Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Discrete Mathematics 307 (2007) 1545 – 1549
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Research Problems from the Fourth Cracow Conference (Czorsztyn, 2002)

Rafał Kalinowski, Mariusz Meszka

Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland

The Research Problems section presents unsolved problems in discrete mathematics. In special issues, these typically are problems collected by the guest editors. In regular issues, they generally consist of problems submitted on an individual basis.

Older problems are acceptable if they are not widely known and the exposition features a new partial result. Concise definitions and commentary (such as motivation or known partial results) should be provided to make the problems accessible and interesting to a broad cross-section of the readership. Problems are solicited from all readers. Ideally, they should be presented in the style below, occupy at most one journal page, and be sent to

Douglas B. West
Mathematics Department, *University of Illinois*, *1409 West Green St.*, *Urbana IL 61801–2975*, *USA*
E-mail: west@math.uiuc.edu

The problems in this issue were presented at the problem session of the fourth Cracow Conference at Czorsztyn, Poland, September 16–20, 2002. They were collected and edited by Rafał Kalinowski and Mariusz Meszka.

Comments and questions of a technical nature about a particular problem should be sent to the correspondent for that problem. Other comments and information about partial or full solutions should be sent to Professor West (for potential later updates).

PROBLEM 453. Nomadic path and circuit decompositions

J. Adrian Bondy
Université Claude Bernard Lyon 1, *France*
E-mail: jabondy@univ-lyon1.fr

A nomad is a person who roams endlessly, with no particular home. We consider nomads who travel along the edges of a directed graph without ever encountering one another.

More precisely, let an *l-circuit decomposition* of a directed graph G on n vertices be a decomposition of G into directed circuits $C_1, \ldots, C_m$ of the same length l; when $l = n$, such a decomposition is called a *Hamilton decomposition*. For $v \in V(C_i)$, let v^{+k} denote the vertex on C_i reached by following C_i from v for k steps. Let f be a function that selects, for each C_i, a root vertex v_i of C_i. An l-circuit decomposition is *nomadic* if there exists some root selection function f such that $v_i^{+k} \neq v_j^{+k}$ whenever $i \neq j$ and $0 \leqslant k < l$. Note that in a nomadic decomposition the number of circuits cannot exceed the order of the graph.

E-mail addresses: kalinows@agh.edu.pl (R. Kalinowski), meszka@agh.edu.pl (M. Meszka).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2006.09.039

Many natural questions arise. In the *complete digraph* with n vertices, each ordered pair of distinct vertices forms an edge. Hence there are $n^2 - n$ edges, and this is divisible by both n and $n-1$. Tillson [2] showed that every complete n-vertex digraph with $n \geqslant 8$ admits a Hamilton decomposition.

Question 1. Does every sufficiently large complete digraph admit a nomadic Hamilton decomposition? If so, is it true that *every* Hamilton decomposition of every sufficiently large complete digraph is nomadic?

Question 2. Does the complete n-vertex digraph admit a nomadic $(n-1)$-circuit decomposition, for sufficiently large n? If so, is it true that, for sufficiently large n, *every* $(n-1)$-circuit decomposition of it is nomadic?

A *tournament* is a digraph obtained by orienting the edges of a complete graph. It is *regular* if at each vertex the indegree and outdegree are equal (in which case the number of vertices is odd). Kelly (see Moon [1], p. 7, Exercise 9) conjectured that every regular tournament of order at least three admits a Hamilton decomposition.

Question 3. Does every sufficiently large regular tournament admit a nomadic Hamilton decomposition? If so, is it true that *every* Hamilton decomposition of every sufficiently large regular tournament is nomadic?

Comment. *Nomadic path decompositions* may be defined similarly, with the nomad starting at the initial vertex of the path. Zsolt Tuza observed that every complete digraph of even order admits a nomadic decomposition into Hamilton paths, namely the decomposition derived from the Walecki decomposition of K_{2n} into Hamilton paths upon replacing each path by two oppositely directed paths.

One may also ask such 'nomadic' questions about Euler tours in eulerian graphs. For example, how many nomads, all following one Euler tour in the same direction, can be placed so that no two occupy the same vertex at any time?

References

[1] J.W. Moon, Topics on Tournaments, Holt, Rinehart and Winston, New York, 1968.
[2] T.W. Tillson, A Hamiltonian decomposition of K^*_{2m}, $2m \geqslant 8$, J. Combin, Theory (B) 29 (1980) 68–74.

PROBLEM 454. Nonrepetitive coloring of graphs

Jarosław Grytczuk
Faculty of Mathematics, Computer Science, and Econometrics, University of Zielona Góra, Zielona Góra, Poland
E-mail: J.Grytczuk@wmie.uz.zgora.pl

A sequence $\{a_j\}_{j \geqslant 1}$ is *nonrepetitive* if no two adjacent blocks of a are exactly the same. For example, 1**2323**21 contains the repetition 2323, while 123132123213 is nonrepetitive. Thue [3] constructed arbitrarily long nonrepetitive sequences using only three symbols.

A natural generalization for graphs is defined as follows. A coloring of the edges of a graph G is *nonrepetitive* if along each path the sequence of colors is nonrepetitive. We call the minimum number of colors in a nonrepetitive coloring the *Thue number* of G and denote it by $\pi(G)$. Every nonrepetitive coloring is a proper edge-coloring, so $\pi(G) \geqslant \chi'(G)$. We seek an analogue of Vizing's Theorem.

Question 1. Does there exist a constant c such that $\pi(G) \leqslant c\Delta(G)$ for every graph G?

Comment. In [1] we proved, using the probabilistic method, that $\pi(G) \leqslant c\Delta(G)^2$ for some absolute constant c. For some classes of graphs, linear upper bounds on $\pi(G)$ were derived by simple explicit colorings. For example, $\pi(K_n) \leqslant 2n-3$, and $\pi(T) \leqslant 4(\Delta(T)-1)$ for every tree T with at least two edges.

There are many exciting generalizations of nonrepetitive sequences and for most of them it also makes sense to study graph-theoretic variants. In principle, any property of sequences can be translated into a property of graphs, via colored paths. In particular, one may take any *avoidable pattern* and study its behavior on graphs. In this way new challenging problems arise relating Graph Theory to Combinatorics on Words.

In [2], the authors study more restrictive conditions than nonrepetitive coloring. A *parity edge-coloring* is an edge-coloring in which no path uses each color an even number of times, and it is a *strong parity edge-coloring* if no open walk uses each color an even number of times. The minimum numbers of colors in such colorings are denoted $p(G)$ and $\hat{p}(G)$, respectively. Every parity edge-coloring is a nonrepetitive coloring, so $\chi'(G) \leqslant \pi(G) \leqslant p(G) \leqslant \hat{p}(G)$. The main result of [2] is that $\hat{p}(K_n) = 2^{\lceil \log_2 n \rceil} - 1$, which improves the bound $\pi(K_n) \leqslant 2n - 3$. Many open problems about parity edge-colorings appear in [2].

Returning to the nonrepetitive condition, one may also consider other types of graph colorings. It is equally interesting to consider nonrepetitive vertex colorings; again repetitions are forbidden along paths. This time the probabilistic upper bound is nearly tight. For some classes of graphs, the *vertex Thue number* $\pi_v(G)$ remains bounded even if $\Delta(G)$ can be arbitrarily large. For example, $\pi_v(T) \leqslant 4$ for every tree T.

Question 2. Is it true that $\pi_v(G)$ is bounded for planar graphs?

References

[1] N. Alon, J. Grytczuk, M. Hałuszczak, O. Riordan, Non-repetitive colorings of graphs, Random Structures Algorithms 21 (2002) 336–346.

[2] D. Bunde, K. Milans, D.B. West, H. Wu, Parity edge-colorings of graphs, submitted for publication.

[3] A. Thue, Die Lösung eines Spezialfalles eines generellen logischen Problems, Christiania Vid. Selsk. Skr. 3 (1910) 7.

PROBLEM 455. Dirac-type theorem for hypergraphs

Gyula Y. Katona (correspondent), H.A. Kierstead

Department of Computer Science, *Budapest University of Technology*, *Hungary*

E-mail: kiskat@cs.bme.hu

Dirac [1] proved that minimum degree at least $n/2$ in an n-vertex graph forces the existence of a hamiltonian cycle (a spanning cycle); the threshold is sharp. We seek a generalization of Dirac's theorem to k-uniform hypergraphs.

In a k-uniform hypergraph, every edge is a k-element subset of the vertex set. A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is a *hamiltonian chain* if every set of k cyclically consecutive vertices is an edge. An ordinary graph is a 2-uniform hypergraph, and this definition reduces to the usual definition of hamiltonian cycle when $k = 2$.

To generalize Dirac's theorem, we need an appropriate notion of degree for hypergraphs. For $l < k$, the *degree* of an l-set S of vertices is the number of edges containing all of S. Although k is fixed, for clarity we denote the degree by $d_k(S)$. Furthermore, $\delta_k^{(l)}(\mathscr{H})$ denotes the minimum degree over all l-tuples in the k-uniform hypergraph $\mathscr{H}$.

Conjecture. If $\mathscr{H}$ is a k-uniform hypergraph on n vertices, and $\delta_k^{(k-1)}(\mathscr{H}) > \frac{1}{2}n + o(n)$, then $\mathscr{H}$ contains a hamiltonian chain.

Comment. A Dirac-type theorem is proved in [2] for all k, but its degree bound is probably not sharp: the result is that $\delta_k^{(k-1)}(\mathscr{H}) > (1 - 1/2k)n + 4 - k - 2/k$ suffices. Ruciński et al. [3] proved that $\delta_3^{(2)}(\mathscr{H}) > \frac{1}{2}n + o(n)$ suffices for 3-uniform hypergraphs when n is sufficiently large n. The proof uses the hypergraph version of Szemerédi's regularity lemma and works only for extremely large n.

A construction in [1] shows that the conjectured bound would be sharp.

References

[1] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2(3) (1952) 69–81.

[2] G.Y. Katona, H.A. Kierstead, Hamiltonian chains in hypergraphs, J. Graph Theory 30 (1999) 205–212.

[3] A. Ruciński, V. Rödl, E. Szemerédi, A Dirac-type theorem for 3-uniform hypergraphs, Combin. Probab. Comput. 15 (2006) 229–251.

PROBLEM 456. Minimum degree, girth, and subdivisions

Wolfgang Mader

Institut fuer Mathematik, *Universitaet Hannover*, *Hannover*, *Germany*

E-mail: mader@math.uni-hannover.de

For finite graphs with minimum degree at least n, sufficiently large girth forces a subdivision of the complete graph K_{n+1}. That is, for each positive n, there is an integer t_n such that every graph with girth at least t_n and minimum degree at least n contains a subdivision of K_{n+1} (proved in [4]). What is the minimum value t_n that suffices?

Conjecture 1 (*see* Mader [5]). Every finite graph of minimum degree n and girth at least 5 contains a subdivision of K_{n+1}.

Comment. The well-known result of Dirac [1] on subdivisions of K_4 states that $t_3 = 3$ suffices. By modifying the last step of the proof in [4], Kühn and Osthus [2] showed that $t_n = 186$ suffices for all n. More recently, in [3], they provided further support for Conjecture 1 by improving this to $t_n = 27$ for all n. The sufficiency of $t_n = 5$ seems very probable at least for $n = 4$ (see [6]). The complete bipartite graph $K_{n,n}$ shows that the conjecture is sharp for $n \geqslant 4$.

References

[1] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc. 27 (1952) 85–92.

[2] D. Kühn, D. Osthus, Topological minors in graphs of large girth, J. Combin. Theory (B) 86 (2002) 364–380.

[3] D. Kühn, D. Osthus, Improved bounds for topological cliques in graphs of large girth, SIAM J. Discrete Math. 20 (2006) 62–78

[4] W. Mader, Topological subgraphs in graphs of large girth, Combinatorica 18 (1998) 405–412.

[5] W. Mader, An extremal problem for subdivisions of K_5^-, J. Graph. Theory 30 (1999) 261–276.

[6] W. Mader, Topological minors in graphs of minimum degree n, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 49, 1999, pp. 199–211.

PROBLEM 457. Forcing subgraphs with high connectivity

Wolfgang Mader

Institut fuer Mathematik, Universitaet Hannover, Hannover, Germany

E-mail: mader@math.uni-hannover.de

Although high minimum degree alone does not force a graph to be highly connected, it forces the existence of a highly connected subgraph. What minimum degree is needed?

Conjecture 1. For $k \geqslant 2$, if G is a finite simple graph with $\delta(G) \geqslant 3k - 4$, then G has a k-connected subgraph, and there are examples without k-connected subgraphs having minimum degree $3k - 5$.

Comment. It was conjectured in [1] that when n is sufficiently large in terms of k, every simple n-vertex graph G with more than $\left(\frac{3}{2}k - 2\right)(n - k + 1)$ edges has a k-connected subgraph (for $k \geqslant 7$, it may suffice that $n > \lfloor (k-1)/2 \rfloor (k-1)$). This conjecture was proved in [1] for $k \leqslant 7$ and remains open for $k \geqslant 8$. For each k, this conjecture implies Conjecture 1. Thus, the sufficiency of $\delta(G) \geqslant 3k - 4$ for the existence of a k-connected subgraph is known for $k \leqslant 7$. For $k \geqslant 8$, we know from [1] that every graph with minimum degree at least $4k - 6$ has a k-connected subgraph.

For $2 \leqslant k \leqslant 7$, the proposer has constructed examples with $\delta(G) = 3k - 5$ that have no k-connected subgraph. For $k \geqslant 8$, it is unknown whether such examples exist with $\delta(G) = 3k - 5$.

Reference

[1] W. Mader, Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte, Abh. Math. Sem. Uni. Hamburg 37 (1972) 86–97.

PROBLEM 458. Packing two trees in the plane

Marc Noy

Departament de Matemàtica Aplicada II, Univ. Politècnica de Catalunya, Barcelona, Spain

E-mail: marc.noy@upc.es

We say that the graphs $H_1, \ldots, H_k$ *pack* into a graph G if G has edge-disjoint subgraphs isomorphic to $H_1, \ldots, H_k$. The problem of packing graphs has been widely studied, especially when G is a complete graph. In particular, it is known that any two n-vertex trees T_1 and T_2 that are not stars pack into the complete graph K_n [1]. The packing of three trees is more difficult; see [2] for a complete solution.

Here we seek a more restricted packing; we want to pack T_1 and T_2 into some planar graph with n vertices. That is, we want to draw both T_1 and T_2 in the plane using exactly n vertices so that no edge is repeated and no two edges cross. It is obviously necessary that neither tree be a star.

Conjecture (Garcia et al. [3]). For any n-vertex trees T_1 and T_2, with neither being a star, it is possible to pack them into some planar graph with n vertices.

Comment. In [3] the conjecture is proved when $T_1 = T_2$ and when one of the trees is a path. Note that if repeated edges are allowed, then the question is trivial: place the n vertices on a circle and draw T_1 inside it and T_2 outside it.

References

[1] S.M. Hedetniemi, S.T. Hedetniemi, P.J. Slater, A note on packing two trees into K_n, Ars Combin. 11 (1981) 149–153.
[2] M. Maheo, J.F. Saclé, M. Woźniak, Edge-disjoint placement of three trees, Eur. J. Combin. 17 (1996) 543–563.
[3] A. Garcia, C. Hernando, F. Hurtado, M. Noy, J. Tejel, Packing trees into planar graphs, J. Graph Theory 40 (2002) 172–181.

PROBLEM 459. Disjoint Hoffman–Singleton graphs

Alexander Rosa
Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada
E-mail: rosa@mcmail.cis.mcmaster.ca

This problem, originally due to van Dam, Klin, and Muzychuk, was conveyed to the poser by Klin. Here, the Hoffman–Singleton graph [1] is the largest known Moore graph; it is the unique strongly regular graph with parameters (50, 7, 0, 1) and is the smallest 7-regular graph with girth 5.

Question. Does the complete graph K_{50} decompose into seven copies of the Hoffman–Singleton graph?

Comment. Recently, Šiagiova and Meszka [3] used methods from topological graph theory to construct a set of five edge-disjoint copies of the Hoffman–Singleton graph in K_{50}. This set is maximal, since the complement of its union is disconnected.

Refining the question somewhat, we may now ask whether the maximum number of edge-disjoint copies of the Hoffman–Singleton graph in K_{50} is 5, 6, or 7.

The analogous question for the Petersen graph was answered by several authors: the maximum number of edge-disjoint Petersen graphs in K_{10} is 2, and the arrangement of these two subgraphs is unique up to isomorphism. See for example [2], whose solution includes references and description of other approaches.

References

[1] A.J. Hoffman, R.R. Singleton, On Moore graphs of diameters 2 and 3, IBM J. Res. Dev. 4 (1960) 497–504.
[2] A.J. Schwenk, Problem 6434, Amer. Math. Monthly 90 (1983) 403; Solution, Amer. Math. Monthly 94 (1987) 885–887.
[3] J. Šiagiová, M. Meszka, A covering construction for packing disjoint copies of the Hoffman–Singleton graph into K_{50}, J. Combin. Des. 11 (2003) 408–412.