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STRUCTURAL STABILITY OF ASYMPTOTIC LINES ON SURFACES IMMERSED IN $\mathbb{R}^3$

BY

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ABSTRACT. – In this paper are studied immersions of surfaces into to $\mathbb{R}^3$ whose nets of asymptotic lines are topologically undisturbed under small perturbations of the immersion. These immersions are called structurally asymptotic stable. Sufficient conditions to belong to this class are established here. These conditions focus on the stable patterns around parabolic points, parabolic separatrix connections, periodic asymptotic lines (including those that intercept the parabolic lines) as well the exclusion of recurrent asymptotic lines. The class of immersions that are structurally stable in this sense is open in the C^5-topology. © Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Dans ce travail sont étudiées les plongements des surfaces dans l'espace $\mathbb{R}^3$ pour lequelles ces réseaux des lignes asymptotiques sont préservées topologiquement pour les pétites déformations du plongement. Ces plongements sont appellés asymptotique structurellement stables. Ces conditions focalisent sur le comportement des lignes asymptotiques dans une voisinage des lignes paraboliques, sur l'absence des connections des séparatrices paraboliques, sur les lignes asymptotiques férmées et aussi sur l'absence des récurrences non triviales des lignes asymptotiques. La classe des plongements structurellement stable est ouverte dans la topologie C^5. © Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

Consider a C^r, $r \geqslant 5$, immersion α of a smooth, compact and oriented, two-dimensional manifold $\mathbb{M}$ into Euclidean space $\mathbb{R}^3$.

The *Fundamental Forms* of α at a point p of $\mathbb{M}$ are the symmetric bilinear forms on $\mathbb{T}_p\mathbb{M}$ defined as follows [26,25]:

The *First Fundamental Form*:

$$I_\alpha(p; v, w) = \langle D\alpha(p; v), D\alpha(p; w)\rangle.$$

The *Second Fundamental Form*:

$$II_\alpha(p; v, w) = -\langle DN_\alpha(p; v), D\alpha(p; w)\rangle.$$

Here, $\langle .\,, .\rangle$ is the Euclidean inner product on $\mathbb{R}^3$ and N_α is the positive normal of the immersion:

$$N_\alpha = \frac{\alpha_u \wedge \alpha_v}{|\alpha_u \wedge \alpha_v|},$$

where (u, v) is a positive chart on $\mathbb{M}$ and $\wedge$ is the vector (wedge) product associated to a once for all fixed orientation on $\mathbb{R}^3$, $\alpha_u = \frac{\partial\alpha}{\partial u}$ and $\alpha_v = \frac{\partial\alpha}{\partial v}$.

A line $\ell = \mathbb{R}.v$, tangent at a point p of $\mathbb{M}$ (i.e., $v \in \mathbb{T}_p\mathbb{M}\backslash 0$), along which the *normal curvature*

$$k_n(p; \ell) = \frac{II_\alpha(p; v, v)}{I_\alpha(p; v, v)}$$

vanishes, is called an *asymptotic direction of* α at p.

A maximal, regular curve $c : (a, b) \to \mathbb{M}$, parametrized by arc length s, whose tangent line is an asymptotic direction is called an *asymptotic line* of α. That is, for every s in (a, b), it holds that $II_\alpha(c(s); c'(s), c'(s)) = 0$.

Through every point p of the *hyperbolic region* $\mathbb{H}_\alpha$ of the immersion α, characterized by the condition that the Gaussian Curvature $\mathcal{K}_\alpha = \det(DN_\alpha)$ is negative, pass two transverse asymptotic lines of α, tangent to the two asymptotic directions through p. This follows from the usual existence and uniqueness theorems on Ordinary Differential Equations. In fact, on $\mathbb{H}_\alpha$ the local line fields are defined by the kernels $\mathcal{L}_{\alpha,1}$, $\mathcal{L}_{\alpha,2}$ of the smooth one-forms $\omega_{\alpha,1}$, $\omega_{\alpha,2}$ which locally split $II_\alpha = \omega_{\alpha,1} \otimes \omega_{\alpha,2}$.

The forms $\omega_{\alpha,i}$ are locally defined up to a non-vanishing factor and a permutation of their indices. Therefore, their kernels and integral foliations are locally well defined only up to a permutation of their indices.

Under the orientability hypothesis imposed on $\mathbb{M}$, it is possible to globalize, to the whole $\mathbb{H}_\alpha$, the definition of the line fields $\mathcal{L}_{\alpha,1}$, $\mathcal{L}_{\alpha,2}$ and of the choice of an ordering between them, as follows:

Consider the field $\mathcal{C}_\alpha$ of tangent cones on $\mathbb{H}_\alpha$, defined by the non-negative part of the second fundamental form, i.e., $I_\alpha(p; v, v) = 1$; $II_\alpha(p; v, v) \geqslant 0$, oriented compatibly with $\mathbb{M}$. Call $\{e_1(p), e_2(p)\}$ a positive basis for $\mathbb{T}_p\mathbb{M}$ consisting of unit asymptotic vectors, positive also for $\mathcal{C}_\alpha(p)$.

This choice of a basis can also be defined as follows:

$D\alpha(p, e_1(p)) \wedge D\alpha(p, e_2(p)) = N_\alpha(p)$ and $II_\alpha(p; v, v) > 0$, for $v = e_1(p) + e_2(p)$.

There is only one other different choice, $\{e'(p), e'_2(p)\}$, for such a basis; both choices define the same *asymptotic line fields* of α:

$\mathcal{L}_{\alpha,1}(p) = \mathbb{R}.e_1(p)) = \mathbb{R}.e'_1(p)$ and $\mathcal{L}_{\alpha,2}(p) = \mathbb{R}.e_2(p) = \mathbb{R}.e'_2(p)$.

These two line fields, called the *asymptotic line fields* of α, are smooth on $\mathbb{H}_\alpha$; they are distinctly defined together with the ordering between them given by the subindices $\{1, 2\}$ which define their *orientation ordering*: "1" for the *first asymptotic line field* $\mathcal{L}_{\alpha,1}$, "2" for the *second asymptotic line field* $\mathcal{L}_{\alpha,2}$. They will be presented as an ordered pair $\mathcal{L}_\alpha = \{\mathcal{L}_{\alpha,1}, \mathcal{L}_{\alpha,2}\}$.

The *asymptotic foliations* of α are the integral foliations $\mathcal{A}_{\alpha,1}$ of $\mathcal{L}_{\alpha,1}$ and $\mathcal{A}_{\alpha,2}$ of $\mathcal{L}_{\alpha,2}$; they fill out the hyperbolic region $\mathbb{H}_\alpha$. The *ordered asymptotic net* of the immersion α is the ordered pair $\mathcal{A}_\alpha = \{\mathcal{A}_{\alpha,1}, \mathcal{A}_{\alpha,2}\}$, the index $i = \{1, 2\}$ will be called the *orientation ordering* of the *asymptotic foliation*.

Clearly, an exchange in the orientations either of $\mathbb{M}$ or of $\mathbb{R}^3$ produces an inversion in the orientation ordering of the asymptotic line fields.

When non-empty, the region $\mathbb{H}_\alpha$ is bounded by the set (generically, i.e., for most $\alpha's$, a regular curve [11,20,6,8]) $\mathbb{P}_\alpha$ of *parabolic* points of α, on which $\mathcal{K}_\alpha$ vanishes. On $\mathbb{P}_\alpha$, the pair of asymptotic directions degenerate into a single one or into the whole tangent plane at points where $II_\alpha = 0$, called *flat umbilic points*.

The parabolic points will be regarded here as the singularities of the asymptotic net. In fact, in the context of Singularity Theory, $\mathbb{P}_\alpha$ is the singular set of the Normal Map N_α from $\mathbb{M}$ to the unit sphere $\mathbb{S}^2$. On the *Elliptic Region* $\mathbb{E}_\alpha$, defined by $\mathcal{K}_\alpha > 0$, the asymptotic directions are imaginary and will not be studied here. Thus the domain for real asymptotic directions and their integral curves in the present work will be the set $\{\mathcal{K}_\alpha \leqslant 0\}$ of non-elliptic points, which generically is either the empty set or a manifold with boundary coincident with $\mathrm{Clos}(\mathbb{H}_\alpha)$.

An immersion α is said to be *C^s-local asymptotic structurally stable at a compact set S* in $\mathrm{Clos}(\mathbb{H}_\alpha)$ if for any sequence α_n converging to α together with its first s derivatives in a compact neighborhood V_S of S there is a sequence of compact subsets S_n and a sequence of homeomorphisms h_n mapping S to S_n, converging to the identity of $\mathbb{M}$ such that on V_S it maps arcs of the asymptotic foliations $\mathcal{A}_{\alpha,i}$ to arcs of that of $\mathcal{A}_{\alpha_n,i}$ for $i = 1, 2$.

An immersion α is said *C^s-global structurally asymptotic stable* if the compact set S above is the closure of the the hyperbolic region $\mathbb{H}_\alpha$.

This implies that the parabolic set must be preserved by the homeomorphism defining the topological equivalence in the case of global structural stability.

Asymptotic lines, together with geodesics and principal curvature lines are studied in Classical Differential Geometry [19,10,12,9,21,7,5,22,24–26].

For geodesics and principal lines, global structural stability and genericity properties have been developed in [1,2,14–16]. Meanwhile, for asymptotic lines the attention has been focused on their description in a small neighborhood of the curve $\mathbb{P}_\alpha$ of parabolic points [3,20,5].

This paper is devoted to the study of the simplest qualitative aspects of asymptotic lines on surfaces immersed into Euclidean space, focusing on their local and global structural stability. The results establish sufficient conditions for an immersion α to be *C^s-global structurally asymptotic stable*, $s \geqslant 5$. This extends the local results for parabolic points and periodic asymptotic lines established in [13] and reviewed below.

2. Preliminares and formulation of the main results

On the projective bundle $\mathbb{PM} = \{\mathbb{TM}\backslash 0\}/\{v = rw, r \neq 0\}$ of $\mathbb{M}$, consider the submanifold $\mathcal{H}_\alpha$ defined by all the asymptotic directions. That is by the zeros of the second fundamental form of α. The first condition to be imposed on α is precisely that 0 is a regular value of the projectivization of II_α, that is $D\mathcal{K}_\alpha \neq 0$ at parabolic points.

The restriction of the projection Π of $\mathbb{PM}$ to $\mathcal{H}_\alpha$ covers $\mathrm{Clos}(\mathbb{H}_\alpha)$. Over $\mathbb{H}_\alpha$ it is a double regular covering. Over $\mathbb{P}_\alpha$ it has a Whitney fold [27,8]. Therefore the Euler–Poincaré characteristic are related by $\chi(\mathcal{H}_\alpha) = 2\chi(\mathbb{H}_\alpha)$.

Lifting to this manifold the line fields $\mathcal{L}_{\alpha,1}$ and $\mathcal{L}_{\alpha,2}$ defines a single line field $\mathcal{L}_{\alpha,I}$ on $\Pi^{-1}(\mathbb{H}_\alpha)$, which under the conditions of regularity uniquely extends to a smooth line field $\mathcal{L}_\alpha$ defined on the whole $\mathcal{H}_\alpha$. Its singularities, when present, are contained in $\mathcal{P}_\alpha = \Pi^{-1}(\mathbb{P}_\alpha\ )$. In a local chart (u, v) the surface $\mathcal{H}_\alpha$ is defined implicitly by the equation,

$$F(u, v, p) = e + 2fp + gp^2 = 0, \quad p = \frac{dv}{du}$$

and the line field $\mathcal{L}_{\alpha,I}$ is locally given by:

$$X\colon \begin{aligned} u' &= F_p, \\ v' &= pF_p, \\ p' &= -(F_u + pF_v). \end{aligned}$$

The submanifold $\mathcal{H}_\alpha$ is a compact and oriented surface and the line field $\mathcal{L}_{\alpha,I}$ is locally defined by a vector field, but in general is not globally orientable.

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When the immersion is of class C^r the line field $\mathcal{L}_{\alpha,I}$ is of class C^{r-3} on the surface $\mathcal{H}_\alpha$.

The integral foliation of this line field is denoted by $\mathcal{F}_{\alpha,I}$. The leaves of $\mathcal{F}_{\alpha,I}$ contains the pullback of the leaves of the pair of asymptotic foliations $\mathcal{A}_\alpha$. The projection of the leaves of $\mathcal{F}_{\alpha,I}$ into $\mathrm{Clos}(\mathbb{H}_\alpha)$ are called the *folded asymptotic lines* of α.

On the surface $\mathcal{H}_\alpha$ there is a canonical involution $\varphi\colon \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ such that $\varphi|_{\mathcal{P}_\alpha} = id$.

If $(u, v, [p : q]) \in \mathcal{H}_\alpha$ then $\varphi_\alpha(u, v, [p_1 : q_1]) = (u, v, [p_2 : q_2])$, i.e., φ_α sends an asymptotic direction $[p_1 : q_1]$ into an asymptotic direction $[p_2 : q_2]$.

Notice that the involution φ_α, $\varphi_\alpha \circ \varphi_\alpha = id$, is a diffeomorphism of $\mathcal{H}_\alpha$ under the regularity hypothesis of the parabolic points.

Now consider the line field on $\mathcal{H}_\alpha$ induced by φ_α. That is, $\mathcal{L}_{\alpha,II} = (\varphi_\alpha)_* \mathcal{L}_{\alpha,I}$.

Denote by $\mathcal{F}_{\alpha,II}$ the integral foliation of $\mathcal{L}_{\alpha,II}$.

These two foliations are transversal in $\mathcal{H}_\alpha$ except at the parabolic points $\mathcal{P}_\alpha$ where they are tangent.

In order to make a distinction between these foliations, we will say that $\mathcal{F}_{\alpha,I}$ is the *first asymptotic foliation* and $\mathcal{F}_{\alpha,II}$ is the *second asymptotic foliation*.

Also, as the singularities of $\mathcal{L}_{\alpha,I}$ are contained in $\mathcal{P}_\alpha$ and the involution have the fixed points formed by $\mathcal{P}_\alpha$ it follows that $\mathcal{L}_{\alpha,I}$ and $\mathcal{L}_{I\alpha,I}$ have the same singular set.

Also, it is clear that the image of $\mathcal{F}_{\alpha,II}$ by the projection $\Pi : \mathcal{H}_\alpha \to \mathbb{M}$ gives the asymptotic foliations $\mathcal{A}_{\alpha,1}$ and $\mathcal{A}_{\alpha,2}$.

The following conditions (inspired in [14,16]) are essential for the formulation of the main stability result of this paper.

(a) *Condition on parabolic points*: Denote by Σ_a the class of immersions α for which the singularities of the line field $\mathcal{L}_{\alpha,I}$, which occur when $\mathcal{L}_{\alpha,I}$ is tangent to $\mathcal{P}_\alpha$, are hyperbolic (non-vanishing real part of eigenvalues). Calculations shows that when the eigenspaces are one-dimensional they are transverse to $\Pi^{-1}(\mathbb{P}_\alpha)$.

There are three cases to consider: the saddle (eigenvalues of opposite sign), the (proper) *node* (i.e., with distinct eigenvalues of the same sign) and the focus (pair of complex conjugate eigenvalues).

These conditions are expressed in terms of the curvature functions of α and will be reviewed in Section 2.

(b) *Condition on hyperbolic closed asymptotic lines*: Denote by Σ_b the class of immersions for which all the regular and folded asymptotic closed lines, i.e., the periodic integral curves of $\mathcal{L}_{\alpha,I}$ are hyperbolic (i.e., the derivative of the return map is different from one).

This condition can be expressed in terms of integral formulas involving the curvature functions of α along the periodic asymptotic line, see Section 3.

(c) *Condition on separatrices*: Denote by Σ_c the class of immersions such that there are no connection between sepatrices of singular points of the foliation $\mathcal{F}_{\alpha,I}$ e consequently of the $\mathcal{A}_{\alpha,1}$ and $\mathcal{A}_{\alpha,2}$.

(d) *Condition on limit sets*: Denote by Σ_d the class of immersions such that for every leave of $\mathcal{F}_{\alpha,I}$ the limit set is a singular point or a closed asymptotic line.

Define $\Sigma^r = \Sigma^r_{(a,b,c,d)} = \Sigma_a \cap \Sigma_b \cap \Sigma_c \cap \Sigma_d$.

Asymptotic lines which violate (c) for being separatrices of two parabolic points or double separatrices of the same parabolic point are called *parabolic connections*; in the second case they are also called *parabolic loops*.

An asymptotic line which violates (d) for being contained in its own limit set, without being an closed asymptotic line, is called *non-trivial recurrent asymptotic line*. An example of this type of lines is given in Section 6.

The main result of this paper is the following,

MAIN THEOREM. – *Let $\alpha : \mathbb{M} \to \mathbb{R}^3$ be an immersion of class C^r, $r \geqslant 5$, of a compact and oriented surface $\mathbb{M}$ of class C^r. Then*:

(i) *The set $\Sigma^r_{(a,b,c,d)}$ is open in $Imm^{r,s}(\mathbb{M}, \mathbb{R}^3)$, $s \geqslant 5$.*
(ii) *If $\alpha \in \Sigma^r_{(a,b,c,d)}$ then α is C^s, $s \geqslant 5$, global structurally asymptotic stable.*

Remark. – In a forthcoming paper, [17], we will prove that the class $\Sigma^r_{(a,b,c,d)}$ is C^2-dense in the space of immersions of compact surfaces. This step will complete the analogy with lines of curvature for which the C^2-density have been proved in [14–16].

3. Asymptotic lines near parabolic points

In this section will be reviewed the local behavior of the asymptotic foliations near parabolic points, in terms of geometric invariants of the immersion α.

Let $c : [0, L] \to \mathbb{M}^2$ be a regular arc of parabolic points, parametrized by arc length u. To fix the notation, suppose that $k_{2|c} = 0$ and $k_{1|c} < 0$, where k_1 and k_2 are the principal curvatures of the immersion α. Let $\varphi(u)$ the angle between $c'(u) = t(u)$ and the principal direction $L_2(\alpha)$, corresponding to k_2, at the point $c(u)$. Denote by $k_g(u)$ the geodesic curvature of c at the point $c(u)$.

THEOREM 3.1. – *Let* $c : [0, L] \to \mathbb{M}$ *be a regular curve of parabolic points as above. Then the following holds*:

1° *If* $\varphi(u) \neq 0$, *the asymptotic foliation, near* $c(u)$, *is as shown in Fig.* 1(a) (*cuspidal type*).

2° *If* $\varphi(u) = 0$ *and* $\varphi'(u) \neq 0$ *there are three cases*:

(a) $k_g(u)/\varphi'(u) < 1$,

(b) $1 < k_g(u)/\varphi'(u) < 9$,

(c) $9 < k_g(u)/\varphi'(u)$.

In cases (a), (b) *and* (c) *above the asymptotic foliation is as shown in the Figs.* 1(b)–(d) *respectively*; *and correspond, respectively, to the folded saddle, focus and node types parabolic points.*

3° *The set of immersions whose parabolic points satisfy conditions* 1° *and* 2° *is open and dense in* C^5*-topology.*

4° *The points described in* 1° *and* 2° *are the only stable locally asymptotic structurally stable parabolic points.*

Remark. – The formulation above, in terms of the geometric invariants of the immersion, is taken from [13]. See also [5,6].

4. Periodic asymptotic lines and their first return maps

In this section will be established an integral expression for the derivative of the first return map of a *folded periodic asymptotic line*. This derivative will be given in terms of curvature functions of the immersion α.

The study of closed asymptotic lines disjoint from parabolic points was carried out in [13].

4.1. Folded periodic asymptotic lines

Here will be established an integral expression for the derivative of the first return map of a folded periodic asymptotic line in terms of the curvature functions of the immersion α.

A *folded periodic asymptotic line* is a closed asymptotic curve $c : [0, L] \to \mathbb{M}$ regular by parts, that is, there exist a finite sequence of numbers a_i, $0 = a_0 < a_1 < \cdots < a_l = L$, such that

$$c_i = c|_{(a_i, a_{i+1})} : (a_i, a_{i+1}) \to Int\, \mathbb{H}$$

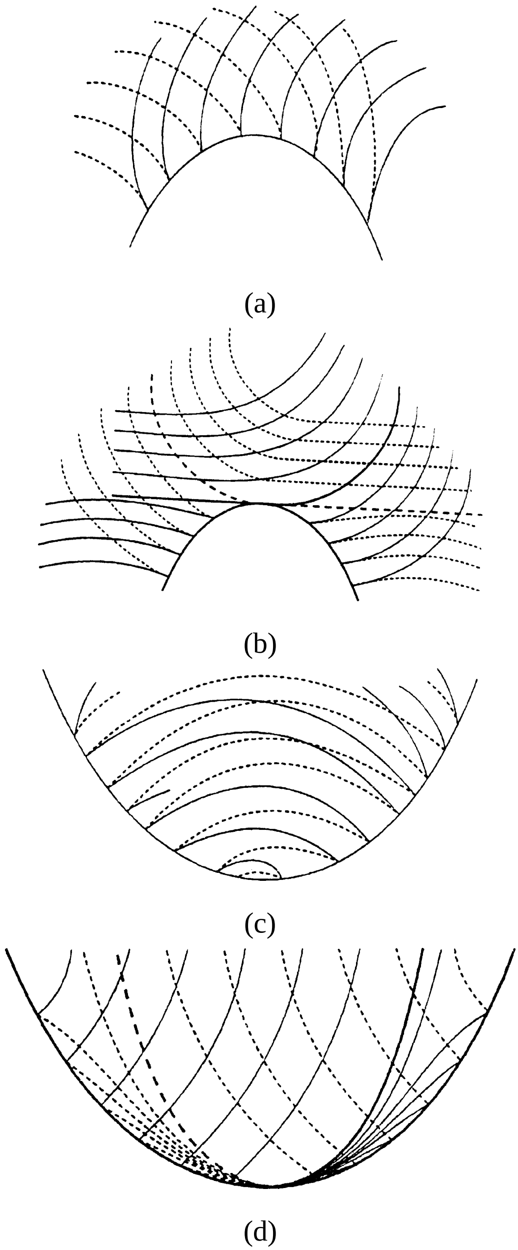


Fig. 1. Asymptotic foliations near parabolic points.

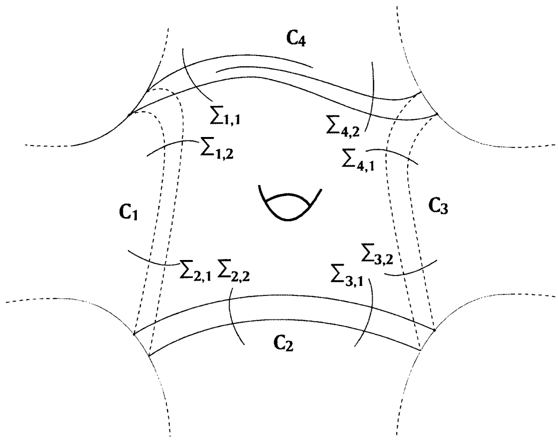


Fig. 2.

is an asymptotic line of α and $p_i = c(a_i) \in \mathbb{P}_\alpha$ for $i = 1, \ldots, l-1$. In other words, a *folded periodic asymptotic line* is the projection of a closed integral curve of the single line field $\mathcal{L}_\alpha$ defined on $\mathcal{H}_\alpha$, which intersects $\mathcal{P}_\alpha$.

Let c be a folded periodic asymptotic line. Near each point p_i, consider two transversal sections to c, $\Sigma_{1,i}$ and $\Sigma_{2,i}$, and the Poincaré map $\sigma_i : \Sigma_{1,i} \to \Sigma_{2,i}$. Denote by $u_i^j = c_i(a_i, a_{i+1}) \cap \Sigma_{j,i}$, $j = 1, 2$. Denote by $\pi_{i+1,i} : \Sigma_{2,i} \to \Sigma_{1,i+1}$ the Poincaré map associated to c_i. It follows that the Poincaré map associated to c, $\Pi : \Sigma_{1,1} \to \Sigma_{1,1}$ is given by:

$$\Pi = \pi_{l-1,1} \circ \sigma_{l-1} \circ \cdots \circ \pi_{i+1,i} \circ \cdots \circ \pi_{2,1} \circ \sigma_1.$$

The next lemma established in [13] it will be useful in what follows.

LEMMA 4.1. – *Let $c : [0, L] \to \mathbb{M}^2$ be an arc of an asymptotic line parametrized by arc length u. Then the expression*:

(1) $\alpha(u, v) = (\alpha \circ c)(u) + v(N \wedge t)(u) + \left[H_\alpha(u)v^2 + A(u, v)v^2\right] N\big(c(u)\big),$

where $A(u, 0) = 0$ and H_α is the Mean Curvature of α, defines a local chart of class C^{r-2} around c.

PROPOSITION 4.2. – *Let $c : [0, L] \to \mathbb{M}^2$ be an arc of an asymptotic line parametrized by arc length u as in the lemma above and two* *transversal sections* $\{u = u_0\}$ *and* $\{u = u_1\}$. *Then the derivative of the holonomy map* Π, *associated to it is given by*:

$$\Pi'(0) = \exp\left[\int_{u_0}^{u_1} \frac{\tau_g' - 2k_g(u)H_\alpha(u)}{2\tau_g(u)}\,du\right],$$

where k_g *is the geodesic curvature of* c *and* $\tau_g = (-\mathcal{K}_\alpha)^{1/2}$ *is the geodesic torsion of* c.

Proof. – The Darboux equations for the positive frame $\{t, N \wedge t, N\}$ are:

$$\begin{aligned} t'(u) &= k_g(u)(N \wedge t)(u), \\ (N \wedge t)'(u) &= -k_g(u)t(u) + \tau_g(u)N(u), \\ N'(u) &= -\tau_g(u)(N \wedge t)(u). \end{aligned} \tag{2}$$

Direct calculation gives that:

$$\begin{aligned} &e(u,0) = 0, \qquad e_v(u,0) = \tau_g' - 2H_\alpha(u)k_g(u), \\ &f(u,0) = \tau_g(u) \qquad g(u,0) = 2H_\alpha(u). \end{aligned} \tag{3}$$

The differential equation of the asymptotic lines in the neighborhood of the line $\{v = 0\}$ is given by:

$$e + 2f\,dv/du + g(dv/du)^2 = 0. \tag{4}$$

Denote by $v(u,r)$ the solution of the (4) with initial condition $v(0,r) = r$. Therefore the return map Π is clearly given by $\Pi(r) = v(L,r)$.

Differentiating (4) with respect to r, it results that:

$$g_r v_r (dv/du)^2 + (2g v_{ur} + 2f_v v_r)(dv/du) + e_v v_r = 0.$$

Evaluating at $v = 0$, it follows that:

$$2f(u,0)v_{ur}(u,0) + e_v(u,0)v_r(u,0) = 0. \tag{5}$$

Therefore, using the expressions for f an e_v found in (3), integration of (5) it is obtained:

$$\ln \Pi'(0) = \int_0^L \frac{-\tau_g' + 2H_\alpha k_g}{2\tau_g}\, du.$$

This ends the proof. $\square$

PROPOSITION 4.3. – *Consider the asymptotic lines near a cuspidal parabolic point and the return map defined in the sections $\sigma_i : \Sigma_{1,i} \to \Sigma_{2,i}$. Then the function σ_i is differentiable.*

Proof. – Near the point p_i take a local chart (U, V) such that the asymptotic lines are given by the differential equation $(dU/dV)^2 = U$, [3,4]. In this system of coordinates $\sigma_i : \{V = \varepsilon\} \to \{V = \varepsilon\}$ is clearly a translation $\sigma_i(u, \varepsilon) = (u + c, \varepsilon)$. Therefore σ_i is differentiable. $\square$

THEOREM 4.4. – *Let $c : [0, L] \to \mathbb{M}$ be a folded closed asymptotic line, parametrized by arc lenght u, of an immersion α.*

If c is hyperbolic, then α is C^s, $s \geqslant 4$, local asymptotic structurally stable at c.

Remark. – When c is disjoint from the parabolic set the condition of hyperbolicity of c is expressed by

$$\int_0^L \frac{k_g H_\alpha}{\sqrt{-\mathcal{K}_\alpha}}\, du \neq 0.$$

See [13].

5. Proof of the Main Theorem

This section will be devoted to the proof of the main stability result of this paper.

5.1. Openness of Σ^r and canonical regions

Let $\alpha \in \Sigma^r_{(a,b,c,d)}$, $r \geqslant 5$. Recall that $\mathcal{F}_{\alpha,I}$ or $\mathcal{F}_{\alpha,II}$ share the same set of singularities each of which is either a *node*, a *saddle point* or else a *focus*. The leaves of $\mathcal{F}_{\alpha,I}$ (respectively $\mathcal{F}_{\alpha,II}$) will be called as *first* (respectively

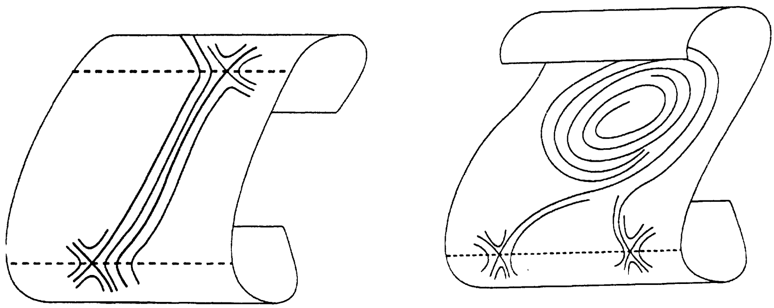


Fig. 3.

second) asymptotic lines. Let $\varphi_\alpha : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ be the diffeomorphism taking $\mathcal{F}_{\alpha,II}$ (respectively $\mathcal{F}_{\alpha,I}$) to $\mathcal{F}_{\alpha,I}$ (respectively to $\mathcal{F}_{\alpha,II}$) and such that $\varphi_\alpha(p) = q$ if, and only if, $\pi(p) = \pi(q)$. In particular, φ_α restricted to $\mathcal{P}_\alpha$ is the identity.

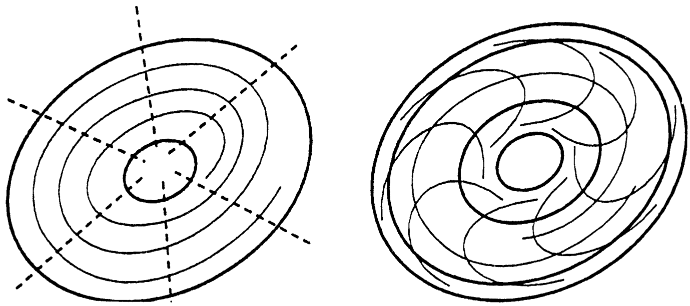
The openness of Σ^r in $\mathrm{Imm}^{r,s}(\mathbb{M}, \mathbb{R}^3)$ follows from the local stability of the singularities together with the local stability of hyperbolic asymptotic closed lines and the continuity (on α) of compact arcs of parabolic separatrices.

A *first* (*respectively second*) *canonical region* of α, with $\alpha \in \Sigma^r$, is a connected component of the complement of the union of the singularities, *first* (respectively *second*) closed asymptotic lines, the *first* (respectively *second*) strong stable separatrices of the *nodes* (with the orientation of attracting *nodes*) and *first* (respectively *second*) parabolic asymptotic separatrices of the *saddle points*.

The canonical regions can be *parallel* or *cylindrical*. In the first case the line field $\mathcal{L}_{\alpha,I}$ (respectively $\mathcal{L}_{\alpha,II}$) restricted to the region is topologically equivalent to $\frac{\partial}{\partial u}$ in $\mathbb{R}^2$, in the second one it is topologically equivalent to $u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$ in $\mathbb{R}^2 \setminus \{0\}$.

Fig. 3 shows some typical examples of cannonical regions. Dotted lines in the pictures of the *canonical regions* represent *cross sections* of the foliations in the region.

Let A be a *second* (respectively *first*) parallel region of α, then $\varphi_\alpha(A)$ is a *first* (respectively *second*) parallel region of α. In either case, if S is a non-empty connected component of $A \cap \mathcal{P}_\alpha$ then S is a cross section for both A and $\varphi_\alpha(A)$; also these are the only *canonical regions* that meet S.



(a) (b)

Fig. 4.

For a cylindrical region of an asymptotic foliation, it can be that either

(a) all the lines of the other foliation cross the region, as in Fig. 4(a), or that

(b) the region contains at least one closed asymptotic line of the other foliation, as in Fig. 4(b).

The *first* (respectively *second*) cylindrical regions of case (a) are called *transversally irreducible first* (respectively *second*) *canonical regions*; those of case (b) are decomposed into the union of a finite number of *transversally irreducible second* (respectively *first*) *canonical regions* and two *semi-transversally irreducible first regions*. The boundary of a *semi-transversally irreducible first* (respectively *second*) region is the union of a *first* closed asymptotic line, to which the *first* asymptotic lines tend, and a *second* closed asymptotic line, to which the foliation is transversal.

In Fig. 4(b) appears a *first* (respectively *second*) canonical cylindrical region decomposed into one *transversally irreducible second* (respectively *first*) *canonical region* and two *semi-transversally irreducible first* (respectively *second*) regions.

It can be found a neighborhood $\mathcal{V}(\alpha)$ of α in the open set Σ^r, $r \geqslant 5$, such that, along a continuous arc α_t, $t \in [0,1]$, in $\mathcal{V}(\alpha)$ joining $\alpha = \alpha_0$ to $\beta = \alpha_1$, there is a unique way to continue the singularities, closed asymptotic lines and asymptotic separatrices of the *saddle points* and nodal (strong separatrices) of both $\mathcal{F}_{\alpha,I}$ and $\mathcal{F}_{\alpha,II}$ in such a way that there is a natural unique continuation of the *canonical regions* of α_0 into those of α_t, which defines a one-to-one correspondence between the *canonical* *regions* of α and those of $\beta \in \mathcal{V}(\alpha)$. Such correspondence preserves the type of *canonical regions*. Let $\varphi_{\alpha_t} : \mathcal{H}_t \to \mathcal{H}_t$ be the diffeomorphism taking $\mathcal{F}_{\alpha_t,II}$ (respectively $\mathcal{F}_{\alpha_t,I}$) to $\mathcal{F}_{\alpha_t,I}$ (respectively to $\mathcal{F}_{\alpha_t,II}$).

The continuation procedure defines uniquely a partial topological equivalence h_t between the singular points of $\mathcal{F}_{\alpha,I}$ and $\mathcal{F}_{\beta,I}$ and the set of points which are simultaneously on a *first* and *second* asymptotic separatrix or closed asymptotic line of α with the similar set of α_t, $t \in [0,1]$.

At this point, by using the method of canonical regions as in [23,14,16], the continuation procedure may be used to define different topological equivalences $(\mathcal{H}_\alpha, \mathcal{F}_{\alpha,I}) \to (\mathcal{H}_\beta, \mathcal{F}_{I,\beta})$ and $(\mathcal{H}_\alpha, \mathcal{F}_{\alpha,II}) \to (\mathcal{H}_\beta, \mathcal{F}_{\beta,II})$ which extend h_t. Below it is indicated how to proceed in order to extend h_t to a topological equivalence between $(\mathcal{H}_\alpha, \mathcal{F}_{\alpha,II}, \mathcal{F}_{\alpha,I})$ and $(\mathcal{H}_\beta, \mathcal{F}_{\beta,II}, \mathcal{F}_{\beta,I})$. This extension is obtained by means of a sequence of partial extensions. Once h_t is defined in (part of) a *canonical region* A, it will necessarily be defined in $\varphi_\alpha(A)$ as the composition $\varphi_{\alpha_t} \circ h_t \circ {\varphi_\alpha}^{-1}$. This will be mentioned explicitly in most of the steps below. As a consequence, h_t will induce a topological asymptotic equivalence $H_t : \mathbb{H}_\alpha \to \mathbb{H}_\beta$ between $(\mathbb{H}_\alpha, \mathcal{A}_{\alpha,1}, \mathcal{A}_{\alpha,2})$ and $(\mathbb{H}_\beta, \mathcal{A}_{\beta,1}, \mathcal{A}_{\beta,2})$.

5.2. Construction of the asymptotic equivalence

Step 1. *Given a parallel* canonical region*, select a specific cross section to it.*

On each *second* parallel *canonical region* R_2 of α, choose—once for all—a cross section S according to the following directions: If R_2 meets $\mathcal{P}_\alpha$, a connected component of $R_2 \cap \mathcal{P}_\alpha$ will be taken as a cross section. If R_2 is disjoint of $\mathcal{P}_\alpha$, a cross section will be taken to be an arc of a *first* asymptotic separatrix (which is always possible). The cross section associated to $\varphi_\alpha(R_2)$ will be $\varphi_\alpha(S)$. In this way, to each parallel region (either *first* or *second*) a cross section has been associated.

Step 2. *Definition of h_t on the cross sections* (*and so on the orbit space*) *of* first *and* second *parallel canonical regions*.

Let R_2 be a *second* parallel *canonical region* of α. Let S be its associated cross section. If S is an arc of a *first* asymptotic separatrix σ, the extremes, **a** and **b** of S have natural continuations $h_t(\mathbf{a})$ and $h_t(\mathbf{b})$; these points define the extremes of the natural continuation S_t of the arc on σ_t, the separatrix on S_t which is the natural continuation of σ. Define

$h_t : S \to S_t$ to be any homeomorphism which extends the correspondence already given on the extremes.

If $S \subset \mathcal{P}_\alpha$, S_t is an arc contained in $\mathcal{P}_{\alpha_t}$, the definition of h_t is similar to the previous one. Recall that if an endpoint of S is a *node* or *saddle point* (respectively belongs to a *second asymptotic separatrix*), its continuation in S_t will also be of the same type.

Let R_2 and S be as above and let $R_1 = \varphi_\alpha(R_2)$; in this case, $h_t : \varphi_\alpha(S) \to \varphi_{\alpha_t}(S_t)$ is the composition $\varphi_{\alpha_t} \circ h_t \circ \varphi_\alpha{}^{-1}$. Recall that $\varphi_{\alpha_t}(S_t)$ is the cross section associated to the *first* parallel region R_{1_t} of α_t.

Step 3. *Definition of h_t on the intersection of* first *and* second *parallel canonical regions*.

Let R_2 and R_1 be arbitrary *second* and *first* parallel regions, respectively. Let S and T be their corresponding cross sections. Denote by $s(\mathbf{p})$ (respectively $t(\mathbf{p})$) the point of intersection of the *second* (respectively *first*) asymptotic line through p with S (respectively T).

On each connected component C of $R_2 \cap R_1$, define h_t of C onto its natural continuation C_t. Notice that it is already defined on the corners of C, which are either singular points or intersections of *first* with *second* separatrices. At a point $\mathbf{p}$ in C, define $h_t(\mathbf{p})$ as the point in C_t which is on the intersection of the *second* asymptotic line which passes through $h_t(s(\mathbf{p})) \in S_t$ with the *first* asymptotic line which passes through $h_t(t(\mathbf{p})) \in T_t$. See Fig. 5.

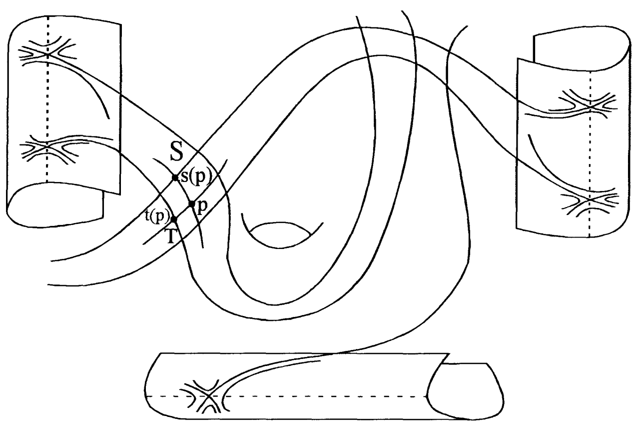


Fig. 5.

Notice that, wherever defined, $\varphi_{\alpha_t} \circ h_t = h_t \circ \varphi_\alpha$ and therefore this procedure already defines an asymptotic topological equivalence between α and α_t, in the case in which there are no closed asymptotic lines.

Step 4. *Selection of a specific cross section associated to a* transversally irreducible second (*respectively* first) *cylindrical region which is not contained in a* first (*respectively* second) *region of type b*).

Let R_2 be a *second transversally irreducible* region of α. The assumptions imply that R_2 meets at least one of the following: $\mathcal{P}_\alpha$ or a *first* asymptotic separatrix or else a *first* closed asymptotic line. Choose -once for all- a cross section S to R_2 according to the following instructions:

If R_2 meets $\mathcal{P}_\alpha$, take (as a cross section) a connected component S of $\mathcal{P}_\alpha \cap R_2$.

If R_2 is disjoint of $\mathcal{P}_\alpha$ but it meets a *first* asymptotic separatrix, say γ, select a connected component S of $\gamma \cap R$.

If R is disjoint of $\mathcal{P}_\alpha$ and of every *first* asymptotic separatrix but it meets a *first* closed asymptotic line , select a connected component S of the intersection of R with this closed asymptotic line.

Now, the associated cross section to $\varphi_\alpha(R)$ will be $\varphi_\alpha(S)$. In this way, it can be associated to each *transversally irreducible* region (either *first* of *second*) a cross section to it.

Step 5. *Definition of h_t on the cross section associated to a transversally irreducible second* (*respectively first*) *canonical region*.

Let R_2 be a *second* region as in the assumption and let S be its associated cross section.

Consider the continuation S_t of S which is a cross section to the continuation R_{2_t} of R_2. The foliation $\mathcal{F}_{\alpha_t,II}|_{R_{2_t}}$ defines a Poincaré map $\pi_t : S_t \to S_t$ with only two fixed points, one attractor and one repellor, in the extremes of S_t. Take a topological conjugation $\theta_t : S \to S_t$, between π_0 and π_t, that is, $\pi_0 = \theta_t^{-1} \circ \pi_t \circ \theta_t$. Define $h_t|_S = \theta_t$. In this way, h_t is a conjugacy between the return map π_0 induced on S by $\mathcal{F}_{\alpha,II}|_{R_2}$ and π_t induced on S_t by $\mathcal{F}_{\alpha_t,II}|_{R_{2_t}}$.

Let R_2 and S be as above and let $R_1 = \varphi_\alpha(R_2)$; in this case, $h_t : \varphi_\alpha(S) \to \varphi_{\alpha_t}(S_t)$ is the composition $\varphi_{\alpha_t} \circ h_t \circ {\varphi_{\alpha_t}}^{-1}$. Recall that $\varphi_{\alpha_t}(S_t)$ is the cross section associated to the *transversally irreducible first canonical region* $R_{1t} = \varphi_{\alpha_t}(R_{2t})$ of α_t.

Step 6. *Definition of h_t on the intersection of a cylindrical transversally irreducible canonical region with a parallel canonical region.*

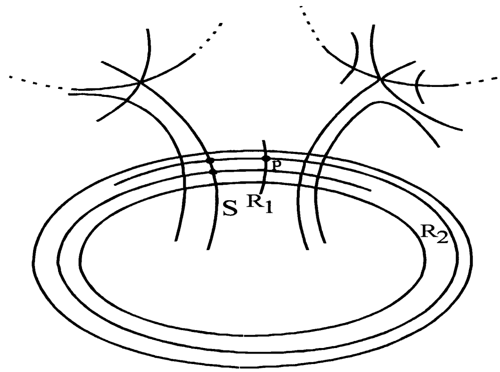


Fig. 6.

Let R_2 (respectively R_1) be a *transversally irreducible second* (respectively parallel *first*) *canonical region*. Let $C = R_2 \cap R_1$. The definition of h_t considered on the cross section associated to R_1 determines a one-to-one correspondence $\sigma_0 \to \sigma_t$ between the leaves of $\mathcal{F}_{\alpha,I}|_C$ and those of $\mathcal{F}_{\alpha_t,I}|_{C_t}$, where C_t is the natural continuation of C.

Extend $h_t : C \to C_t$, so that—keeping the notations of Step 5—the intersection of C with the arc $[\mathbf{p}, \pi_0(\mathbf{p})]$ of asymptotic line in $\mathcal{F}_{\alpha,II}$ (with $\mathbf{p} \in S$) is mapped onto the intersection of C_t with the arc $[h_t(\mathbf{p}), \pi_t(h_t(\mathbf{p}))]$ of asymptotic line in $\mathcal{F}_{\alpha_t,II}$, preserving the correspondence $\sigma_0 \to \sigma_t$ indicated right above; see Fig. 6. Now, extend h_t to $\varphi_\alpha(C)$ so that, rectricted to C, $h_t \circ \varphi_\alpha = \varphi_{\alpha_t} \circ h_t$.

Step 7. *Definition of h_t on the intersection of two cylindrical transversally irreducible canonical regions*.

Let R_1 and R_2 be such *first* and *second canonical region*, respectively, which intersect each other. Let λ_0 (respectively σ_0) be the cross section associated to R_1 (respectively R_2). Recall that $h_t : \lambda_0 \to \lambda_t$ is a conjugacy between the return maps induced on λ_0 by the *first asymptotic foliation* of α and that induced on λ_t by the *first asymptotic foliation* $\mathcal{F}_{\alpha_t,I}$ of α_t. Here R_{1t} and λ_t are the corresponding natural continuations of R_1 and λ_0. The analogous statement is true for $h_t : \sigma_0 \to \sigma_t$.

Define h_t on a connected component of $R_1 \cap R_2$ by the same procedure of Step 6: Given $\mathbf{p}$ in $R_1 \cap R_2$, it is on an asymptotic line $\gamma_1(\mathbf{p})$ of $\mathcal{F}_{\alpha,I}$ and $\gamma_2(\mathbf{p})$ of $\mathcal{F}_{\alpha,II}$, which intersects, respectively, λ_0 and σ_0 on orbits of the respective return maps. The asymptotic lines of $\gamma_{1t}(\mathbf{p})$ of $\mathcal{F}_{\alpha_t,I}$ and $\gamma_{2t}(\mathbf{p})$ of $\mathcal{F}_{\alpha_t,II}$ determined by the h_t-images of these orbits, on λ_t and σ_t,

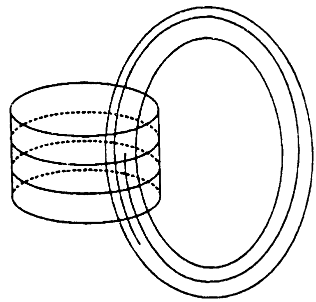


Fig. 7.

intersect on continuous curves (depending on the parameter t) only one of which, $\mathbf{p}_t$, passes through $\mathbf{p}$ at $t = 0$. Define $h_t(\mathbf{p}) = \mathbf{p}_t$. See Fig. 7.

Notice that the *first* cylindrical *canonical region* R_1 can only intersect *second* cylindrical *canonical regions* R_2 of the kind being considered in the present step as well as *second parallel canonical regions*.

Step 8. *Definition of* h_t *on the intersection of a first* (*respectively second*) *cylindrical canonical region with a closed asymptotic line of the second* (*respectively first*) *asymptotic foliation.*

Similar to that of Step 7.

Step 9. *Definition of* h_t *on a closed asymptotic line of the second* (*respectively first*) *asymptotic foliation and on the first* (*respectively second*) *cylindrical canonical regions of type* (a) *which are contained in second* (*respectively first*) *cylindrical canonical regions of type* (b).

Take a *second cylindrical canonical region* R_2 of type b) for α. Define the homeomorphism h_t on a closed asymptotic line γ_0 of $\mathcal{F}_{\alpha,I}$ contained in R_2 and its natural continuation γ_t in R_{2t}. This defines a one to one correspondence between the lines of $\mathcal{F}_{\alpha,II}$, in R_2 and those of $\mathcal{F}_{\alpha_t,II}$ in R_{2t}. Now, if R_2 contains *first* cylindrical regions of type (a), define h_t on them following the procedure is Step 6, conjugating their return maps. See Fig. 8.

Let R_2, R_{2t} be as above and let A (respectively A_t) be the subset of R_2 (respectively of R_{2t}) where h_t has already been defined. In this case, extend $h_t : \varphi_\alpha(A) \to \varphi_{\alpha_t}(A_t)$ as the composition $\varphi_{\alpha_t} \circ h_t \circ \varphi_\alpha{}^{-1}$. In this way, h_t has been defined on the closed asymptotic lines and on the *cylindrical canonical regions* of type (a) satisfying the assumptions for this step.

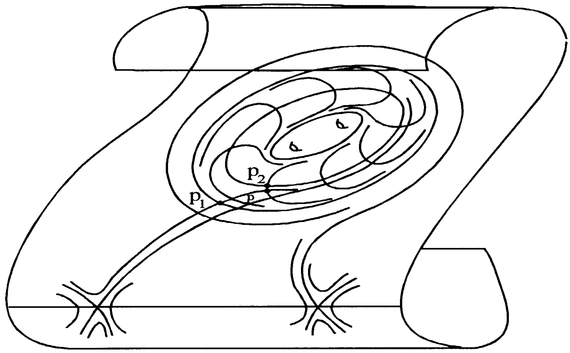


Fig. 8.

Step 10. *Definition of h_t on the semi-transversally irreducible second* (*respectively first*) *regions which are contained in second* (*respectively first*) *cylindrical canonical regions of type* (b).

Let R_2 be such a *second semi-transversally irreducible* region contained in a cylindrical region of type (b). Notice that a *second* closed asymptotic line of ∂R_2 is contained either in the union of *first parallel canonical regions* or in a *first cylindrical canonical region* of type (b). This implies that h_t must have already been defined in ∂R_2, by previous steps.

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be points, respectively, on the closed asymptotic lines in ∂R_2 of the first and second asymptotic foliations. It may certainly be assumed that $\mathbf{p}_{1t} = h_t(\mathbf{p}_1)$ as well as $\mathbf{p}_{2t} = h_t(\mathbf{p}_2)$ depend continuously on $(\mathbf{p}_1, t)$ and $(\mathbf{p}_2, t)$.

Let $\gamma_{1t}(\mathbf{p}_1)$ and $\gamma_{2t}(\mathbf{p}_2)$ denote respectively the curves of $\mathcal{F}_{\alpha_t,I}$ and $\mathcal{F}_{\alpha_t,II}$ passing through $\mathbf{p}_{1t}$ and $\mathbf{p}_{2t}$, then for any $\mathbf{p}$ in $\gamma_{10}(\mathbf{p}_1) \cap \gamma_{20}(\mathbf{p}_2)$ there is a unique $\mathbf{p}_t$ in $\gamma_{1t}(\mathbf{p}_1) \cap \gamma_{2t}(\mathbf{p}_2)$ which is its natural continuation. Define $h_t(\mathbf{p}) = \mathbf{p}_t$.

In this way, all the possibilities for *canonical regions* have been considered for the definition of the extension of the asymptotic equivalence between $\alpha \in \Sigma_\alpha^r$, $r \geqslant 4$, and $\beta \in \mathcal{V}(\alpha)$, when $\mathcal{V}(\alpha)$ is small enough. This finishes the proof of the theorem. $\square$

6. On a class of dense asymptotic lines

The goal of this section is to present examples of folded recurrent asymptotic lines.

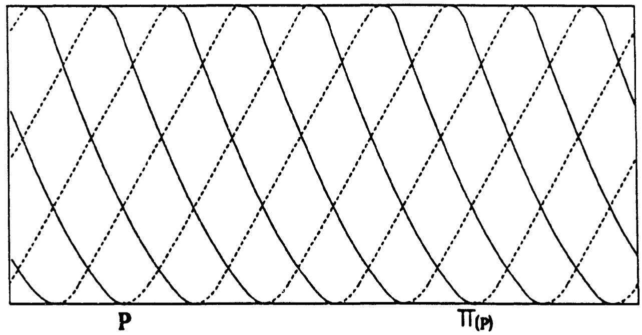


Fig. 9.

PROPOSITION 6.1. – *Let T^2 be the torus of revolution, obtained by the rotation of the circle $(x-R)^2+z^2=r^2$, $r<R$, around the z axis. Then the qualitative behavior of the asymptotic lines is as shown in Fig.* 9.

Moreover the return map $\Pi:\mathbb{S}(R)\to\mathbb{S}(R)$, where $\mathbb{S}(R)=\{(x,y,z)\colon x^2+y^2=R^2, z=-r\}$, is a rotation by an angle equal to $4RT(r/R)$, where

$$T\left(\frac{r}{R}\right)=\sum_{n=0}^{\infty}\frac{2a_n}{n!}\left(\frac{r}{R}\right)^n,$$

with

$$a_n=\frac{1\times 3\times\cdots\times(2n-1)}{2^n}\frac{\Gamma(\frac{1}{2})\Gamma(2n+\frac{1}{4})}{\Gamma(2n+\frac{3}{4})}.$$

Proof. – Consider the following parametrization of the torus of revolution:

$$(u,v)\to(\cos v(R+r\cos u),\sin v(R+r\cos u),r\sin u).$$

Performing the calculation of the second fundamental form, it is obtained that,

$$e(u,v)=R^2,\qquad f(u,v)=0,$$
$$g(u,v)=R(R+r\cos u)\cos u.$$

Therefore the differential equation of the asymptotic lines is:

$$F(u,v,du/dv)=R(du/dv)^2+\cos u(R+r\cos u)=0.$$

Writing $p = du/dv$, consider the vector field

$$X\colon \begin{cases} u' = F_p, \\ v' = pF_p, \\ p' = -(F_u + pF_v). \end{cases}$$

After multiplying X by $1/p$ it results that:

$$X\colon \begin{cases} u' = 2Rp, \\ v' = 2R, \\ p' = R\sin u + r\sin 2u. \end{cases}$$

Consider also the projected vector field,

$$Y\colon \begin{cases} u' = 2Rp, \\ p' = R\sin u + r\sin 2u. \end{cases}$$

Notice that the orbit of Y through $(\frac{\pi}{2}, 0)$ reaches $(\frac{3\pi}{2}, 0)$.

In fact, from the first integral of Y,

$$G(u, p) = Rp^2 + R\cos u + \frac{r}{2}\cos 2u$$

it follows that $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$ are in the same connected component of $G^{-1}(\frac{-r}{2})$.

The time spent by an orbit that starts at $(\frac{\pi}{2}, 0)$ to reach the point $(\frac{3\pi}{2}, 0)$ can be calculated as follows:

From $G(u, p) = \frac{r}{2}$ it results that:

$$p = \left\{ \frac{[-r(1 + \cos 2u) - 2R\cos u]}{2R} \right\}^{1/2}.$$

As $du/dt = 2Rp$, it follows that:

$$T = R^{1/2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{du}{[-\cos u(r\cos u + R)]^{1/2}}$$

$$= 2\int_{0}^{\frac{\pi}{2}} \frac{du}{[\sin u(1 - \frac{r}{R}\sin u)]^{1/2}}.$$

It follows from [18, pp. 369 and 950] that the analytic function $T(\frac{r}{R})$ has the following expansion in series

$$T\left(\frac{r}{R}\right) = \sum_{n=0}^{\infty} \frac{2a_n}{n!}\left(\frac{r}{R}\right)^n,$$

where

$$a_n = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n} \frac{\Gamma(\frac{1}{2})\Gamma(2n+\frac{1}{4})}{\Gamma(2n+\frac{3}{4})}.$$

Therefore, from $dv/dt = 2R$, it follows that an arc of the asymptotic line that starts at the point $(\frac{\pi}{2}, v_0)$ ends at the point $(\frac{3\pi}{2}, v_1)$, where v_1 is given by $v_1 = 2RT \mp v$.

So the return map $\Pi : \{v = \frac{-\pi}{2}\} \to \{v = \frac{-\pi}{2}\}$ is given by $\Pi(v_0) = v_0 + 4RT(\frac{r}{R})$.

As T is clearly non-constant, it is possible to select r and R such that the rotation number of Π is irrational. □

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