal{V}(\alpha) \) is small enough. This finishes the proof of the theorem. \( \square \)

6. On a class of dense asymptotic lines

The goal of this section is to present examples of folded recurrent asymptotic lines.
Proposition 6.1. – Let $T^2$ be the torus of revolution, obtained by the rotation of the circle $(x - R)^2 + z^2 = r^2$, $r < R$, around the $z$ axis. Then the qualitative behavior of the asymptotic lines is as shown in Fig. 9.

Moreover the return map $\Pi : \mathbb{S}(R) \to \mathbb{S}(R)$, where $\mathbb{S}(R) = \{(x, y, z) : x^2 + y^2 = R^2, z = -r\}$, is a rotation by an angle equal to $4RT(r/R)$, where

$$T \left( \frac{r}{R} \right) = \sum_{n=0}^{\infty} \frac{2a_n}{n!} \left( \frac{r}{R} \right)^n,$$

with

$$a_n = \frac{1 \times 3 \times \cdots \times (2n - 1) \Gamma\left(\frac{1}{2}\right) \Gamma\left(2n + \frac{1}{2}\right)}{2^n \Gamma(2n + \frac{3}{2})}.$$ 

Proof. – Consider the following parametrization of the torus of revolution:

$$(u, v) \to (\cos v(R + r \cos u), \sin v(R + r \cos u), r \sin u).$$

Performing the calculation of the second fundamental form, it is obtained that,

$$e(u, v) = R^2,$$
$$f(u, v) = 0,$$
$$g(u, v) = R(R + r \cos u) \cos u.$$

Therefore the differential equation of the asymptotic lines is:

$$F(u, v, du/dv) = R(du/dv)^2 + \cos u(R + r \cos u) = 0.$$
Writing \( p = du/dv \), consider the vector field

\[
X: \begin{cases}
  u' = F_p, \\
  v' = p F_p, \\
  p' = -(F_u + p F_v).
\end{cases}
\]

After multiplying \( X \) by \( 1/p \) it results that:

\[
X: \begin{cases}
  u' = 2Rp, \\
  v' = 2R, \\
  p' = R \sin u + r \sin 2u.
\end{cases}
\]

Consider also the projected vector field,

\[
Y: \begin{cases}
  u' = 2Rp, \\
  p' = R \sin u + r \sin 2u.
\end{cases}
\]

Notice that the orbit of \( Y \) through \((\frac{\pi}{2}, 0)\) reaches \((\frac{3\pi}{2}, 0)\).

In fact, from the first integral of \( Y \),

\[
G(u, p) = Rp^2 + R \cos u + \frac{r}{2} \cos 2u
\]

it follows that \((\frac{\pi}{2}, 0)\) and \((\frac{3\pi}{2}, 0)\) are in the same connected component of \( G^{-1}(\frac{\pi}{2}) \).

The time spent by an orbit that starts at \((\frac{\pi}{2}, 0)\) to reach the point \((\frac{3\pi}{2}, 0)\) can be calculated as follows:

From \( G(u, p) = \frac{\pi}{2} \) it results that:

\[
p = \left\{ \frac{-r(1 + \cos 2u) - 2R \cos u)}{2R} \right\}^{1/2}.
\]

As \( du/dt = 2Rp \), it follows that:

\[
T = R^{1/2} \int_{\pi/2}^{3\pi/2} \frac{du}{\left[ -\cos u(r \cos u + R) \right]^{1/2}}
\]

\[
= 2 \int_{0}^{\pi} \frac{du}{\left[ \sin u(1 - \frac{r}{R} \sin u) \right]^{1/2}}.
\]
It follows from [18, pp. 369 and 950] that the analytic function \( T(\frac{r}{R}) \) has the following expansion in series

\[
T \left( \frac{r}{R} \right) = \sum_{n=0}^{\infty} \frac{2a_n}{n!} \left( \frac{r}{R} \right)^n,
\]

where

\[
a_n = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(2n + \frac{1}{2})}{\Gamma(2n + \frac{3}{2})}.
\]

Therefore, from \( dv/dt = 2R \), it follows that an arc of the asymptotic line that starts at the point \( (\frac{r}{2}, v_0) \) ends at the point \( (\frac{r}{2}, v_1) \), where \( v_1 \) is given by \( v_1 = 2RT \div v \).

So the return map \( \Pi : \{ v = \frac{v_0}{2} \} \to \{ v = \frac{v_1}{2} \} \) is given by \( \Pi(v_0) = v_0 + 4RT \left( \frac{v}{R} \right) \).

As \( T \) is clearly non-constant, it is possible to select \( r \) and \( R \) such that the rotation number of \( \Pi \) is irrational. \( \square \)

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