Applied Mathematics Letters 22 (2009) 994–997

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Lower and upper orientable strong diameters of graphs satisfying the Ore condition[☆]

Meirun Chen [a,b], Xiaofeng Guo [a,*], Hao Li [b,c]

[a] School of Mathematical Sciences, Xiamen University, Xiamen Fujian 361005, China

[b] Laboratorie de Recherche en Informatique, Université Paris-Sud, Bât. 490, 91405 Orsay Cedex, France

[c] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

ARTICLE INFO

Article history:
Received 11 October 2007
Received in revised form 29 September 2008
Accepted 6 January 2009

Keywords:
Lower and upper orientable strong diameter
Lower and upper orientable strong radius
Strong distance
Strong eccentricity
Strong radius and strong diameter
The Ore condition

ABSTRACT

Let D be a strong digraph. The strong distance between two vertices u and v in D, denoted by $sd_D(u, v)$, is the minimum size (the number of arcs) of a strong sub-digraph of D containing u and v. For a vertex v of D, the strong eccentricity $se(v)$ is the strong distance between v and a vertex farthest from v. The minimum strong eccentricity among all vertices of D is the strong radius, denoted by $srad(D)$, and the maximum strong eccentricity is the strong diameter, denoted by $sdiam(D)$. The lower (resp. upper) orientable strong radius $srad(G)$ (resp. $SRAD(G)$) of a graph G is the minimum (resp. maximum) strong radius over all strong orientations of G. The lower (resp. upper) orientable strong diameter $sdiam(G)$ (resp. $SDIAM(G)$) of a graph G is the minimum (resp. maximum) strong diameter over all strong orientations of G. In this work, we determine a bound of the lower orientable strong diameters and the bounds of the upper orientable strong diameters for graphs $G = (V, E)$ satisfying the Ore condition (that is, $\sigma_2(G) = \min\{d(x) + d(y) \mid \forall\, xy \notin E(G)\} \geq n$), in terms of girth g and order n of G.

1. Introduction

In [1], Chartrand et al. defined the strong distance $sd_D(u, v)$ (or simply $sd(u, v)$) between two vertices u and v in a strong digraph D as the minimum size (the number of arcs) of a strong sub-digraph of D containing u and v. The definition is motivated by the lack of symmetry of the familiar distance in directed graphs and by the definition of distance between two vertices in undirected graphs as the minimum size of a connected subgraph containing both vertices. It was shown in [1] that the strong distance is a metric on the vertex set of D. A (u, v)-geodesic is a strong sub-digraph of D of size $sd(u, v)$ containing u and v. Fig. 1 shows a strong digraph with $sd(w, v) = 3$, $sd(u, w) = 5$ and $sd(u, x) = 6$.

The strong eccentricity $se(v)$ of a vertex v in a strong digraph D is

$$se(v) = \max\{sd(v, x) \mid x \in V(D)\}.$$

The strong radius $srad(D)$ of D is

$$srad(D) = \min\{se(v) \mid v \in V(D)\},$$

while the strong diameter $sdiam(D)$ of D is

$$sdiam(D) = \max\{se(v) \mid v \in V(D)\}.$$

☆ The project was supported by NSFC (No. 10831001).

* Corresponding author.
E-mail address: xfguo@xmu.edu.cn (X. Guo).

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.01.026

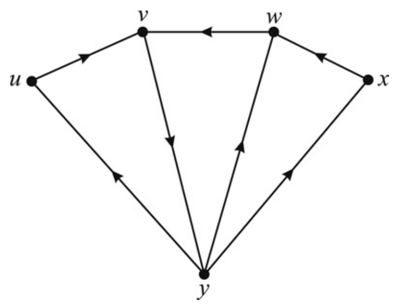

Fig. 1. Strong distance in a strong digraph.

In [2], for a connected graph G, Lai et al. defined the lower orientable strong radius $srad(G)$ of G as

$$srad(G) = \min\{srad(D) \mid D \text{ is a strong orientation of } G\},$$

while the upper orientable strong radius $SRAD(G)$ of G is

$$SRAD(G) = \max\{srad(D) \mid D \text{ is a strong orientation of } G\},$$

they also defined the lower orientable strong diameter $sdiam(G)$ of G as

$$sdiam(G) = \min\{sdiam(D) \mid D \text{ is a strong orientation of } G\},$$

while the upper orientable strong diameter $SDIAM(G)$ of G is

$$SDIAM(G) = \max\{sdiam(D) \mid D \text{ is a strong orientation of } G\}.$$

The strong radius and strong diameter of a strong digraph satisfy the following inequality.

Theorem 1 ([1]). *For every strong digraph D,*

$$srad(D) \le sdiam(D) \le 2srad(D).$$

In [1], Chartrand et al. gave an upper bound on the strong diameter of a strong oriented graph D.

Theorem 2 ([1]). *If D is a strong oriented graph of order $n \ge 3$, then*

$$sdiam(D) \le \left\lfloor \frac{5(n-1)}{3} \right\rfloor.$$

In [3], Dankelmann et al. investigated the structure of a (u, v)-geodesic for $u, v \in V(D)$, where D is a strong digraph, and gave the following theorem.

Theorem 3 ([3]). *Let D be a strong digraph. For $u, v \in V(D)$, let D_{uv} be a (u, v)-geodesic. Then $D_{uv} = P \cup Q$, where P and Q are a directed (u, v)-path and a directed (v, u)-path, respectively, in D_{uv}. There exist directed cycles $C_1, C_2, \ldots, C_k \subset D_{uv}$ such that*

(i) $u \in V(C_1)$, $v \in V(C_k)$;
(ii) $\bigcup_{i=1}^{k} C_i = D_{uv}$;
(iii) *each C_i contains at least one arc that is in P but not in Q, and at least one arc that is in Q but not in P;*
(iv) *$C_i \cap C_{i+1}$ is a directed path for $i = 1, 2, \ldots, k-1$;*
(v) *$V(C_i) \cap V(C_j) = \emptyset$ for $1 \le i < j - 1 \le k - 1$.*

In [3], Dankelmann et al. presented upper bounds for the strong diameter of D in terms of order n, directed girth $g \ge 2$, and strong connectivity $\kappa(D)$.

Theorem 4 ([3]). *If D is a strong digraph of order n and directed girth $g \ge 2$, then*

$$sdiam(D) \le \left\lfloor \frac{(n-1)(g+2)}{g} \right\rfloor.$$

Theorem 5 ([3]). *Let D be a strong oriented graph and $\kappa(D) = \kappa$. Then*

$$sdiam(D) \le \frac{5}{3}\left(1 + \frac{n-2}{\kappa}\right).$$

In [3], Dankelmann et al. also gave an upper bound on the strong radius of a strong oriented graph D.

Theorem 6 ([3]). *For any strong oriented graph D of order n, $srad(D) \le n$, and this bound is sharp.*

In [2], Lai et al. gave a lower bound on the strong radius of a strong oriented graph D.

Theorem 7 ([2]). *Let $G = (V, E)$ be a connected graph with n vertices, girth $g(G)$ and D be a strong orientation of G. Then $srad(D) \geq g(G)$.*

Some known results of strong distance, strong radius and strong diameter can be found in [2,4–7]. For graph theoretic notation and terminology not described here, the readers are referred to [8].

In this work, we determine a bound on the lower orientable strong diameters and the bounds on the upper orientable strong diameters for graphs $G = (V, E)$ satisfying the Ore condition (that is, $\sigma_2(G) = \min\{d(x) + d(y) | \forall xy \notin E(G)\} \geq n$), in terms of girth g and order n of G.

2. Main result

Theorem 8. *Let $G = (V, E)$ be a simple graph with girth g and order n. If $\sigma_2(G) = \min\{d(x) + d(y) | \forall xy \notin E(G)\} \geq n$, then $sdiam(G) \geq g$, $n \leq SDIAM(G) \leq n + 1$ and the bounds are sharp.*

Proof. By Theorems 1 and 7, we have $sdiam(D) \geq srad(D) \geq g$ for any strong orientation D of G. Thus $sdiam(G) \geq g$. The bound is realized by the complete bipartite graphs $K_{m,m}$ (see [6, Theorem 2.4]), where $m \geq 2$.

For any strong orientation $D = (V(D), A(D))$ of G, let $u, v \in V(D)$ be two vertices such that $sd_D(u, v) = sdiam(D)$. To prove $SDIAM(G) \leq n + 1$, it suffices to prove that $sd_D(u, v) \leq n + 1$. We consider two cases.

Case 1. $uv \in E(G)$. Then without loss of generality, assume that $(v, u) \in A(D)$. Let P be a shortest directed (u, v)-path and $C = P + (v, u)$. Clearly, C is a directed cycle containing u and v. So $sd_D(u, v) \leq |A(C)| \leq n$.

Case 2. $uv \notin E(G)$. Let D_{uv} be a (u, v)-geodesic in D. By Theorem 3, we have $D_{uv} = P \cup Q$, where P is a directed (u, v)-path and Q is a directed (v, u)-path in D_{uv}. Furthermore, there exist directed cycles $C_1, C_2, \ldots, C_k$ in D_{uv} such that $D_{uv} = \bigcup_{i=1}^{k} C_i$ satisfying (i)–(v) in Theorem 3.

Subcase 2.1. $k = 1$. Then $sd_D(u, v) = |A(C_1)| = |V(C_1)| \leq n$.

Subcase 2.2. $k = 2$. Then $sd_D(u, v) = |A(C_1 \cup C_2)| = |V(C_1 \cup C_2)| + 1 \leq n + 1$.

Subcase 2.3. $k \geq 3$.

Let $\Gamma(x)$ denote the neighbor set of a vertex x of G.

Claim 1. $\Gamma(u) \cap V\left(\bigcup_{i=3}^{k} C_i\right) = \Gamma(v) \cap V\left(\bigcup_{j=1}^{k-2} C_j\right) = \emptyset$.

By Theorem 3 (iv), $C_i \cap C_{i+1}$ is a directed path for $i = 1, 2, \ldots, k - 1$. Let a_i (resp. b_i) be the starting (resp. terminating) vertex of the directed path $C_i \cap C_{i+1}$ (possibly, $a_i = b_i$). Denote by $P[x, y]$ (resp. $Q[x, y]$) the sub-directed path of P (resp. Q) starting from x and terminating at y. Assume that there is a vertex w in $\Gamma(u) \cap V(C_l)$, where $3 \leq l \leq k$. Let $(u, w) \in A(D)$ (resp. $(w, u) \in A(D)$). If $w \in V(P)$ (resp. $V(Q)$), let $P' = (u, w) \cup P[w, v]$ (resp. $Q' = Q[v, w] \cup (w, u)$). Otherwise $w \in V(Q) \setminus V(P)$ (resp. $V(P) \setminus V(Q)$); let $P' = (u, w) \cup Q[w, a_{l-1}] \cup P[a_{l-1}, v]$ (resp. $Q' = Q[v, b_{l-1}] \cup P[b_{l-1}, w] \cup (w, u)$). Consider the strong digraph $D'_{uv} = P' \cup Q$ (resp. $D'_{uv} = P \cup Q'$). By Theorem 3 (iii), it is not difficult to verify that $|A(D'_{uv})| < |A(D_{uv})|$, which contradicts the minimality of D_{uv}. Hence, $\Gamma(u) \cap V\left(\bigcup_{i=3}^{k} C_i\right) = \emptyset$. By the same arguments, we have that $\Gamma(v) \cap V\left(\bigcup_{j=1}^{k-2} C_j\right) = \emptyset$.

Claim 2. If $k = 3$, then any vertex of $V(D_{uv}) - u - v$ is adjacent to at most one vertex of u and v.

Suppose $w \in \Gamma(u) \cap \Gamma(v) \cap V(D_{uv})$. Then, by claim 1, w must be an internal vertex of $Q[b_2, a_1]$ or $P[b_1, a_2]$, say $Q[b_2, a_1]$. If $(v, w) \in A(D)$ (resp. $(w, u) \in A(D)$), let $Q'' = (v, w) \cup Q[w, u]$ (resp. $Q'' = Q[v, w] \cup (w, u)$). Consider the strong digraph $D''_{uv} = P \cup Q''$. By Theorem 3 (iii) and the assumption that w is an internal vertex of $Q[b_2, a_1]$, we have $|A(D''_{uv})| < |A(D_{uv})|$, which contradicts the minimality of D_{uv}. Otherwise $(u, w) \in A(D)$ and $(w, v) \in A(D)$. Let $P'' = (u, w) \cup (w, v)$ and $D'''_{uv} = P'' \cup Q$. By Theorem 3 (iii), we also have $|A(D'''_{uv})| < |A(D_{uv})|$, again a contradiction.

By the above two claims and the fact that $\sigma_2(G) \geq n$, we have

$$n \leq d(u) + d(v) \leq n - |V(D_{uv})| + |\Gamma_{D_{uv}}(u)| + n - |V(D_{uv})| + |\Gamma_{D_{uv}}(v)|$$
$$\leq \begin{cases} 2n - 2|V(D_{uv})| + |V(D_{uv})| - 2, & \text{if } k = 3; \\ 2n - 2|V(D_{uv})| + |V(D_{uv})| - 2 - |V(C_2 \cap C_3)|, & \text{if } k = 4; \\ 2n - 2|V(D_{uv})| + |V(D_{uv})| - 2 - \left|V\left(\bigcup_{i=3}^{k-2} C_i\right)\right|, & \text{if } k \geq 5. \end{cases}$$

That is,

$$|V(D_{uv})| \leq \begin{cases} n - 2, & \text{if } k = 3; \\ n - 2 - |V(C_2 \cap C_3)|, & \text{if } k = 4; \\ n - 2 - \left|V\left(\bigcup_{i=3}^{k-2} C_i\right)\right|, & \text{if } k \geq 5. \end{cases} \tag{1}$$

Clearly, $|V(C_2 \cap C_3)| \geq 1$. By Theorem 3 (v), the cycles $C_3, C_5, \ldots$ (resp. $C_2, C_4, C_6, \ldots$) are disjoint; thus

$$\left| V\left(\bigcup_{i=3}^{k-2} C_i\right)\right| \geq \begin{cases} |V(C_3)| + |V(C_5)| + \cdots + |V(C_{k-2})| \geq \dfrac{3(k-3)}{2}, & \text{if } k \geq 5 \text{ and is odd;} \\ |V(C_2 \cap C_3)| + |V(C_4)| + \cdots + |V(C_{k-2})| \geq 1 + \dfrac{3(k-4)}{2}, & \text{if } k \geq 6 \text{ and is even.} \end{cases} \quad (2)$$

Combining (1) and (2), we obtain $sd_D(u, v) = |A(D_{uv})| = |V(D_{uv})| + k - 1 \leq n$.

Now, we know that $sd_D(u, v) \leq n + 1$ in all cases.

The bound is attained by $K_2 \times K_r$ with $r \geq 3$, which satisfies the Ore condition and $SDIAM(K_2 \times K_r) = 2r+1$ (see Theorem 14 in [5]).

Finally, we prove $SDIAM(G) \geq n$. It is well-known that G contains a hamiltonian cycle, since $\sigma_2 \geq n$. Let $C = v_1v_2 \ldots v_nv_1$ be a hamiltonian cycle of G, and D a strong orientation of G with arc set $A(D) = \{(v_i, v_{i+1})|1 \leq i \leq n-1\} \cup \{(v_j, v_i)|j \geq i+2 \text{ and } v_iv_j \in E(G)\}$. Since all vertices of D lie on the directed hamiltonian cycle C, it follows that $sdiam(D) \leq n$. Consider the vertices v_1 and v_n. Certainly, there exists a directed path of length 1 from v_n to v_1. However, the shortest directed path from v_1 to v_n is $v_1v_2 \ldots v_n$. Therefore, $sd_D(v_1, v_n) = n$ and $sdiam(D) = n$. Hence $SDIAM(G) \geq n$.

The bound is attained by the complete bipartite graph $K_{m,m}$ with $m \geq 2$, which satisfies the Ore condition and $SDIAM(K_{m,m}) = 2m$ (see Theorem 4 in [2]).

The proof is completed. □

References

[1] G. Chartrand, D. Erwin, M. Raines, P. Zhang, Strong distance in strong digraphs, J. Combin. Math. Combin. Comput. 31 (1999) 33–44.

[2] Yung-Ling Lai, Feng-Hsu Chiang, Chu-He Lin, Tung-Chin Yu, Strong distance of complete bipartite graphs, in: The 19th Workshop on Combinatorial Mathematics and Computation Theory, 2002, pp. 12–16.

[3] Peter Dankelmann, Henda C. Swart, David P. Day, On strong distance in oriented graphs, Discrete Math. 266 (2003) 195–201.

[4] G. Chartrand, D. Erwin, M. Raines, P. Zhang, On strong distance in strong oriented graphs, Congr. Numer. 138 (1999) 49–63.

[5] Meirun Chen, Xiaofeng Guo, Lower and upper orientable strong radius and strong diameter of cartesian product of complete graphs, Ars Combin. (in press).

[6] Huifang Miao, Xiaofeng Guo, Lower and upper orientable strong radius and strong diameter of complete k-partite graphs, Discrete Appl. Math. 154 (11) (2006) 1606–1614.

[7] Huifang Miao, Xiaofeng Guo, Strong distance in strong oriented complete k-partite graphs, Ars Combin. (in press).

[8] J. Bang-Jensen, G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer, London, 2000.