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The number of removable edges in a 4-connected graph ☆

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Abstract

Let G be a 4-connected graph. For an edge e of G, we do the following operations on G: first, delete the edge e from G, resulting the graph $G-e$; second, for all the vertices x of degree 3 in $G-e$, delete x from $G-e$ and then completely connect the 3 neighbors of x by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by $G\ominus e$. If $G\ominus e$ is still 4-connected, then e is called a removable edge of G. In this paper we prove that every 4-connected graph of order at least six (excluding the 2-cyclic graph of order six) has at least $(4|G|+16)/7$ removable edges. We also give the structural characterization of 4-connected graphs for which the lower bound is sharp.

MSC: 05C40; 05C38; 05C75

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1. Introduction

All graphs considered here are finite and simple. For notations and terminology not defined here, we refer the reader(s) to [1]. The concepts of contractible edges and

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removable edges of graphs are very important in studying the structures of graphs and in proving some properties of graphs by induction. In 1961, Tutte [3] gave the structural characterization for 3-connected graphs by using the existence of contractible edges and removable edges. He proved that every 3-connected graph with order at least 5 contains contractible edges. Perhaps, this is the earliest result concerning the concepts of contractible edges and removable edges.

Removable edges and contractible edges in 3-connected graphs have been studied extensively in literature. In this paper we shall focus on the study of only removable edges in 4-connected graphs. First of all, we give the definition of a removable edge for a 4-connected graph. Let G be a 4-connected graph and e an edge of G. Consider the graph $G-e$ obtained by deleting the edge e from G. If $G-e$ has vertices of degree 3, we do the following operations on $G-e$. For all vertices x of degree 3 in $G-e$, delete x from $G-e$ and then completely connect the three neighbors of x by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by $G \ominus e$. Note that if there is no vertex of degree 3 in $G-e$, then $G \ominus e$ is simply the graph $G-e$.

Definition 1.1. For a 4-connected graph G and an edge e of G, if $G \ominus e$ is still 4-connected, then the edge e is called *removable*; otherwise, it is called *unremovable*. The set of all removable edges of G is denoted by $E_R(G)$; whereas the set of unremovable edges of G is denoted by $E_N(G)$. The number of removable edges and the number of unremovable edges of G is denoted by $e_R(G)$ and $e_N(G)$, respectively.

The aim to introduce the concept of removable edges in 4-connected graphs is to find a new method to construct 4-connected graphs and a new method to prove some properties of 4-connected graphs. In [4], Yin proved that there always exist removable edges in 4-connected graphs G unless G is a 2-cyclic graph with order 5 or 6, where a 2-cyclic graph is the graph of the square of a cycle [2]. He showed that a 4-connected graph can be obtained from a 2-cyclic graph by the following four operations: (i) adding edges, (ii) splitting vertices, (iii) adding vertices and removing edges, and (iv) extending vertices. In this paper we shall obtain a sharp lower bound for the number of removable edges in a 4-connected graph, and moreover, we shall give the structural characterization of the 4-connected graphs for which the lower bound is sharp.

Without specific statement, in the following G always denotes a 4-connected graph. The vertex set and edge set of G is denoted, respectively, by $V(G)$ and $E(G)$. The order and size of G is denoted, respectively, by $|G|$ and $|E(G)|$. For $x \in V(G)$, we simply write $x \in G$. The neighborhood of $x \in G$ is denoted by $\Gamma_G(x)$ and the degree of x is denoted by $d_G(x) = |\Gamma_G(x)|$. If no confusion, we simply write $d(x)$ for $d_G(x)$. If x and y are the two end-vertices of an edge e, we write $e = xy$. For a nonempty subset F of $E(G)$, or N of $V(G)$, the induced subgraph by F or N in G is denoted by $[F]$ or $[N]$. Let $A, B \subset V(G)$ such that $A \neq \emptyset \neq B$ and $A \cap B = \emptyset$, define $[A, B] = \{xy \in E(G) \mid x \in A, y \in B\}$. If H is a subgraph of G, we say that G contains H. For a subset S of $V(G)$, $G-S$ denotes the graph obtained by deleting all the vertices in S

from G together with all the incident edges. If $G - S$ is disconnected, we say that S is a vertex-cut of G. If $|S| = s$ for such an S, we say that S is an s-vertex-cut. A cycle of G with length l is simply called an l-cycle of G. We denote the 2-cyclic graphs of order 5 and 6 by C_5^2 and C_6^2, respectively. For $e \in E(G)$ and $S \subset V(G)$ such that $|S| = 3$, if $G - e - S$ has exactly two (connected) components, say A and B, such that $|A| \geqslant 2$ and $|B| \geqslant 2$, then we say that (e, S) is a *separating pair* and $(e, S; A, B)$ is a *separating group*, in which A and B are called the *edge-vertex-cut fragments*.

2. Some known results

First of all, we list some known results on removable edges of 4-connected graphs, which can be found in [4] and will be used in the sequel.

Theorem 2.1. *Let G be a 4-connected graph with $|G| \geqslant 7$. An edge e of G is unremovable if and only if there is a separating pair (e, S), or a separating group $(e, S; A, B)$ in G.*

Theorem 2.2. *Let G be a 4-connected graph with $|G| \geqslant 8$ and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$ and $|A| \geqslant 3$. Then, every edge in $[\{x\}, S]$ is removable.*

Corollary 2.3. *Let G be a 4-connected graph with $|G| \geqslant 8$. Then, every 3-cycle of G contains at least one removable edge.*

Theorem 2.4. *Let G be a 4-connected graph with $|G| \geqslant 8$. If for an unremovable edge xy, i.e., $xy \in E_N(G)$, there is a separating group $(xy, S; A, B)$, then all the edges in $E([S])$ are removable, i.e., $E([S]) \subseteq E_R(G)$.*

In the subsequent sections we shall obtain a sharp lower bound for the number of removable edges in a 4-connected graph.

3. Terminology and notations for subgraphs with special structures

For convenience, we introduce the following special terminology and notations for some subgraphs with special structures in a graph G.

Definition 3.1. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. If H satisfies the following conditions:

(i) $d(a) = d(x_i) = 4$ for $i = 1, 2, 3, 4$,
(ii) $ax_1, ax_2, ax_3, ax_4 \in E_N(G)$ and $x_1x_2, x_2x_3, x_3x_4, x_4x_1 \in E_R(G)$,

then H is called a *helm*, and the edges ax_i for $i = 1, 2, 3, 4$ are called *inner edges* of H.

Definition 3.2. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{a, b, x_1, x_2, \ldots, x_{l+3}\}$ and $E(H) = \{x_1x_2, x_2x_3, \ldots, x_{l+2}x_{l+3}, ax_2, ax_3, \ldots, ax_{l+2}, bx_2, bx_3, \ldots, bx_{l+2}\}$ with $l \geqslant 1$. If H satisfies the following conditions:

(i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2, \ldots, l+2$,
(ii) $ax_j, bx_j \in E_R(G)$ for $j = 2, 3, \ldots, l+2$,
(iii) $d(x_j) = 4$ for $j = 2, 3, \ldots, l+2$,

then H is called an *l-bi-fan*.

An l-bi-fan H is said to be maximal if $\Gamma_G(x_1) \neq \{a, b, x_2, u\}$ and $\Gamma_G(x_{l+3}) \neq \{a, b, x_{l+2}, v\}$ for any $u, v \in G$. The edges x_jx_{j+1} for $j = 2, 3, \ldots, l+1$ of an l-bi-fan or a maximal l-bi-fan H are called *inner edges* of H.

Definition 3.3. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, \ldots, x_{l+2}, y_1, y_2, \ldots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1x_2, x_2x_3, \ldots, x_{l+1}x_{l+2}, y_1y_2, y_2y_3, \ldots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \ldots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}\}$. Then, H is called an *l-belt* if the following conditions are satisfied

(i) $E_1(H) \subseteq E_N(G)$ and $E_2(H) \subseteq E_R(G)$,
(ii) $d(x_i) = d(y_j) = 4$ for $i = 2, 3, \ldots, l+1; j = 2, 3, \ldots, l+1$.

An l-belt H is said to be maximal if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(x_{l+2}) \neq \{x_{l+1}, y_{l+1}, y_{l+2}, v\}$ for any $u, v \in G$. The edges x_ix_{i+1}, y_jy_{j+1} for $i = 2, 3, \ldots, l+1; j = 1, 2, \ldots, l$ of an l-belt or a maximal l-belt H are called *inner edges* of H.

Definition 3.4. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, \ldots, x_{l+2}, x_{l+3}, y_1, y_2, \ldots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1x_2, x_2x_3, \ldots, x_{l+1}x_{l+2}, x_{l+2}x_{l+3}, y_1y_2, y_2y_3, \ldots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \ldots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}, x_{l+2}y_{l+2}\}$. Then, H is called an *l-co-belt* if the following conditions are satisfied:

(i) $E_1(H) \subseteq E_N(G)$ and $E_2(H) \subseteq E_R(G)$,
(ii) $d(x_i) = d(y_j) = 4$ for $i = 2, 3, \ldots, l+2; j = 2, 3, \ldots, l+1$.

An l-co-belt H is said to be maximal if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(y_{l+2}) \neq \{x_{l+2}, y_{l+1}, x_{l+3}, v\}$ for any $u, v \in G$. The edges x_ix_{i+1}, y_jy_{j+1} for $i = 2, 3, \ldots, l+1; j = 1, 2, \ldots, l+1$ of an l-co-belt or a maximal l-co-belt H are called *inner edges* of H.

Definition 3.5. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then, H is called a *W-framework* if the following conditions are satisfied:

(i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2$,

(ii) $d(x_2) = d(y_2) = d(y_3) = 4$,
(iii) $y_2y_3, x_1y_2, x_2y_2, x_2y_3, x_3y_3 \in E_R(G)$.

The edges x_1x_2, x_2x_3 of a W-framework H are called *inner edges* of H.

Definition 3.6. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, x_1x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then, H is called a W'*-framework* if the following conditions are satisfied:

(i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2$,
(ii) $d(x_2) = d(x_3) = d(y_2) = d(y_3) = 4$ and $d(x_1) \geqslant 5$,
(iii) $y_2y_3, x_1y_2, x_2y_3, x_3y_3, x_1x_3 \in E_R(G), x_2y_2 \in E_N(G)$.

The edges x_1x_2, x_2x_3, x_2y_2 of a W'-framework H are called *inner edges* of H.

For convenience, some special notations are introduced as follows.

By L_1 we denote the maximal 1-belt such that $V(L_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and $E(L_1') = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_1x_2, x_2y_2, y_2x_3\}$. We say that x_2x_3, y_1y_2 are inner edges of L_1.

By L_2 we denote the maximal 2-belt such that $V(L_2) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$ and $E(L_2') = \{x_1x_2, x_2x_3, x_3x_4, y_1y_2, y_2y_3, y_3y_4, y_1x_2, x_2y_2, y_2x_3, x_3y_3, y_3x_4\}$. We say that $x_2x_3, x_3x_4, y_1y_2, y_2y_3$ are inner edges of L_2.

By $L_1{}'$ we denote the maximal 1-co-belt such that $V(L_1{}') = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}$ and $E(L_1{}') = \{x_1x_2, x_2x_3, x_3x_4, y_1y_2, y_2y_3, y_1x_2, x_2y_2, y_2x_3, x_3y_3\}$. We say that x_2x_3, y_1y_2, y_2y_3 are inner edges of $L_1{}'$.

By $L_2{}'$ we denote the maximal 2-co-belt such that $V(L_2{}') = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4\}$ and $E(L_2{}') = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5, y_1y_2, y_2y_3, y_3y_4, y_1x_2, x_2y_2, y_2x_3, x_3y_3, y_3x_4, x_4y_4\}$. We say that $x_2x_3, x_3x_4, y_1y_2, y_2y_3, y_3y_4$ are inner edges of $L_2{}'$.

By F we denote the maximal 1-bi-fan such that $V(F) = \{a, b, x_1, x_2, x_3, x_4\}$ and $E(F) = \{x_1x_2, x_2x_3, x_3x_4, ax_2, ax_3, bx_2, bx_3\}$. We say that x_2x_3 is the inner edge of F.

By W we denote the W-framework such that $V(W) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(W) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. We say that x_1x_2, x_2x_3 are inner edges of W.

By W' we denote the W'-framework such that $V(W') = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(W') = \{x_1x_2, x_2x_3, x_1x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, y_2x_2, x_2y_3, y_3x_3\}$. We say that x_1x_2, x_2x_3, x_2y_2 are inner edges of W'.

By H we denote the helm such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. We say that the edges ax_i for $i = 1, 2, 3, 4$ are inner edges of H.

The set of all the above mentioned subgraphs with special notations L_1, L_2, $L_1{}'$, $L_2{}'$, F, W, W' and H of a graph G is denoted by $\mathfrak{R}$. Then, we have the following result.

Lemma 3.7. *There is no common inner edge between any two different subgraphs of G in $\mathfrak{R}$.*

Proof. By contradiction. Suppose that there are two different subgraphs H and H' of G in $\mathfrak{R}$ that have a common inner edge. Then, we discuss the following cases.

(1). H is a maximal 1-belt L_1. Then, x_2x_3 and y_1y_2 are the inner edges of H. Without loss of generality, we may assume that x_2x_3 is also an inner edge of H'. Similarly, we can treat the case that y_1y_2 is a common inner edge of H and H'. We discuss the following subcases for H':

(1.1). H' is a maximal 1-belt. Let $V(H') = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ and $E(H') = \{u_1u_2, u_2u_3, v_1v_2, v_2v_3, v_1u_2, u_2v_2, v_2u_3\}$, and let the inner edges of H' be u_2u_3, v_1v_2. If $x_2x_3 = u_2u_3$, then we have $x_2 = u_2, x_3 = u_3$ or $x_2 = u_3, x_3 = u_2$. If $x_2 = u_2, x_3 = u_3$, then $H = L_1 = H'$. If $x_2 = u_3, x_3 = u_2$, then we have $d(x_3) = 4$ and $x_3y_3 \in E(G)$ or we have $d(y_1) = 4$ and $x_1y_1 \in E(G)$. However, this contradicts to that $H = L_1$ is a maximal 1-belt.

(1.2). Obviously, a similar argument can lead to that H' is not a maximal 1-co-belt, a maximal 2-belt or a maximal 2-co-belt. And vice versa.

(1.3). H' is a maximal 1-bi-fan. Then, we have that $x_3y_1 \in E(G)$ or $x_1x_3 \in E(G)$. If $x_1x_3 \in E(G)$, then from the definition of the maximal 1-bi-fan, we have that $x_1x_2 \in E_R(G)$, which contradicts to the definition of the maximal 1-belt $H = L_1$. If $x_3y_1 \in E(G)$, since $y_1y_2 \in E_N(G)$, we take the corresponding separating group $(y_1y_2, S; A, B)$ such that $y_1 \in A, y_2 \in B$. Since $y_1y_2x_2y_1, y_1y_2x_3y_1$ are 3-cycles of G, we have that $x_2x_3 \in E([S])$. From Theorem 2.4 we have that $x_2x_3 \in E_R(G)$, which contradicts to the definition of the maximal 1-belt $H = L_1$. Therefore, any inner edge of the maximal 1-belt cannot be inner edge of any maximal 1-bi-fan. And vice versa.

(1.4). H' is a W-framework or a W'-framework. Then, we have that $y_1y_2 \in E_R(G)$, which contradicts to the definition of the maximal 1-belt $H = L_1$. Hence, any inner edge of the maximal 1-belt cannot be inner edge of any W-framework or W'-framework. And vice versa.

(1.5). H' is a helm. Then, either x_2 or x_3 is incident with four unremovable edges in G. Obviously, it is impossible since x_2x_3 is an inner edge of the maximal 1-belt $H = L_1$. Therefore, any inner edge of the maximal 1-belt cannot be inner edge of any helm, and vice versa.

(2). H is a maximal 2-belt L_2. Without loss of generality, we may assume that x_2x_3 is a common inner edge of H and H'. We discuss the following subcases for H':

(2.1). H' is also a maximal 2-belt. Let $V(H') = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ and $E(H') = \{u_1u_2, u_2u_3, u_3u_4, v_1v_2, v_2v_3, v_3v_4, v_1u_2, u_2v_2, v_2u_3, u_3v_3, v_3u_4\}$, and let u_2u_3, u_3u_4, v_1v_2, v_2v_3 be the inner edges of H'. If $x_2x_3 = u_2u_3$, then one of the following things holds: (i) $H = L_2 = H'$; (ii) $d(y_1) = 4$ and $x_1y_1 \in E(G)$, which contradicts to that $H = L_2$ is a maximal 2-belt. If $x_2x_3 = v_1v_2$, it is easy to see that $u_1v_1 \in E(G)$ and $d(v_1) = 4$, which contradicts to that H' is a maximal 2-belt. By symmetry, for the other cases, we may employ a similar argument to show that the conclusion holds.

(2.2). Since a maximal 1-co-belt is a subgraph of a maximal 2-belt, it is easy to see that x_2x_3 or y_1y_2 is not an inner edge of a maximal 1-co-belt. Otherwise, it would lead to a contradiction to the definition of the maximal 1-co-belt. Similarly, a maximal 2-belt and a maximal 2-co-belt do not have any common inner edge.

(2.3). Obviously, it is impossible that an inner edge of a maximal 2-belt is an inner edge of the following subgraphs: maximal 1-bi-fan, W-framework, W'-framework or helm. And vice versa.

(3). H is a maximal 2-co-belt. It is easy to see that an argument similar to that used in (2). can be employed to deduce contradictions.

(4). H is a maximal 1-bi-fan. If H' is also a maximal 1-bi-fan F', it is easy to see that this is true only if $F = F'$ holds. Obviously, it is impossible that the inner edge x_2x_3 of H is an inner edge of the following subgraphs: W-framework, W'-framework or helm.

(5). H is a W-framework, or a W'-framework, or a helm. Obviously, no matter whatever H' is, we always can deduce contradictions. The details are omitted, and the proof is complete. □

4. Preliminary results

In order to obtain the sharp lower bound for the number of removable edges in a 4-connected graph, we need to prove the following preliminary results.

Theorem 4.1. *Let G be a 4-connected graph and F a maximal l-bi-fan of G with $l \geqslant 2$. Then, there exists an edge e' in F such that $e' \in E_R(G)$ and $e_R(G) \geqslant e_R(G \ominus e') + 1$.*

Proof. Let F be defined as in Definition 3.2. First, we claim that $d(a) \geqslant 5, d(b) \geqslant 5$. Otherwise, we may assume that $d(a) = 4$ and let $\Gamma_G(a) = \{x_2, x_3, x_4, v\}$. Obviously, $v \neq b$, otherwise, $\{x_2, x_4, b\}$ would be a 3-vertex-cut of G, a contradiction. Let $A = \{a, x_3\}, S = \{x_2, x_4, v\}, e = bx_3, B = G - e - A - S$, then $(bx_3, S; A, B)$ is a separating group of G, and therefore, $bx_3 \in E_N(G)$, which contradicts to that F is l-bi-fan.

Let $e' = ax_3, H = G \ominus e'$. We will show that for any edge $e \neq x_2x_4$ in H, if $e \in E_R(H)$, then we have $e \in E_R(G)$.

By contradiction. Assume that there exists an edge $e \in E_R(H)$, but $e \in E_N(G)$. Let $e = xy$. Since $xy \in E_N(G)$, from Theorem 2.1 we can take its corresponding separating group $(e, T; C, D)$ such that $x \in C, y \in D$. We distinguish the following cases to proceed the proof:

Case 1: $a, x_3 \in T$.

Since $d(x_3) = 4$ and $ax_3 \in E(G)$, we have that $|\Gamma_G(x_3) \cap C| = 1$ or $|\Gamma_G(x_3) \cap D| = 1$. Without loss of generality, we may assume that $|\Gamma_G(x_3) \cap C| = 1$. Let $\Gamma_G(x_3) \cap C = \{v_1\}, T = \{a, x_3, w\}$. If $|C| \geqslant 3$, let $T' = \{a, v_1, w\}$, $C' = C - \{v_1\}$ and $D' = H - xy - T' - C'$. We claim that $v_1 \neq x$. Otherwise, we have that $\{a, w, v_1\}$ is a 3-vertex-cut of G, which contradicts to that G is 4-connected. It is easy to see that $(e, T'; C', D')$ is a separating group of H, and therefore $e \in E_N(H)$, a contradiction. If $|C| = 2$, then $v_1x \in E(G)$. Since $d(b) \geqslant 5$ and obviously $v_1 \neq b$, we have $v_1 \in \{x_2, x_4\}$. If $v_1 = x_2$, then $x = x_1$. Since $\Gamma_G(x_2) = \{b, x_1, x_3, a\}$, we have that $w = b$ and $\Gamma_G(x_1) = \{a, b, x_2, y\}$. Obviously, $\{ax_1, bx_1\} \subset E_R(G)$ and $x_1y \in E_N(G)$, which contradicts to the definition of a maximal l-bi-fan of G. If $v_1 = x_4$, then $w = b$, and therefore

$\Gamma_G(x) = \{a, b, x_4, y\}$, and so $x = x_5$. Let $C' = \{x_4, x\}, e = xy, T' = \{a, b, x_2\}$, $D' = H - e - C' - T'$. Then, we have that $(e, T'; C', D')$ is a separating group of H, and so $e \in E_N(H)$, which contradicts to that $e \in E_R(H)$.

Case 2: $a \in T, x_3 \in C$.

So, $\Gamma_G(x_3) = \{a, b, x_2, x_4\}$. If $|C| \geqslant 3$, then it is easy to see that $(e, T; C - \{x_3\}, D)$ is a separating group of H, and hence $e \in E_N(H)$, which contradicts to that $e \in E_R(H)$. Therefore, $|C| = 2$, and so $x \in \Gamma_G(x_3)$. If $x = b$, then $T = \{a, x_2, x_4\}$, $\Gamma_G(b) = \{a, x_2, x_3, x_4, y\}, \Gamma_G(x_2) \cap D = \{x_1\}$. Since $x_1 x_4 \notin E(G)$ and $x_1 \neq y$, we have that $|D| \geqslant 3$. Let $T' = \{a, x_1, x_4\}, D' = D - \{x_1\}, C' = H - xy - T' - D'$, then $(xy, T'; C', D')$ is a separating group of H, and so $e \in E_N(H)$, a contradiction. If $x = x_2$, then $y = x_1$. Obviously, if we let $e = x_2 x_1, C' = \{x_2, x_4\}$, $T' = \{a, b, x_5\}, D' = H - e - C' - T'$, then $(e, T'; C', D')$ is a separating group of H, and so $x_2 x_1 \in E_N(H)$, a contradiction. If $x = x_4$, then we have that $y = x_5$. Let $C' = \{x_2, x_4\}, T' = \{a, b, x_1\}, D' = H - x_4 x_5 - T' - C'$, then $(x_4 x_5, T'; C', D')$ is a separating group of H, and so $x_4 x_5 \in E_N(H)$, a contradiction to the assumption.

Case 3: $a \in C, x_3 \in T$.

If $|C| = 2$, then $a = x$, and so $C - \{a\} = \{x_2\}$ or $C - \{a\} = \{x_4\}$. If $C - \{a\} = \{x_2\}$, then $b \in T$. Since $x_3 x_4 \in E_N(G)$, from Theorem 2.4 we have $x_4 \notin T$. If $x_4 \in D - \{y\}$, then $a x_4 \notin E(G)$, a contradiction. If $C - \{a\} = \{x_4\}$, a similar argument can lead to a contradiction, and therefore $|C| \geqslant 3$. Since $a \in C$, we have that $x_2, x_4 \in C \cup T$. Noticing that $\Gamma_G(x_3) \cap D \neq \emptyset$, we have $b \in D$, and so $x_2, x_4 \in T$. Here, $\{x_2, x_4, x\}$ is a 3-vertex-cut of H, a contradiction.

Case 4: $a, x_3 \in C$.

Obviously, here we have that $|C| \geqslant 3$, a similar argument can lead to $e \in E_N(H)$ if $e \in E_N(G)$.

Based on the above arguments, we know that if $e \in E_R(H)$ and $e \neq x_2 x_4$, then $e \in E_R(G)$. Noticing that $a x_3, b x_3 \in E_R(G)$, but $a x_3, b x_3 \notin E(H)$, we have that $e_R(G) \geqslant e_R(G \ominus e) + 1$. The proof is now complete. $\square$

Theorem 4.2. *Let G be a 4-connected graph and L a maximal l-belt of G with $l \geqslant 3$. Then, there exists an edge e' in G such that $e_R(G) \geqslant e_R(G \ominus e') + 2$.*

Proof. Let L be defined as in Definition 3.3. Take $e' = x_3 y_3$ and let $H = G \ominus e'$. Then, we delete three removable edges $y_2 x_3, y_3 x_3, y_3 x_4$ from G and add three edges $y_2 x_4, x_2 x_4, y_2 y_4$ to get H. Let $A' = \{y_2, x_2\}, e_1 = y_2 y_4, S' = \{x_1, y_1, x_4\}$ and $B' = G - e_1 - S' - A'$, then $(e_1, S'; A', B')$ is a separating group of H, and hence $y_2 y_4 \in E_N(G)$. A similar argument can lead to $x_2 x_4 \in E_N(H)$. Here, we only need to show that for any $e \in E(H)$ and $e \neq y_2 x_4$, if $e \in E_R(H)$ then $e \in E_R(G)$.

By contradiction. Assume that there exists an edge $e \in E_R(H)$, but $e \in E_N(G)$. Let $e = xy$. From Theorem 2.1 we take its corresponding separating group $(e, S; A, B)$ such that $x \in A, y \in B$. Next we will distinguish the following cases to proceed the proof:

Case 1: $x_3, y_3 \in S$.

Let $S=\{x_3,y_3,w\}, w\in G$ and $U=\{x_2,x_4,y_2,y_4\}$. From $\Gamma_G(x_3)=\{x_2,x_4,y_2,y_3\}$ and $\Gamma_G(y_3)=\{x_3,x_4,y_2,y_4\}$, we claim that $|A\cap U|=2=|B\cap U|$. Otherwise, we may assume that $|A\cap U|=1$. Let $A\cap U=\{v_1\}$, then $\{x,v_1,w\}$ would be a 3-vertex-cut of G, which contradicts to that G is 4-connected. If $|A|=3$, since $l\geqslant 3$, obviously we have that $|G|\geqslant 10$, and so $|B|\geqslant 4$. Let $B\cap U=\{v_1,v_2\}$. Then, if we let $S_1=\{v_1,v_2,w\}$, $B_1=B-\{v_1,v_2\}, A_1=H-e-S_1-B_1$, then $(e,S_1;A_1,B_1)$ is a separating group of H, and so $e\in E_N(H)$, a contradiction to the assumption. If $|A|\geqslant 4$, let $A\cap U=\{u_1,u_2\}, S_1=\{u_1,u_2,w\},\ A_1=A-\{u_1,u_2\}, B_1=H-e-S_1-A_1$. Then $(e,S_1;A_1,B_1)$ is a separating group of H, and so $e\in E_N(H)$, which contradicts to the assumption.

Case 2: $x_3\in A, y_3\in S$.

Subcase 2.1: If $|A|=2$, then $x\in\Gamma_G(x_3)$. If $x=x_2$, then $S=\{y_2,y_3,x_4\}$. Since $x_2y_3, x_2x_4\notin E(G)$, we have that $d(x_2)<4$, a contradiction. If $x=x_4$, a similar argument can lead to $d(x_4)<4$, a contradiction. If $x=y_2$, then $y=y_1$. Let $A_1=\{y_2,x_4\}, e=y_1y_2, S_1=\{x_2,x_5,y_4\}$, $B_1=H-e-A_1-S_1$, then $(e,S_1;A_1,B_1)$ is a separating group of H, and so $e\in E_N(H)$, which contradicts to the assumption.

Subcase 2.2: If $|A|\geqslant 3$, since $x_3\in A$, it is easy to see that $B\cap\Gamma_G(y_3)=\{y_4\}$. If $|B|\geqslant 3$, let $B_1=B-\{y_4\}, S_1=\{y_4\}\cup S-\{y_3\}, A_1=H-e-S_1-B_1$. Then $(e,S_1;A_1,B_1)$ is a separating group of H, and so $e\in E_N(H)$. If $|B|=2$, since $\Gamma_G(y_4)=\{y_3,y_5,x_4,x_5\}$, then we have $y\in\{x_4,x_5,y_5\}$. If $y=x_4$, then this is true only if $x=x_3$ holds, a contradiction. If $y=x_5$, since $y_3x_5\notin E(G)$, we have that $d(x_5)=4$ and $S=\{y_3,y_5,x_4\}$. Let $A_1=A-\{y_2\}, S_1=\{y_2,y_5,x_4\},\ B_1=H-e-S_1-A_1$, then $(e,S_1;A_1,B_1)$ is a separating group of H, and hence $e\in E_N(H)$. If $y=y_5$, then $S=\{x_4,x_5,y_3\}$. Note that $y_3y_5, x_4y_5\notin E(G)$. So, $d(y_5)<4$, a contradiction.

To sum up, from the above arguments we know that in Case 2 we always have $e\in E_N(H)$.

Case 3: $x_3\in S, y_3\in A$.

By symmetry, an argument analogous to that used in Case 2 can lead to that $e\in E_N(H)$.

Case 4: $x_3,y_3\in A$.

If $|A|\geqslant 4$, obviously, $e\in E_N(H)$, a contradiction to the assumption. So, $|A|\leqslant 3$. Obviously, $x_3\neq x, y_3\neq x$. Therefore, we have that $|A|=3$. Since A is a connected subgraph of G, we may assume that $x_3x\in E(G)$. If $x=x_4$, then $xy=x_4x_5$. Let $S_1=\{y_1,y_4,x_2\}, A_1=\{y_2,x_4\}, B_1=H-e-S_1-A_1$, then $(e,S_1;A_1,B_1)$ is a separating group of H, and so $e\in E_N(H)$. If $x=y_2$, then $y=y_1$. Let $e=y_2y_1, A_1=\{y_2,x_4\}, S_1=\{x_2,x_5,y_4\}, B_1=H-e-S_1-A_1$, then $(e,S_1;A_1,B_1)$ is a separating group of H, and so $e\in E_N(H)$. If $x=x_2$, then $S=\{y_2,y_4,x_4\}$. It is easy to see that $d(x_2)<4$, a contradiction.

Based on all the above arguments, we have that $E_R(H)\subseteq E_R(G)$ except the edge y_2x_4. Noticing that $y_2x_3, x_3y_3, x_4y_3\in E_R(G)$, we have that $e_R(G)\geqslant e_R(G\ominus e)+2$. The proof is now complete. $\square$

Lemma 4.3. *Let G be a 4-connected graph and $(xy,S;A,B)$ a separating group of G such that $x\in B, y\in A$. If there exists another edge $yz\in E_N(G)$ and its corresponding* *separating group* $(yz, S'; A', B')$ *such that* $y \in A', z \in B'$ *which satisfy the following conditions*:

(i) $A \cap A' = \{y\}, A \cap B' = \{z\}, A \cap S' = \{a\}, A' \cap S = \{b\}, B' \cap S = \{u, v\}$ *such that* $a, b, u, v \in G$,
(ii) $\{zu, zv\} \cap E_N(G) \neq \emptyset, ab \in E_N(G)$,

then we have that au, av *cannot belong to* $E(G)$ *simultaneously*.

Proof. By contradiction. Assume that $au, av \in E(G)$. Without loss of generality, we may assume that $zu \in E_N(G)$. So, there is a corresponding separating group $(zu, T_1; C_1, D_1)$ such that $z \in C_1, u \in D_1$. Then, we have that $z \in C_1 \cap B', u \in B' \cap D_1$. Since $azua$ is a 3-cycle of G, we have $a \in T_1$, and so $a \in S' \cap T_1$. Let

$$Y_1 = (A' \cap T_1) \cup (S' \cap T_1) \cup (C_1 \cap S'),$$

$$Y_2 = (C_1 \cap S') \cup (S' \cap T_1) \cup (B' \cap T_1),$$

$$Y_3 = (B' \cap T_1) \cup (S' \cap T_1) \cup (S' \cap D_1),$$

$$Y_4 = (D_1 \cap S') \cup (S' \cap T_1) \cup (A' \cap T_1).$$

Obviously, $y \in A' \cap C_1$ or $y \in A' \cap T_1$. Next we will distinguish the following cases to proceed the proof.

Case 1: If $y \in A' \cap C_1$, then Y_1 is a vertex-cut of $G - yz$. Since G is 4-connected, we have that $|Y_1| \geqslant 3$. By a similar argument, we can deduce that $|Y_3| \geqslant 3$. Since $|Y_1| + |Y_3| = |S'| + |T_1| = 6$, we have that $|Y_1| = |Y_3| = 3$, and so $|A' \cap T_1| = |S' \cap D_1|, |S' \cap C_1| = |B' \cap T_1|$. Since $a \in S' \cap T_1$ and $ab \in E_N(G)$, from Theorem 2.4 we have that $b \notin T_1$ and $b \notin S'$. Since $by \in E(G)$, we have that $b \in A' \cap C_1$. From $zv \in E(G)$ and $v \in B'$, we know that $v \in B' \cap (C_1 \cup T_1)$. Hence, we have that $|A' \cap T_1| = |S' \cap D_1| = 0, 1$ or 2.

Now we discuss the following subcases:

Subcase 1.1: If $|A' \cap T_1| = |D_1 \cap S'| = 2$, then noticing that $|T_1| = |S'| = 3$ and $a \in S' \cap T_1$, we have that $|S' \cap C_1| = |B' \cap T_1| = 0$. Since $avza$ is a 3-cycle of G, we have that $v \in B' \cap C_1$, and so $|B' \cap C_1| \geqslant 2$. Then, $\{a, z\}$ would be a 2-vertex-cut of G, which contradicts to that G is 4-connected.

Subcase 1.2: If $|A' \cap T_1| = |D_1 \cap S'| = 1$, then $|S' \cap T_1| \leqslant 2$. First, we claim that $B' \cap D_1 = \{u\}$. Otherwise, if $|B' \cap D_1| \geqslant 2$, since $\Gamma_G(a) = \{y, z, u, v, b\}$, by the foregoing argument we have that $\Gamma_G(a) \cap (B' \cap D_1) = \{u\}$. Then, $\{u\} \cup (Y_3 - \{a\})$ would be a 3-vertex-cut of G, a contradiction. Hence, $D_1 \cap B' = \{u\}$. Let $D_1 \cap S' = \{u_1\}$. If $S \cap T_1 = \{a\}$, then $|Y_4| = 3$. Since G is 4-connected, we have that $D_1 \cap A' = \emptyset$. Then, $u_1 \in \Gamma_G(a)$. However, it is easy to see that $u_1 \notin \{y, z, b, u, v\}$, a contradiction. Therefore, $|S' \cap T_1| = 2$. It is easy to see that $\Gamma_G(a) \cap (A' \cap D_1) = \emptyset$. If $A' \cap D_1 \neq \emptyset$, then $Y_4 - \{a\}$ would be a 3-vertex-cut of G, a contradiction. If $A' \cap D_1 = \emptyset$, then it is easy to see that $au_1 \in E(G)$. However, $u_1 \notin \{b, u, v, y, z\}$, a contradiction.

Subcase 1.3: If $|A' \cap T_1| = |D_1 \cap S'| = 0$, since D_1 is a connected subgraph of G, we have that $A' \cap D_1 = \emptyset$. From $|D_1| \geqslant 2$, we have that $|D_1 \cap B'| \geqslant 2$. By an analogous argument we can deduce that $\Gamma_G(a) \cap (D_1 \cap B') = \{u\}$. Since $|Y_3| = |T_1| = 3$, $\{u\} \cup (Y_3 - \{a\})$ would be a 3-vertex-cut, a contradiction.

Case 2: $y \in A' \cap T_1$.

Since $yz \in E_N(G)$, from Theorem 2.2 we have that $|C_1| = 2$. Since C_1 is a connected subgraph of G, we have that $A' \cap C_1 = \emptyset$. If $S' \cap C_1 \neq \emptyset$, from $|C_1| = 2$ we have that $|S' \cap C_1| = 1$. Since $a \in S' \cap T_1$, we have that $|D_1 \cap S'| \leqslant 1$. Since Y_3 is a vertex-cut of $G - zu$, we have that $|Y_3| \geqslant 3$, and so $|B' \cap T_1| \geqslant 1$. Noticing that $|T_1| = 3$, we have that $A' \cap T_1 = \{y\}$ and $|Y_4| = 3$. Since G is 4-connected, we have that $A' \cap D_1 = \emptyset$, and therefore, we have that $A' = \{y\}$, which contradicts to that $|A'| \geqslant 2$. If $S' \cap C_1 = \emptyset$, then $|B' \cap C_1| = 2$. Since $A' \cap T_1 \neq \emptyset$, we have that $|Y_2| = |T_1 \cap (B' \cup S')| \leqslant 2$, and so $\{z\} \cup Y_2$ would be a vertex-cut of G. However, $|\{z\} \cup Y_2| < 4$, a contradiction.

From all the above arguments we have that au, av cannot belong to $E(G)$ simultaneously. The proof is now complete. □

A 4-connected graph G is said to have property ($\bigstar$) if there does not exist any edge $xy \in E_R(G)$ such that both $d(x) \geqslant 5$ and $d(y) \geqslant 5$.

Theorem 4.4. *Let G be a 4-connected graph with property ($\bigstar$), $|G| \geqslant 8$, and C' be a cycle of G. If C' does not contain any removable edges of G, then G has one of the following structures as its subgraph: l-belt, l-bi-fan ($l \geqslant 1$), W-framework, W'-framework or helm, such that it intersects C' at its some inner edge*(s).

Proof. For every edge $e = xy$ in C', from Theorem 2.1 there exists a separating group $(e, S; A, B)$ of G, in which we always choose A and B such that $\min\{|A|, |B|\}$ is as small as possible. Without loss of generality, we may assume $|A| \leqslant |B|$ such that $y \in A, x \in B$. Then, we take $f = yz \in E(C'), z \neq x$, and its corresponding separating group $(f, T; C, D)$ such that $y \in C, z \in D$ in G. Let

$$X_1 = (S \cap C) \cup (S \cap T) \cup (A \cap T),$$

$$X_2 = (A \cap T) \cup (S \cap T) \cup (S \cap D),$$

$$X_3 = (S \cap D) \cup (S \cap T) \cup (B \cap T),$$

$$X_4 = (B \cap T) \cup (S \cap T) \cup (S \cap C).$$

It is easy to see that the edge $e = xy$ is the unique edge connecting A and B, and the edge $f = yz$ is the unique edge connecting C and D, and so $x \notin D, z \notin B$. Since X_1 is a vertex-cut of $G - yx - yz$ and G is 4-connected, we have that $|X_1| \geqslant 2$.

Next we will distinguish the following cases to proceed the proof:

Case 1: $x \in B \cap C, z \in D \cap S$.

From Theorem 2.2 we have that $|A| = 2$. Since $A \cap C \neq \emptyset$ and A is a connected subgraph of G, we have that $A \cap D = \emptyset$, and so $|A \cap T| \leqslant 1$. If $|A \cap T| = 0$, then $|A \cap C| = 2$. Since $S \cap D \neq \emptyset$, by noticing that $|S| = 3$, we have that $|X_1| = |(S \cap C) \cup (S \cap T)| \leqslant 2$, and thus $X_1 \cup \{y\}$ would be a vertex-cut of G. However, $|X_1 \cup \{y\}| < 4$, which contradicts to that G is 4-connected. Therefore, $|A \cap T| = 1, A \cap C = \{y\}$. Since X_4 is a vertex-cut of $G - xy$, we have that $|X_4| \geqslant 3$, and hence

$|S\cap C|\geqslant|A\cap T|=1, |B\cap T|\geqslant|S\cap D|\geqslant 1$. So, $S\cap T=\emptyset$ or $|S\cap T|=1$. We claim that $S\cap T=\emptyset$. Otherwise, if $|S\cap T|=1$, then $|X_3|=3$, and so $B\cap D=\emptyset$. Since $A\cap D=\emptyset$, it is easy to see that $D=D\cap S=\{z\}$, which contradicts to that $|D|\geqslant 2$, and thus $S\cap T=\emptyset$. Noticing that $|T|=3$, we have that $|B\cap T|=2$. If $|S\cap C|=2$, then $|S\cap D|=1$. A similar argument can be used to get that $D=\{z\}$, which contradicts to that $|D|\geqslant 2$. Therefore, $|C\cap S|=1$, and so $|D\cap S|=2$.

Let $A\cap T=\{a\}, S\cap C=\{b\}, S\cap D=\{z,c\}$. It is easy to see that $\Gamma_G(y)=\{x,z,a,b\}$, $\Gamma_G(a)=\{y,z,b,c\}$. Next we will show that $ay, az, by\in E_R(G)$ by contradiction.

(1). Assume that $ay\in E_N(G)$ and we take a separating group $(ay, U; A', B')$ such that $a\in A', y\in B'$. Since $ayza, abya$ are 3-cycles of G, we have that $z, b\in U$. Since $yz\in E_N(G)$, from Theorem 2.2 we have that $|B'|=2$. Let $B'=\{v_1, y\}$, then byv_1b is a 3-cycle of G and $v_1\neq a$, and so this is true only if $v_1=x$ holds. However, $xz\notin E(G)$, and so $d(x)<4$, a contradiction.

(2). Assume that $az\in E_N(G)$ and we take its separating group $(az, U; A', B')$ such that $a\in A', z\in B'$ in G. Since $ayza$ is a 3-cycle of G, we have that $y\in U$. Since $yz\in E_N(G)$, from Theorem 2.2 we have that $|B'|=2$. Let $B'=\{z, v_1\}$, then yzv_1y is a 3-cycle of G and $v_1\neq a$, which is impossible to hold in G. Therefore, $az\in E_R(G)$.

(3). Assume that $by\in E_N(G)$. First, let $A'=C\cap(B\cup S), S'=\{y\}\cup(B\cap T)$, $B'=G-ab-A'-S'$, then $(ab, S'; A', B')$ is a separating group of G, and hence $ab\in E_N(G)$. Since $by\in E_N(G)$, we take its separating group $(by, U; A', B')$ such that $b\in A', y\in B'$. Since $abya$ is a 3-cycle of G, we have that $a\in S'$. Since $ab\in E_N(G)$, from Theorem 2.2 we have that $|A'|=2$. Let $A'=\{b, v_1\}$. Then abv_1a is a 3-cycle of G and $v_1\neq y$, which is impossible in G, and therefore, we have $by\in E_R(G)$.

Let $A'=\{a,y\}, S'=\{b,z,x\}$, $B'=G-ac-S'-A'$, then $(ac, S'; A', B')$ is a separating group of G, and so $ac\in E_N(G)$. It is easy to see that $(ab, B\cap T\cup\{y\})$ is a separating pair of G, so $ab\in E_N(G)$.

Obviously, yz is an inner edge of an l-belt or l-co-belt with $l\geqslant 1$, and so the conclusion holds.

Case 2: $z\in S\cap D, x\in B\cap T$.

From Theorem 2.2 we have that $|A|=|C|=2$. Since A and C are two connected subgraphs of G, we have that $A\cap D=\emptyset=B\cap C$. First, we claim that $|A\cap C|=1$. Otherwise, $|A\cap C|=2$, and so $A\cap T=\emptyset=S\cap C$. Since $B\cap T\neq\emptyset\neq S\cap D$, we have that $|X_1|=|S\cap T|\leqslant 2$, and so $X_1\cup\{y\}$ would be a vertex-cut of G. However, $|X_1\cup\{y\}|<4$, which contradicts to that G is 4-connected. Therefore, $|A\cap T|=1, |S\cap C|=1$. Second, we claim that $S\cap T=\emptyset$. Otherwise, $|S\cap T|=1$. Then, $|X_3|=3$, and so $B\cap D=\emptyset$. Hence, $D=D\cap S=\{z\}$, which contradicts to that $|D|\geqslant 2$. Therefore, we have that $|B\cap T|=|S\cap D|=2$.

Let $A\cap T=\{a\}, S\cap C=\{b\}, D\cap S=\{z,v\}, B\cap T=\{x,u\}$, then $\Gamma_G(y)=\{x,z,a,b\}, \Gamma_G(a)=\{x,z,b,v\}, \Gamma_G(b)=\{x,y,a,u\}$.

Next we will show $az\in E_R(G)$. By contradiction, assume that $az\in E_N(G)$ and we take the corresponding separating group $(az, U; A', B')$ such that $a\in A', z\in B'$. Since $azya$ is a 3-cycle of G, we have that $y\in U$. Since $yz\in E_N(G)$, from Theorem 2.2 we have that $|B'|=2$. Let $B'=\{z, v_1\}$, then yzv_1y is a 3-cycle of G and $v_1\neq a$, and so this is true only if $v_1=x$ holds. Since $bx\in E(G)$, we have $b\in U$. Then, $(U-\{y\})\cup\{a\}$

would be a 3-vertex-cut of G, a contradiction. Therefore, $az \in E_R(G)$ holds. By symmetry, we have that $bx \in E_R(G)$. Let $A' = \{a, y\}$, $S' = \{x, z, b\}$, $B' = G - av - S' - A'$, then $(av, S'; A', B')$ is a separating group of G, and so $av \in E_N(G)$. A similar argument can lead to $bu \in E_N(G)$.

Now we discuss the following subcases:

Subcase 2.1: $xz \notin E(G)$. We will show that $ay, by \in E_R(G)$. By contradiction, assume that $ay \in E_N(G)$ and we take its separating group $(ay, U; A', B')$ such that $a \in A', y \in B'$. Since $ayza$ is a 3-cycle of G, we have that $z \in U$. Since $yz \in E_N(G)$, from Theorem 2.2 we have that $|B'| = 2$. Let $B' = \{y, v_1\}$, then yzv_1y is a 3-cycle of G. Obviously, $v_1 \neq a$. Note that $xz \notin E(G)$, and so $v_1 \neq x$, which is impossible in G. Therefore, we have that $ay \in E_R(G)$. By symmetry, we have that $by \in E_R(G)$. It is easy to see that if $ab \in E_N(G)$, then G contains an l-belt or an l-co-belt with $l \geqslant 1$ such that yz is its an inner edge. If $ab \in E_R(G)$, then G contains a W-framework such that yz is its an inner edge. Therefore, the conclusion holds.

Subcase 2.2: $xz \in E(G)$. Since $xy, yz \in E_N(G)$, from Corollary 2.3 we have $xz \in E_R(G)$. Since G has property ($\bigstar$), we have that $d(x) = 4$ or $d(z) = 4$.

Subsubcase 2.2.1: $d(x) = 4, d(z) \geqslant 5$. Let $\Gamma_G(x) = \{y, z, b, w\}$. Since $|G| \geqslant 8$, we have that $B \cap D \neq \emptyset$, and so $w \in B \cap D$. Let $A' = \{x, y\}$, $U = \{w, z, b\}$, $B' = G - ay - U - A'$. Then $(ay, U; A', B')$ is a separating group of G, and so $ay \in E_N(G)$. We claim that $ab \in E_R(G)$. Otherwise, $ab \in E_N(G)$. Then, we take a separating group $(ay, T_1; C_1, D_1)$ of G such that $a \in C_1, y \in D_1$. Obviously, $z, b \in T_1$. Since $ab, yz \in E_N(G)$, from Theorem 2.2 we have that $|C_1| = |D_1| = 2$, which contradicts to that $|G| \geqslant 8$, and so $ab \in E_R(G)$. We claim that $by \in E_R(G)$. Otherwise, $by \in E_N(G)$, and we take its separating group $(by, T_1; C_1, D_1)$ such that $b \in C_1, y \in D_1$. Since $byxb$ is a 3-cycle of G, we have $x \in T_1$. Since $xy \in E_N(G)$, from Theorem 2.2 we have that $|D_1| = 2$. Let $D_1 = \{y, v_1\}$, then yxv_1y is a 3-cycle of G, and hence this is true only if $v_1 = z$ holds. However, $d(v_1) = 4$, which contradicts to that $d(z) \geqslant 5$. Therefore, $by \in E_R(G)$. Obviously, here xy, yz are inner edges of a W'-framework in G. The conclusion holds.

Subsubcase 2.2.2: $d(x) \geqslant 5, d(z) = 4$. By symmetry, from an argument similar to that used in Subsubcase 2.2.1 we can get the conclusion.

Subsubcase 2.2.3: $d(x) = d(z) = 4$. Let $\Gamma_G(x) = \{y, z, b, w\}$. Let $A' = \{x, y\}$, $U = \{w, z, b\}$, $B' = G - ay - U - A'$, then $(ay, U; A', B')$ is a separating group of G, and so $ay \in E_N(G)$. By symmetry, we have that $by \in E_N(G)$. Since $xy, yz \in E_N(G)$, from Corollary 2.3 we have that $ab, bx, xz, za \in E_R(G)$. Obviously, G contains a helm as a subgraph such that xy, yz are its inner edges. Therefore, the conclusion holds.

Case 3: $z \in A \cap D, x \in B \cap T$.

From Theorem 2.2 we have that $|C| = 2$. Since $|A| \leqslant |C|$, we have that $|A| = 2$, and hence $A = \{y, z\}$, $A \cap T = \emptyset$. Since $A \cap D \neq \emptyset$, we have that $|X_2| \geqslant 3$. Noticing that $|S| = 3$, we have that $|A \cap T| \geqslant |S \cap C|$, and so $|S \cap C| = 0$. Since C is a connected subgraph of G and $|C| = 2$, from $A = \{y, z\}$ we can get that $A \cap C = \{y\}$. Therefore, $C \cap S \neq \emptyset$, a contradiction. So, Case 3 cannot occur.

Case 4: $z \in A \cap D, x \in B \cap C$.

So, $A \cap D \neq \emptyset \neq B \cap C$, and therefore $|X_2| \geqslant 3, |X_4| \geqslant 3$. Since $|X_2| + |X_4| = |S| + |T| = 6$, we have that $|X_2| = |X_4| = 3$, and so $|A \cap T| = |S \cap C|, |B \cap T| = |S \cap D|$.

First, we claim that $A \cap D = \{z\}$. Otherwise, $|A \cap D| \geqslant 2$. Let $U' = X_2, A' = A \cap D, B' = G - yz - U' - A'$, then $(yz, U'; A', B')$ is a separating group of G, and $yz \in E(C')$, $|A'| < |A|$, which contradicts to that $|A|$ is as small as possible. Therefore, $A \cap D = \{z\}$. Since D is a connected subgraph of G and $|D| \geqslant 2$, we have that $D \cap S \neq \emptyset \neq B \cap T$, and so $|S \cap T| \leqslant 2$. If $|S \cap T| = 1$, we claim that $S \cap C \neq \emptyset \neq A \cap T$. Otherwise, $|X_1| = 1$. Obviously, $|A \cap C| \geqslant 2$, and so $\{y\} \cup (S \cap T)$ would be a 2-vertex-cut of G, a contradiction. Therefore, $|S \cap C| = |A \cap T| = 1, |D \cap S| = |B \cap T| = 1$, and hence $|X_3| = 3$. Then, we have that $B \cap D = \emptyset$ and $|D| = 2$. However, $|A| \geqslant 3$. Then, $|D| < |A|$, which contradicts to that $|A|$ is as small as possible. Therefore, $|S \cap T| = 0$ or $|S \cap T| = 2$.

Next we will show that $|S \cap T| \neq 0$. Assume that $|S \cap T| = 0$. Then, $|B \cap T| = |S \cap D| = 2$ and $|A \cap T| = |S \cap C| = 1$ must hold. We claim that $A \cap C = \{y\}$. Otherwise, $|A \cap C| \geqslant 2$. Then, $X_1 \cup \{y\}$ would be a 3-vertex-cut of G, which contradicts to that G is 4-connected, and so $d(y) = 4$. Let $A \cap T = \{a\}, S \cap C = \{b\}, S \cap D = \{u, v\}$. First, let $A' = \{a, z\}, S' = \{y\} \cup (S \cap D), B' = G - ab - S' - A'$, then $(ab, S'; A', B')$ is a separating group of G, and so $ab \in E_N(G)$. Second, we claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, we take the separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $y \in S'$. Since $yz \in E_N(G)$, from Theorem 2.2 we have that $|B'| = 2$, say $B' = \{z, v_1\}$. Then, zv_1yz is a 3-cycle of G and $v_1 \neq a$, which is impossible to hold, so $az \in E_R(G)$. Since C' is a cycle of G, we have that $\{zu, zv\} \cap E_N(G) \neq \emptyset$. From Lemma 4.3 we have that au, av cannot belong to $E(G)$ simultaneously. Without loss of generality, we may assume that $au \notin E(G)$. Let $S' = (S - \{u\}) \cup \{z\}$, $A' = A - \{z\}, B' = B \cup \{u\}$, then $(xy, S'; A', B')$ is a separating group of G, and $|A'| < |A|$, which contradicts to that $|A|$ is as small as possible. Therefore, $S \cap T \neq \emptyset$, and so $|S \cap T| = 2$. Then, we have that $|S \cap D| = |B \cap T| = 1, |A \cap T| = |S \cap C| = 0, A \cap C = \{y\}$.

Let $S \cap T = \{a, b\}, S \cap D = \{u\}$. It is easy to see that $\Gamma_G(y) = \{x, a, b, z\}, \Gamma_G(z) = \{y, a, b, u\}$.

First, we will show that the conclusion of the theorem holds if $az \in E_N(G)$. From Theorem 2.1 we take its corresponding separating group $(az, S_1; A_1, B_1)$ such that $a \in B_1, z \in A_1$. Since $ayza$ is a 3-cycle of G, we have $y \in S_1$, and so $y \in S_1 \cap C$, $a \in B_1 \cap T$. From Theorem 2.2 we have that $|A_1| = |D| = 2$. If $|A_1 \cap D| = 2$, since $S_1 \cap C \neq \emptyset$, then $|S_1 \cap T| \leqslant 2$, and so $\{z\} \cup (S_1 \cap T)$ would be a vertex-cut with cardinality less than 4, a contradiction. Therefore, $|A_1 \cap D| = 1$. Since $b \in T$ and $bz \in E(G)$, we have that $b \in A_1 \cap T$. Since D is a connected subgraph of G and $|D| = 2$, it is easy to see that $|D \cap S_1| = 1$. Since $zu \in E(G)$, we have that $D \cap S_1 = \{u\}$. We claim that $S_1 \cap T = \emptyset$. Otherwise, $|S_1 \cap T| = 1$. Then, $|S_1 \cap C| = |B_1 \cap T| = 1$. Obviously, $|(S_1 \cap C) \cup (S_1 \cap T) \cup (B_1 \cap T)| = 3$. Since G is 4-connected, we have that $B_1 \cap C = \emptyset$. Therefore, $|C| = |C \cap S_1| = 1$, which contradicts to that $|C| \geqslant 2$. Hence, $S_1 \cap T = \emptyset$, and therefore, $|S_1 \cap C| = |B_1 \cap T| = 2$. Here we need to discuss the following cases:

(1). If $d(y) = 4, d(a) \geqslant 5$, an argument similar to that used in Subsubcase 2.2.1 can lead to that G contains a W'-framework such that yz is its an inner edge. Then, the conclusion holds.

(2). If $d(y) = d(a) = 4$, an argument similar to that used in Subsubcase 2.2.3 can lead to that G contains a helm such that yz is its an inner edge. The conclusion holds.

If $bz \in E_N(G)$, by the symmetry of az and bz, a similar argument can be used to get the conclusion. Therefore, we may assume that $az, bz \in E_R(G)$.

Next we consider ay. Assume $ay \in E_N(G)$. From Theorem 2.1 we take its separating group $(ay, S_1;\ A_1, B_1)$ such that $a \in A_1, y \in B_1$. It is easy to see that $z \in S_1 \cap D$, $y \in B_1 \cap C$ and $a \in A_1 \cap T$. Since $ay, yz \in E_N(G)$, from Theorem 2.2 we have that $|C| = 2 = |B_1|$, and so $C = \{y, x\}$. By an argument analogous to that used in Case 2, we can get that $|B_1 \cap T| = |S_1 \cap C| = 1, B_1 \cap C = \{y\}$, $|A_1 \cap T| = |D \cap S_1| = 2$. Then, $S_1 \cap C = \{x\}$. Since $byzb$ is a 3-cycle of G, it is easy to see that $B_1 \cap T = \{b\}$ and $d(x) = d(b) = d(z) = 4$. Here we need to discuss the following cases:

(1). If $d(a) \geqslant 5$, an argument analogous to that used in Subsubcase 2.2.1 can lead to that G contains a W'-framework such that xy, yz are its inner edges. Then, the conclusion holds.

(2). If $d(a) = 4$, an argument analogous to that used in Subsubcase 2.2.3 can lead to that G contains a helm such that xy, yz are its inner edges. Then, the conclusion holds.

Thus, we may assume that $ay, by \in E_R(G)$. Then, according to the definition of the l-bi-fan, $(l \geqslant 1)$, G contains a l-bi-fan such that yz is its an inner edge. The proof is now complete. $\square$

Lemma 4.5. *Let G be a 4-connected graph with property $(\bigstar)$, and let $P = y_1y_2\cdots y_k$ be a path of $[E_N(G)]$ with $k \geqslant 3$ and take a set D such that $\emptyset \neq D \subset V(G)$. Suppose that $(y_1y_2, U'; X', Y')$ is a separating group of G such that $y_1 \in Y', y_2 \in X'$ and $D \cap Y' \neq \emptyset$. We choose $i \in \{1, 2, \ldots, k\}$ and a separating group $(y_iy_{i+1}, S; A, B)$ satisfying $y_i \in B, y_{i+1} \in A, D \cap B \neq \emptyset$ such that $|A|$ is as small as possible. If $i \leqslant k-2$, we take another separating group $(y_{i+1}y_{i+2}, S'; A', B')$ such that $y_{i+1} \in B', y_{i+2} \in A'$, Then, one of the following conclusions holds*:

- (i) $A \cap B' = \{y_{i+1}\}, A \cap A' = \{y_{i+2}\}, A \cap S' = \{a\}$, $B' \cap S = \{b\}, S \cap S' = \emptyset, y_i \in B \cap B'$, $|B \cap S'| = |A' \cap S| = 2$, $A' \cap S = \{u, v\}$, where $y_{i+2}u, y_{i+2}v, y_{i+2}a \in E_R(G)$ and $a, b, u, v \in G$.
- (ii) $A \cap A' = \{y_{i+2}\}, y_{i+1} \in A \cap B', S \cap S' = \emptyset = A' \cap B$, $B \cap S' = \{d\} = D \cap B, D \cap B' = \emptyset, A' \cap S = \{c\}$, $|B' \cap S| = |A \cap S'| = 2, y_i \in B \cap B'$, where $d, c \in G$.
- (iii) $A \cap A' = \{y_{i+2}\}$, $y_{i+1} \in A \cap B', S \cap S' = \{w\}$, $D \cap B = \{d\} = B \cap S', D \cap B' = \emptyset = B \cap A', A' \cap S = \{c\}$, $|B' \cap S| = |A \cap S'| = 1, y_i \in B \cap B'$, where $d, c, w \in G$.
- (iv) G contains one of the following structures: l-belt, $(l \geqslant 1)$, helm, W-framework, W'-framework, l-bi-fan, $(l \geqslant 1)$, as its subgraph, such that it intersects P at its some inner edge(s).

Proof. Let

$$X_1 = (A \cap S') \cup (S \cap S') \cup (B' \cap S),$$

$$X_2 = (A \cap S') \cup (S \cap S') \cup (A' \cap S),$$

$$X_3 = (A' \cap S) \cup (S \cap S') \cup (B \cap S'),$$

$$X_4 = (B' \cap S) \cup (S \cap S') \cup (B \cap S').$$

We will distinguish the following cases to proceed the proof.

Case 1: $y_i \in B \cap B', y_{i+2} \in A \cap A'$.

Since $B \cap B' \neq \emptyset$, then X_4 is a vertex-cut of $G - y_i y_{i+1}$. Since G is 4-connected, we have that $|X_4| \geqslant 3$. By a similar argument we can deduce that $|X_2| \geqslant 3$. Since $|X_2| + |X_4| = |S| + |S'| = 6$, we have that $|X_2| = |X_4|$, and so $|A \cap S'| = |B' \cap S|$, $|A' \cap S| = |B \cap S'|$.

First, we claim that $A' \cap (B \cup S) \neq \emptyset$. Otherwise, $A' \cap (B \cup S) = \emptyset$. Since $|A' \cap S| = 0$, we have that $S' \cap B = \emptyset$. Since B is a connected subgraph of G, we have that $B = B \cap B'$. Therefore, we have that $\emptyset \neq D \cap B = D \cap (B \cap B') \subset D \cap B'$. For the separating group $(y_{i+1}y_{i+2}, S'; A', B')$ of G, we have that $y_{i+1} \in B'$, $y_{i+2} \in A', D \cap B' \neq \emptyset$, and $A' \subset A, |A'| < |A|$, which contradicts to that $|A|$ is as small as possible, and so $A' \cap (B \cup S) \neq \emptyset$. Since A' is a connected subgraph of G and $A \cap A' \neq \emptyset \neq A' \cap (B \cup S)$, we have that $A' \cap S \neq \emptyset \neq B \cap S'$. If $|A' \cap S| = 3$, then $|X_1| = 0$, and so $\{y_i, y_{i+2}\}$ would be a 2-vertex-cut of G, a contradiction. Therefore, $|A' \cap S| = 2$ or $|A' \cap S| = 1$.

Next we will discuss the following subcases.

Subcase 1.1: $|A' \cap S| = |S' \cap B| = 2$. Let $A' \cap S = \{u, v\}$. Since G is 4-connected and X_1 is a vertex-cut of $G - y_i y_{i+1} - y_{i+1} y_{i+2}$, we have that $|X_1| \geqslant 2$. Noticing that $|S| = |S'| = 3$, it is easy to see that $|A \cap S'| = |B' \cap S| = 1, |S \cap S'| = 0$. Let $A \cap S' = \{a\}, B' \cap S = \{b\}$. First, we claim that $A \cap B' = \{y_{i+1}\}$. Otherwise, $|A \cap B'| \geqslant 2$, and so $X_1 \cup \{y_{i+1}\}$ would be a 3-vertex-cut of G, a contradiction. Second, we claim that $A \cap A' = \{y_{i+2}\}$. Otherwise, $|A \cap A'| \geqslant 2$. Let $A_1 = A \cap A'$, $S_1 = X_2, B_1 = G - y_{i+1}y_{i+2} - S_1 - A_1$. It is easy to see that $D \cap B_1 \neq \emptyset$. Then, $(y_{i+1}y_{i+2}, S_1; A_1, B_1)$ is a separating group of G such that $y_{i+1} \in B_1, y_{i+2} \in A_1$ and $D \cap B_1 \neq \emptyset$. However, $|A_1| < |A|$, which contradicts to that $|A|$ is as small as possible. Therefore, $A \cap A' = \{y_{i+2}\}$. Obviously, (ab, S_1) is a separating pair of G such that $S_1 = \{y_{i+1}, u, v\}$, and so $ab \in E_N(G)$. We claim that $y_{i+2}u, y_{i+2}v \in E_R(G)$. Otherwise, $\{y_{i+2}u, y_{i+2}v\} \cap E_N(G) \neq \emptyset$. From Lemma 4.3 we have that au, av cannot belong to $E(G)$ simultaneously. Without loss of generality, we may assume that $au \notin E(G)$. Let $A_1 = A - \{y_{i+2}\}, S_1 = \{y_{i+2}\} \cup (S - \{u\}), B_1 = G - y_i y_{i+1} - S_1 - A_1$, then $(y_i y_{i+1}, S_1; A_1, B_1)$ is a separating group of G such that $D \cap B_1 \neq \emptyset$. However, $|A_1| < |A|$, which contradicts to that $|A|$ is as small as possible. Therefore, $y_{i+2}u, y_{i+2}v \in E_R(G)$. We claim that $ay_{i+2} \in E_R(G)$. Otherwise, $ay_{i+2} \in E_N(G)$, and we take its separating group $(ay_{i+2}, T'; C', D')$ such that $a \in C', y_{i+2} \in D'$. Since $ay_{i+1}y_{i+2}a$ is a 3-cycle of G, we have that $y_{i+1} \in T'$. Since $y_{i+1}y_{i+2} \in E_N(G)$, from Theorem 2.2 we have that $|D'| = 2$. Let $D' = \{y_{i+2}, v_1\}$, then $v_1 y_{i+1} y_{i+2} v_1$ is a 3-cycle of G and $v_1 \neq a$. Obviously, it is impossible to hold in G, and hence, $ay_{i+2} \in E_R(G)$. Then, the conclusion (i) holds.

Subcase 1.2: $|A' \cap S| = |B \cap S'| = 1$.

Let $A' \cap S = \{c\}, B \cap S' = \{d\}$. Then, we will discuss the following subsubcases.

Subsubcase 1.2.1: $|S \cap S'| = 0, |B' \cap S| = |A \cap S'| = 2$.

It is easy to see that $|X_3| = 2$. Since G is 4-connected, we have that $A' \cap B = \emptyset$ and $|X_2| = 3$. We claim that $A \cap A' = \{y_{i+2}\}$. Otherwise, $|A \cap A'| \geqslant 2$. Let $A_1 = A \cap A', S_1 = X_2, B_1 = G - y_{i+1}y_{i+2} - S_1 - A_1$, then $(y_{i+1}y_{i+2}, S_1; A_1, B_1)$ is a separating group. Obviously, $D \cap B_1 \neq \emptyset$ and $|A_1| < |A|$, which contradicts to that $|A|$ is as small as possible. Therefore, $A \cap A' = \{y_{i+2}\}$, and so $A' = \{y_{i+2}, c\}, |A'| = 2 < |A|$. By the minimum property of $|A|$, we have that $B' \cap D = \emptyset$, and therefore, $B \cap D = B \cap S' = \{d\}$ and $|B \cap D| = 1$. Then, conclusion (ii) holds.

Subsubcase 1.2.2: $|S \cap S'| = 1, |B' \cap S| = |A \cap S'| = 1$.

Let $A' \cap S = \{c\}$, $S \cap S' = \{w\}, B \cap S' = \{d\}$. Since $|X_3| = 3 < 4$, we have that $B \cap A' = \emptyset$. An argument similar to that used in Subsubcase 1.2.1 can lead to that $A \cap A' = \{y_{i+2}\}, y_{i+1} \in A \cap B'$. Since $|A'| = 2 < |A|$, by an argument similar to that used in Subsubcase 1.2.1, we have that $B' \cap D = \emptyset$, and so $D \cap B = B \cap S' = \{d\}$. Then, conclusion (iii) holds.

Subsubcase 1.2.3: $|S \cap S'| = 2, |B' \cap S| = |A \cap S'| = 0$,

Let $S \cap S' = \{a, b\}$. We claim that $A \cap B' = \{y_{i+1}\}$. Otherwise, $|A \cap B'| \geqslant 2$. Then, $\{y_{i+1}, a, b\}$ would be a 3-vertex-cut of G, which contradicts to that G is 4-connected. It is easy to see that $|X_2| = 3$. An argument similar to that used in Subsubcase 1.2.1 can lead to that $A \cap A' = \{y_{i+2}\}$. From Corollary 2.3 we have that $\{ay_{i+1}, ay_{i+2}\} \cap E_R(G) \neq \emptyset$, $\{by_{i+1}, by_{i+2}\} \cap E_R(G) \neq \emptyset$. Next we discuss the following cases.

(1). If $ay_{i+2} \in E_N(G)$, then $A' \cap B = \emptyset$ and we take the corresponding separating group $(ay_{i+2}, S_1; A_1, B_1)$ such that $y_{i+2} \in A_1, a \in B_1$. Since $ay_{i+1}y_{i+2}a$ is a 3-cycle of G, we have that $y_{i+1} \in S_1$, and so $y_{i+1} \in S_1 \cap B'$. Since $a \in S'$, we have that $a \in S' \cap B_1$. Obviously, $d(y_{i+1}) = d(y_{i+2}) = 4$. By an argument analogous to that used in Subcase 2.2 of Theorem 4.4, we can get that $y_{i+1}y_{i+2}$ is an inner edge of a W'-framework or a helm, and so conclusion (iv) holds. For $by_{i+2} \in E_N(G)$, we may employ a similar argument to get conclusion (iv). Hence, we may assume that $ay_{i+2}, by_{i+2} \in E_R(G)$.

(2). If $ay_{i+1} \in E_N(G)$, we take the corresponding separating group $(ay_{i+1}, S_1; A_1, B_1)$ such that $y_{i+1} \in A_1, a \in B_1$. Then, $y_{i+1} \in A_1 \cap B', a \in B_1 \cap S'$. Since $ay_{i+1}y_{i+2}a$ is a 3-cycle of G, we have that $y_{i+2} \in S_1$, and so $y_{i+2} \in A' \cap S_1$. Since $ay_{i+2} \in E(G)$ and $d(y_{i+2}) = 4$, by an argument analogous to that used in Subcase 2.2 of Theorem 4.4 we can get that $y_{i+1}y_{i+2}$ is an inner edge of a W'-framework or a helm, and hence, conclusion (iv) holds. For $by_{i+1} \in E_N(G)$, we may employ a similar argument to get conclusion (iv).

Based on the above arguments, we may assume that $ay_{i+1}, by_{i+1}, ay_{i+2}, by_{i+2} \in E_R(G)$, and so G contains a l-bi-fan such that $y_{i+1}y_{i+2}$ is its an inner edge. Therefore, conclusion (iv) holds.

Case 2: $y_{i+2} \in A \cap A', y_i \in B \cap S'$.

Since $y_iy_{i+1} \in E_N(G)$, from Theorem 2.2 we have that $|B'| = 2$. Since B' is a connected subgraph of G, we have that $B \cap B' = \emptyset$. Because G is 4-connected and X_1 is a vertex-cut of $G - y_iy_{i+1} - y_{i+1}y_{i+2}$, we have that $|X_1| \geqslant 2$. A similar argument can lead to that $|X_2| \geqslant 3$. We claim that $A \cap B' = \{y_{i+1}\}$. If not, i.e., $|A \cap B'| = 2$, from $B \cap S' \neq \emptyset$ and $|S'| = 3$ we have that $|X_1| \leqslant 2$, and so $X_1 \cup \{y_{i+1}\}$ is a vertex-cut of G with cardinality less than 4, which contradicts to that G is 4-connected. Therefore, $|A \cap B'| = |B' \cap S| = 1$. If $|B \cap S'| = 1$, then $|X_3| = 3$, and so $A' \cap B = \emptyset$. Then, we have that $|B| = |B \cap S'| = 1$, which contradicts to that $|B| \geqslant 2$, and so $|B \cap S'| \geqslant 2$. If $|B \cap S'| = 3$, then we have that $A \cap S' = \emptyset = S \cap S'$, and so $|X_1| = 1$, which contradicts to that $|X_1| \geqslant 2$. Therefore, $|B \cap S'| = 2$ and $|S \cap S'| \leqslant 1$. If $|S \cap S'| = 1$, then $A \cap S' = \emptyset$ and $|A' \cap S| = 1$, and hence $|X_2| = 2$, which contradicts to that $|X_2| \geqslant 3$. Then, we can conclude that $S \cap S' = \emptyset$ and $|A \cap S'| = 1$. From $|S| = 3$ we know that $|A' \cap S| = 2, |X_2| = 3$. We claim that $A \cap A' = \{y_{i+2}\}$. If not, i.e., $|A \cap A'| \geqslant 2$, then we take $A_1 = A \cap A', S_1 = X_2, B_1 = G - y_{i+1}y_{i+2} - S_1 - A_1$, and so $(y_{i+1}y_{i+2}, S_1; A_1, B_1)$ is a separating group of G. It is easy to see that $B_1 \cap D \neq \emptyset$. However, we have that $|A_1| < |A|$, which contradicts to that $|A|$ is as small as possible. Therefore, $A \cap A' = \{y_{i+2}\}$.

Let $A \cap S' = \{a\}, B' \cap S = \{b\}$. Next we will show that $by_i, by_{i+1}, ay_{i+1} \in E_R(G)$ by contradiction.

(1). If $by_i \in E_N(G)$, we take its corresponding separating group $(by_i, T; C, K)$ of G such that $b \in C, y_i \in K$. Since $by_iy_{i+1}b$ is a 3-cycle of G, we have that $y_{i+1} \in T$. Since $y_iy_{i+1} \in E_N(G)$, from Theorem 2.2 we can get $|K| = 2$, say $K = \{y_i, v_1\}$. Then, $v_1y_{i+1}y_iv_1$ is a 3-cycle of G and $v_1 \neq b$, which is impossible in G, and hence $by_i \in E_R(G)$.

(2). If $by_{i+1} \in E_N(G)$, similarly we take its corresponding separating group $(by_{i+1}, T;\ C, K)$ of G such that $b \in C, y_{i+1} \in K$. It is easy to see that $\{a, y_i\} \subset T$. Since $y_iy_{i+1} \in E_N(G)$, from Theorem 2.2 we have that$|K| = 2$, say $K = \{y_{i+1}, v_1\}$. Then, $v_1 \in \Gamma_G(y_i) \cap \Gamma_G(y_{i+1}) \cap \Gamma_G(a)$, which is impossible in G, and so $by_{i+1} \in E_R(G)$.

(3). If $ay_{i+1} \in E_N(G)$, again similarly we take its corresponding separating group $(ay_{i+1}, T; C, K)$ such that $a \in C, y_{i+1} \in K$. Since $ay_{i+1}y_{i+2}a$ is a 3-cycle of G, we have $y_{i+2} \in T$. Since $y_{i+1}y_{i+2} \in E_N(G)$, from Theorem 2.2 we have that $|K| = 2$. Let $K = \{y_{i+1}, v_1\}$, then $y_{i+1}v_1y_{i+2}y_{i+1}$ is a 3-cycle of G, and $v_1 \neq a$, which is impossible in G, and so $ay_{i+1} \in E_R(G)$.

Let $A_1 = \{a, y_{i+2}\}, S_1 = S \cap A' \cup \{y_{i+1}\}$ and $B_1 = G - ab - S_1 - A_1$, then $(ab, S_1; A_1, B_1)$ is a separating group of G, and so $ab \in E_N(G)$.

Noticing that $d(b) = d(y_{i+1}) = 4$, by the definition of an l-belt we know that G contains an l-belt such that y_iy_{i+1} is its an inner edge. Therefore, conclusion (iv) holds.

Case 3: $y_i \in B \cap S', y_{i+2} \in A' \cap S$.

From Theorem 2.2 we have that $|A| = 2, |B'| = 2$. Since A and B' are connected subgraphs of G, we have that $A \cap A' = \emptyset = B \cap B'$. If $|A \cap B'| = 2$, then $B' \cap S = \emptyset = A \cap S'$. Since $B \cap S' \neq \emptyset \neq A' \cap S$, by noticing that $|S| = |S'| = 3$, we have that $|S \cap S'| \leqslant 2$, and so $\{y_{i+1}\} \cup (S \cap S')$ is a vertex-cut of G with cardinality less than 4, which contradicts to that G is 4-connected. Therefore, $A \cap B' = \{y_{i+1}\}$, and so $|B' \cap S| = |A \cap S'| = 1$. If $|A' \cap S| = 1$, then $A' \cap B \neq \emptyset$. Then, X_3 is a vertex-cut of G, and so $|X_3| \geqslant 4$. Then, $1 = |A' \cap S| > |A \cap S'| = 1$, a contradiction. Hence, $|A' \cap S| = 2$, and so $S \cap S' = \emptyset, |B \cap S'| = 2$. By an argument similar to that used in Case 2 of Theorem 4.4, we know that conclusion (iv) of the lemma holds.

Case 4: $y_i \in B \cap B', y_{i+2} \in A' \cap S$.

An argument analogous to that used in Case 1 of Theorem 4.4 can show that G contains an l-belt such that $y_{i+1}y_{i+2}$ is its an inner edge. Therefore, conclusion (iv) of the lemma holds. The proof of the lemma is complete. □

Theorem 4.6. *Let G be a* 4*-connected graph with property* $(\bigstar)$. *Suppose that H is a helm of G such that H is defined as in Definition* 3.1. *Let* $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ *and* $P = y_1 y_2 \cdots y_h$ *is a path in* $[E_N(G)]$ *with* $h \geqslant 2$ *such that* $a \notin V(P)$ *and* $\{y_1, y_h\} \subset \{x_1, x_2, x_3, x_4\}$. *Then, G contains one of the following structures H_1 as its subgraph: l-belt, l-bi-fan,* $(l \geqslant 1)$, *W-framework, W′-framework or helm, such that at least one inner edge of H_1 belongs to* $E(P \cup H)$, *and H and H_1 do not have any common inner edge.*

Proof. Without loss of generality, we assume that $y_1 = x_1$, then it is easy to see that $y_2 = v_1$. Let $k = h + 1, y_k = a$, then $P' = y_1 y_2 \cdots y_k$ is also a path of $[E_N(G)]$ where $k \geqslant 3$. Let $D = \{a\}$. We take the separating group $(x_1 v_1, S_1; A_1, B_1)$ such that $S_1 = \{x_2, x_3, x_4\}, B_1 = \{x_1, a\}, A_1 = G - x_1 v_1 - S_1 - B_1$. Obviously, $D \cap B_1 \neq \emptyset$.

We take the separating group $(y_i y_{i+1}, S; A, B)$ of G, where $i = 1, 2, \ldots, k-1$, such that $y_i \in B, y_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is as small as possible. We claim that $i + 1 \leqslant k - 1$ holds. Otherwise, $y_{i+1} = y_k$, i.e., $y_{i+1} = a$. Then, $a \in A \cup S$, which contradicts to that $D \cap B \neq \emptyset$. Therefore, $i + 1 \leqslant k - 1$.

We take another separating group $(y_{i+1} y_{i+2}, S'; A', B')$ such that $y_{i+1} \in B', y_{i+2} \in A'$, and $|A'|$ is as small as possible. From Lemma 4.5 we know that one of the four conclusions of Lemma 4.5 holds. Now we discuss them as follows.

(1). Conclusion (i) of Lemma 4.5 holds. It is easy to see that $P' + a x_1$ is a cycle of $[E_N(G)]$. Then, each vertex of P is incident with at least two unremovable edges of G. However, from conclusion (i) we have that $d(y_{i+2}) = 4$ and y_{i+2} is incident with three removable edges of G. Therefore, conclusion (i) cannot hold.

(2). Conclusion (ii) of Lemma 4.5 holds. Then, $B \cap S' = \{d\} = \{a\} = D \cap B$, $c \in \{x_1, x_2, x_3, x_4\}$, and $ac (= dc)$ is not in any 3-cycle of G. However, from the definition of the helm, we know that $ac (= a x_j)$ for each $j = 1, 2, 3, 4$ is in two 3-cycles of G, a contradiction.

(3). Conclusion (iii) of Lemma 4.5 holds. Then, $\{d\} = B \cap S' = \{a\} = D \cap B$. Since $ac \in E(G)$, we have $c \in \{x_1, x_2, x_3, x_4\}$. Then, we have that ac is in two 3-cycles of G. However, this is impossible to hold in G. Therefore, conclusion (iii) cannot hold.

(4). If conclusion (iv) of Lemma 4.5 holds, then the theorem holds. The proof is complete. □

Theorem 4.7. *Let G be a* 4*-connected graph with property* $(\bigstar)$ *and L_1 a maximal* 1*-belt of G defined as in Definition* 3.3 *such that* $V(L_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. *Suppose that* $P = l_1 l_2 \cdots l_h$ *is a path of* $[E_N(G)]$ *such that* $\{l_1, l_h\} \subset \{x_1, x_3, y_1, y_3\}$ *and* $\{x_2, y_2\} \cap V(P) = \emptyset$. *Then, G contains one of the following structures L' as its subgraph: l-belt,* $(l \geqslant 1)$, *helm, W-framework, W′-framework or l-bi-fan,* $(l \geqslant 1)$, *such that at least one inner edge of L' belongs to* $E(P \cup L_1)$.

Proof. We distinguish the following cases.

Case 1: If $l_h = y_3$, by letting $k = h + 1, l_k = y_2$, then $P' = l_1 l_2 \cdots l_k$ is also a path of $[E_N(G)]$. Let $D = \{x_2, y_2\}$, and take a separating group $(l_1 l_2, S_1; A_1, B_1)$ of G such that $l_1 \in B_1, l_2 \in A_1$. Next we will show that $B_1 \cap D \neq \emptyset$. We discuss the following subcases:

Subcase 1.1: If $l_1 = x_1$, we claim that $x_2 \in B_1$. Otherwise, $x_2 \in S_1$. Since $x_1x_2 \in E_N(G)$, from Theorem 2.2 we have that $|B_1| = 2$. Let $B_1 = \{l_1, v_1\}$, then $v_1 \in \Gamma_G(x_1) \cap \Gamma_G(x_2)$. If $v_1 = y_1$, then $\Gamma_G(y_1) = \{x_1, x_2, y_2, w\}$, where $w \in V(G)$, which contradicts to that L_1 is a maximal 1-belt. If $v_1 = x_3$, then $\Gamma_G(x_3) = \{x_2, y_2, x_1, w\}$. It is easy to see that (x_2y_1, T) is a separating pair of G such that $T = \{w, y_2, x_1\}$, and so $x_2y_1 \in E_N(G)$, which contradicts to the definition of the l-belt. Therefore, $x_2 \in B_1$ holds, i.e., $D \cap B_1 \neq \emptyset$.

Subcase 1.2: If $l_1 = y_1$, then if $y_2 \in S_1$, since $y_1y_2 \in E_N(G)$, from Theorem 2.2 we have that $|B_1| = 2$. It is easy to see that $B_1 = \{y_1, x_2\}$, and so $D \cap B_1 \neq \emptyset$. If $y_2 \in B_1$, then $D \cap B_1 \neq \emptyset$.

Subcase 1.3: If $l_1 = x_3$, we claim that $D \cap B_1 \neq \emptyset$. Otherwise, $D \cap B_1 = \emptyset$. From $x_3y_2, x_3x_2 \in E(G)$ we have that $x_2, y_2 \in S_1$. Since $x_2x_3 \in E_N(G)$, from Theorem 2.2 we have that $|B_1| = 2$. Let $B_1 = \{x_3, v_1\}$, then it is easy to see that $v_1 \in \Gamma_G(x_2) \cap \Gamma_G(y_2) \cap \Gamma_G(x_3)$. Then $v_1 = y_1$ holds, i.e., $y_1x_3 \in E(G)$. Since $x_2x_3 \in E_N(G)$, we take the separating group $(x_2x_3, T_1; C_1, D_1)$ such that $x_2 \in C_1, x_3 \in D_1$. Then $y_1, y_2 \in T_1$. From Theorem 2.4, we have that $y_1y_2 \in E_R(G)$, which contradicts to the definition of the l-belt. Therefore, $D \cap B_1 \neq \emptyset$.

We take the separating group $(l_il_{i+1}, S; A, B)$ of G such that $l_i \in B, l_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is as small as possible. We claim that $i + 1 \leqslant k - 1$. Otherwise, $i + 1 = k$ holds. Then, $l_k = y_2$. From $x_2y_2 \in E(G)$ we have that $\{x_2, y_2\} \subset A \cup S$, which contradicts to that $D \cap B \neq \emptyset$. Therefore, $i + 1 \leqslant k - 1$ holds.

Case 2: If $l_h = x_3$, we take the separating group $(l_1l_2, S_1; A_1, B_1)$ of G such that $l_1 \in B_1, l_2 \in A_1$. Let $D = \{x_2, y_2\}$. Similarly, we need to show that $D \cap B_1 \neq \emptyset$.

Subcase 2.1: If $l_1 = y_1$, from $y_1y_2 \in E(G)$ we have that $y_2 \in B_1 \cup S_1$. If $y_2 \in S_1$, since $y_1y_2 \in E_N(G)$, from Theorem 2.2 we have that $|B_1| = 2$. Let $B_1 = \{y_1, v_1\}$. Then, $y_1y_2v_1y_1$ is a 3-cycle of G. It is easy to see that $v_1 = x_2$. Then, $D \cap B_1 \neq \emptyset$.

By the symmetry of the maximal 1-belt, for the other cases we may employ a similar argument.

We take the separating group $(l_il_{i+1}, S; A, B)$ such that $l_i \in B, l_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is small as possible, where $i = 1, 2 \cdots, h - 1$. We claim that $i + 1 \leqslant h - 1$. Otherwise, $l_h = x_3 \in A$. From $x_2x_3, y_2x_3 \in E(G)$ we have that $x_2, y_2 \in A \cup S$, which contradicts to that $D \cap B \neq \emptyset$.

We take the separating group $(l_{i+1}l_{i+2}, S'; A', B')$ of G such that $l_{i+1} \in B', l_{i+2} \in A'$ and $|A'|$ is as small as possible. From Lemma 4.5 we have that one of the four conclusions of Lemma 4.5 holds. Here we will discuss them as follows:

(1). It is easy to see that each vertex of P is incident with at least two unremovable edges, and so conclusion (i) of Lemma 4.5 cannot hold.

(2). If conclusion (ii) of Lemma 4.5 holds, then we have that $B \cap S' = D \cap B = \{d\} \subset \{x_2, y_2\}$. By the symmetry of x_2 and y_2, without loss of generality, we may assume that $d = x_2$. For $d = y_2$, we may employ a similar argument.

From Lemma 4.5, we know that $A \cap A' = \{l_{i+2}\}, l_{i+1} \in A \cap B'$. Let $A \cap S' = \{v_1, v_2\}$. If $v_1l_{i+2} \in E_N(G)$, we take the corresponding separating group $(v_1l_{i+2}, T; C, K)$ such that $v_1 \in C, l_{i+2} \in K$, and so $v_1 \in S' \cap C$.

(2.1). If $l_{i+1} \in B' \cap K$, by the argument analogous to that used in Case 1 of Theorem 4.4, we can get that $|A'| = 2, |K \cap A'| = |A' \cap T| = 1, |C \cap S'| = 2, |S' \cap K| = 1$. Let $K \cap S' = \{b\}, A' \cap T = \{a\}, S' \cap C = \{v_1, w\}$. Then, by an argument analogous to that used in Case 1 of Theorem 4.4, we have that $al_{i+2}, av_1 \in E_R(G), bl_{i+2} \in E_R(G)$, $ab \in E_N(G), d(a) = d(l_{i+2}) = 4$. It is easy to see that the l-belt is a subgraph of G, where $l \geqslant 1$, and $\Gamma_G(l_{i+2}) = \{l_{i+1}, v_1, a, b\}$. We claim that l_{i+2} is not an end-vertex of P. Otherwise, we have $l_{i+2} \in \{x_1, x_3, y_1, y_3\}$. Since $B \cap S' = \{x_2\}$, and $x_1, x_3, y_1 \in \Gamma_G(x_2)$, then this is true only if $l_{i+2} = y_3$ holds. Let $A' \cap S = \{k\}$. Noticing that (kx_2, T') will be the separating pair of G such that $T' = \{l_{i+1}\} \cup (S' - \{x_2\})$, we have that $k \in \{x_3, x_1\}$. If $k = x_3$, then we will have that $x_3y_3 \in E(G)$ and $d(x_3) = 4$, which contradicts to the definition of the maximal 1-belt. If $k = x_1$, noticing that $y_2 \notin V(P)$, then $l_{i+1} \neq y_2$, and so we will have that $x_1y_2 \in E(G)$, a contradiction. Therefore, we have that l_{i+2} is not an end-vertex of P. From $al_{i+2}, bl_{i+2} \in E_R(G)$ we have that $l_{i+2}v_1 \in E(P)$ and $l_{i+2}v_1$ is an inner edge of the l-belt. Hence, the theorem holds.

(2.2). If $l_{i+1} \in B' \cap T$, then by an argument analogous to that used in Case 2 of Theorem 4.4, we have that $l_{i+1}l_{i+2}$ is an inner edge of one of the following subgraphs of G: helm, W'-framework, W-framework or l-belt. Therefore, the theorem holds.

So, we may assume that $v_1l_{i+2} \in E_R(G)$. If $v_2l_{i+2} \in E_N(G)$, we may employ a similar argument. So, we may assume that $v_2l_{i+2} \in E_R(G)$. Let $A' \cap S = \{c\}$. Since P is a path of $[E_N(G)]$, and l_{i+2} is not an end-vertex of P, we have that $l_{i+2}c \in E_N(G) \cap E(P)$. If $cv_1 \in E_N(G)$, we take the corresponding separating group $(cv_1, T'; C', D')$ of G such that $v_1 \in C', c \in D'$. Obviously, $l_{i+2} \in T'$. Since $cl_{i+2} \in E_N(G)$, from Theorem 2.2 we have that $|D'| = 2$, and so $D' = \{c, v_2\}$. Then, $|\Gamma_G(c) \cap \Gamma_G(v_2)| \geqslant 2$. Noticing that $v_1 \in C_1$, obviously it is impossible to hold in G. So, $cv_1 \in E_R(G)$. By an analogous argument, we have that $cv_2 \in E_R(G)$. It is easy to see that cl_{i+2} is an inner edge of an l-bi-fan, and so the theorem holds.

(3). If conclusion (iii) of Lemma 4.5 holds, then we have that $B \cap S' = D \cap B = \{d\} \subset \{x_2, y_2\}$. By the symmetry of x_2 and y_2, we may assume that $d = y_2$. Let $A \cap S' = \{v_1\}, \quad S \cap S' = \{w\}, A' \cap S = \{c\}$, then $\Gamma_G(c) = \{l_{i+2}, v_1, w, y_2\}$. Since $cw \in E([S])$, from Theorem 2.4 we have that $cw \in E_R(G)$. By an analogous argument used in (2.1). we can get that l_{i+2} is not an end-vertex of P.

(3.1). If $l_{i+2}v_1 \in E_N(G)$, we take the corresponding separating group $(l_{i+2}v_1, T; C, K)$ such that $l_{i+2} \in K, v_1 \in C$. Then, $l_{i+2} \in A' \cap K, v_1 \in C \cap S', l_{i+1} \in B'$. We claim that $l_{i+1} \notin B' \cap K$. Otherwise, $l_{i+1} \in B' \cap K, A' = \{l_{i+2}, c\}$. By an argument analogous to that used in Case 1 of Theorem 4.4, we can get that $A' \cap K = l_{i+2}\}, A' \cap T|! = c\}, T \cap S' = \emptyset, |T \cap B'| = |C \cap S'| = 2, |K \cap S'| = 1$. Since $wl_{i+2} \in E(G)$, we have $w \in K \cap S'$. Let $A_2 = (K \cap B') \cup \{w\}, S_2 = (T \cap B') \cup \{l_{i+2}\}$, $B_2 = G - cw - S_2 - A_2$, then $(cw, S_2; A_2, B_2)$ is a separating group of G. So, $cw \in E_N(G)$, which contradicts to that $cw \in E_R(G)$. Hence, $l_{i+1} \notin B' \cap K$, and so $l_{i+1} \in B' \cap T$. By an argument analogous to that used in Case 2 of Theorem 4.4, we have that $|A'| = |K| = 2$ and $|K \cap S'| = |A' \cap T| = 1$. Noticing that $c \in A', w \in S', \Gamma_G(l_{i+2}) = \{l_{i+1}, c, w, v_1\}$, it is easy to see that $K \cap S' = \{w\}, A' \cap T = \{c\}$. By an argument analogous to that used in Case 2 of Theorem 4.4, and noticing that $cw \in E_R(G)$, we have that $l_{i+1}l_{i+2}$ is an inner edge of one of the following subgraphs of G: W'-framework, W-framework or helm. Therefore, the theorem holds.

So, we may assume that $l_{i+2}v_1 \in E_R(G)$.

(3.2). If $wl_{i+2} \in E_N(G)$, we take the corresponding separating group $(wl_{i+2}, T'; C', D')$ of G such that $w \in C', l_{i+2} \in D'$. Then, $w \in S' \cap C'$.

(3.2.1). If $l_{i+1} \in B' \cap D'$, by an argument analogous to that used in Case 1 of Theorem 4.4, we know that wl_{i+2} is an inner edge of an l-belt, where $l \geqslant 1$, and $cl_{i+2} \in E_R(G)$. Since l_{i+2} is incident with only two unremovable edges $l_{i+1}l_{i+2}, wl_{i+2}$, and l_{i+2} is not an end-vertex of P, we have $wl_{i+2} \in E(P)$. Hence, the theorem holds.

(3.2.2). If $l_{i+1} \in B' \cap T'$, then by an argument analogous to that used in Case 2 of Theorem 4.4, we know that $l_{i+1}l_{i+2}$ is an inner edge of one of the following subgraphs of G: l-belt, W-framework, W'-framework or helm, and so the theorem holds.

Therefore, next we may assume that $wl_{i+2} \in E_R(G)$.

Since $E(P) \subset E_N(G)$, we have $cl_{i+2} \in E_N(G)$. If $cv_1 \in E_N(G)$, we take the corresponding separating group $(cv_1, T'; C', D')$ such that $v_1 \in C', c \in D'$. Obviously, $l_{i+2} \in T'$. Since $cl_{i+2} \in E_N(G)$, from Theorem 2.2 we have that $|D'| = 2$. Let $D' = \{u, c\}$, then $cul_{i+2}c$ is a 3-cycle of G, and so this is true only if $u = w$ holds. From $cy_2(= cd) \in E(G)$ we have that $y_2 \in T'$, and so $wy_2 \in E(G)$. We take the separating group $(cl_{i+2}, T_1; C_1, D_1)$ such that $c \in C_1, l_{i+2} \in D_1$. Since $cv_1l_{i+2}c$ is a 3-cycle of G, we have $v_1 \in T_1$. Then, we have that $l_{i+2} \in D_1 \cap T', v_1 \in C' \cap T_1, c \in D' \cap C_1$. By an argument analogous to that used in Case 2 of Theorem 4.4, and by noticing that $d(l_{i+2}) = 4$, and $v_1l_{i+2} \in E(G)$, we can get that cl_{i+2} is an inner edge of one of the following subgraphs of G: W'-framework or helm. Therefore, the theorem holds.

So, we may assume that $cv_1 \in E_R(G)$. It is easy to see that G contains an l-bi-fan such that cl_{i+2} is its an inner edge, where $l \geqslant 1$. An analogous argument can lead to that $cl_{i+2} \in E(P)$. So, the theorem holds.

(4). If conclusion (iv) of Lemma 4.5 holds, then the Theorem holds. The proof is now complete. $\square$

Corollary 4.8. *Let G be a* 4*-connected graph with property* ($\bigstar$) *and ${L_1}'$ a maximal* 1*-co-belt of G defined as in Definition* 3.4. $V({L_1}') = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3\}$. *Suppose that $P = l_1l_2\cdots l_h$ is a path of $[E_N(G)]$ such that $\{x_2, x_3, y_2\} \cap V(P) = \emptyset$ and $\{l_1, l_h\} \subset \{x_1, x_4, y_1, y_3\}$. Then, G contains one of the following structures as its subgraph*: *l-belt*, $(l \geqslant 1)$, *W-framework*, *W'-framework*, *helm or l-bi-fan*, $(l \geqslant 1)$, *such that it has some inner edge*(s) *belonging to $E(P)$.*

Proof. We distinguish the following cases:

Case 1: If $l_h = x_4$, by letting $k = h + 1, l_k = x_3$, then $P' = l_1l_2\cdots l_k$ is also a path of $[E_N(G)]$. Let $D = \{x_2, x_3, y_2\}$, and take a separating group $(l_1l_2, S_1; A_1, B_1)$ of G such that $l_1 \in B_1, l_2 \in A_1$. Next we will show that $B_1 \cap D \neq \emptyset$. We discuss the following subcases:

Subcase 1.1: If $l_1 = x_1$, we claim that $x_2 \in B_1$. Otherwise, $x_2 \in S_1$. Since $x_1x_2 \in E_N(G)$, from Theorem 2.2 we have that $|B_1| = 2$. Let $B_1 = \{l_1, v_1\}$, then $v_1 \in \Gamma_G(x_1) \cap \Gamma_G(x_2)$. If $v_1 = y_1$, then $\Gamma_G(y_1) = \{x_1, x_2, y_2, w\}$, where $w \in V(G)$, which contradicts to that L_1' is a maximal 1-co-belt. Obviously, $v_1 \notin \{x_3, y_2\}$, and therefore $x_2 \in B_1$ holds, i.e., $D \cap B_1 \neq \emptyset$.

Subcase 1.2: If $l_1 = y_1$, then if $y_2 \in S_1$, since $y_1 y_2 \in E_N(G)$, from Theorem 2.2 we have that $|B_1| = 2$. It is easy to see that $B_1 = \{y_1, x_2\}$, and so $D \cap B_1 \neq \emptyset$. If $y_2 \in B_1$, then $D \cap B_1 \neq \emptyset$.

Subcase 1.3: If $l_1 = y_3$, we claim that $D \cap B_1 \neq \emptyset$. Otherwise, $D \cap B_1 = \emptyset$. From $x_3 y_3, y_2 y_3 \in E(G)$ we have that $x_3, y_2 \in S_1$. Since $y_2 y_3 \in E_N(G)$, from Theorem 2.2 we have that $|B_1| = 2$. Let $B_1 = \{y_3, v_1\}$, then it is easy to see that $v_1 \in \Gamma_G(y_2) \cap \Gamma_G(y_3) \cap \Gamma_G(x_3)$, which is impossible to hold in G. Therefore, $D \cap B_1 \neq \emptyset$.

We take the separating group $(l_i l_{i+1}, S; A, B)$ of G such that $l_i \in B, l_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is as small as possible. We claim that $i + 1 \leqslant k - 1$. Otherwise, $i + 1 = k$. Then, $l_k = x_3$. From $x_2 x_3, y_2 x_3 \in E(G)$ we have that $\{x_2, x_3, y_2\} \subset A \cup S$, which contradicts to that $D \cap B \neq \emptyset$. Therefore, $i + 1 \leqslant k - 1$ holds.

Case 2: If $l_h = y_3$, by letting $k = h + 1, l_k = y_2$, then $P' = l_1 l_2 \cdots l_k$ is also a path of $[E_N(G)]$. Let $D = \{x_2, x_3, y_2\}$. We take the separating group $(l_1 l_2, S_1; A_1, B_1)$ of G such that $l_1 \in B_1, l_2 \in A_1$. Similarly, we need to show that $D \cap B_1 \neq \emptyset$.

Subcase 2.1: If $l_1 = y_1$, from $y_1 y_2, y_1 x_2 \in E(G)$ we have that $x_2, y_2 \in B_1 \cup S_1$. If $x_2, y_2 \in S_1$, since $y_1 y_2 \in E_N(G)$, from Theorem 2.2 we have that $|B_1| = 2$. Let $B_1 = \{y_1, v_1\}$. Then, $v_1 = \Gamma_G(y_1) \cap \Gamma_G(y_2) \cap \Gamma_G(x_2)$, which is impossible to hold in G. Then, $D \cap B_1 \neq \emptyset$.

By the symmetry of the maximal 1-co-belt, for the other cases we may employ a similar argument.

We take the separating group $(l_i l_{i+1}, S; A, B)$ such that $l_i \in B, l_{i+1} \in A, D \cap B \neq \emptyset$ and $|A|$ is small as possible, where $i = 1, 2 \cdots, k - 1$. We claim that $i + 1 \leqslant k - 1$. Otherwise, $l_k = y_2 \in A$. From $x_2 y_2, y_2 x_3 \in E(G)$ we have that $x_2, x_3, y_2 \in A \cup S$, which contradicts to that $D \cap B \neq \emptyset$.

We take the separating group $(l_{i+1} l_{i+2}, S'; A', B')$ of G such that $l_{i+1} \in B', l_{i+2} \in A'$ and $|A'|$ is as small as possible. From Lemma 4.5 we have that one of the four conclusions of Lemma 4.5 holds. Here we will discuss them as follows:

(1). It is easy to see that each vertex of P is incident with at least two unremovable edges, and so conclusion (i) of Lemma 4.5 cannot hold.

(2). If conclusion (ii) of Lemma 4.5 holds, then we have that $B \cap S' = D \cap B = \{d\} \subset \{x_2, x_3, y_2\}$.

First, we claim that l_{i+2} is not the end-vertex of P, otherwise, we assume that $l_{i+2} \in \{x_1, x_4, y_1, y_3\}$ holds. Let $A' \cap S = \{k\}$. Noticing that (kd, T') is a separating pair of G such that $T' = \{l_{i+1}\} \cup (S' - \{d\})$, so $kd \in E_N(G)$. If $d = x_2$, from $x_1 x_2, x_2 y_1 \in E(G)$, we have that $l_{i+2} \in \{y_3, x_4\}$: (1). If $l_{i+2} = x_4$, it is easy to see that $k \in \{x_1, x_3\}$, if $k = x_1$, noticing that $x_3 \notin V(P)$, then $l_{i+1} \neq x_3$, then we will have that $x_1 x_3 \in E(G)$, a contradiction; if $k = x_3$, then we will have that $|\Gamma_G(x_3) \cap \Gamma_G(x_4)| = 2$, which is impossible to hold in G. (2). If $l_{i+2} = y_3$, we claim that $k \neq x_3$, otherwise, we will have that $y_3 x_4 \in E(G)$ and $d(y_3) = 4$,which contradicts to the definition of maximal 1-co-belt. Then only $k = x_1$ holds, then we will have that $|\Gamma_G(x_1) \cap \Gamma_G(y_3)| = 2$, $x_1 y_3 \in E(G)$ and $d(x_1) = d(y_3) = 4$ holds, which is impossible to hold in G. Therefore, $d \neq x_2$. By the symmetry of x_2 and x_3, we have that $d \neq x_3$.

Therefore, $d = y_2$ holds, then we have that $l_{i+2} \in \{x_1, x_4\}$ and $k \in \{y_1, y_3\}$. (1). If $l_{i+2} = x_1$: We claim that $k \neq y_1$, otherwise, we will have that $x_1y_1 \in E(G), d(y_1) = 4$, which contradicts to the definition of the maximal 1-co-belt, so $k = y_3$ holds, then we will have that $|\Gamma_G(x_1) \cap \Gamma_G(y_3)| = 2$ and $x_1y_3 \in E(G), d(x_1) = d(y_3) = 4$ holds, which is impossible to hold in G. (2). If $l_{i+2} = x_4$, by the symmetry of x_1 and x_4, we may employ a similar argument to get that the assumption is not true.

From the above argument, we have that l_{i+2} is not the end-vertex of P.

We may employ an argument similar to that used in (2) of Theorem 4.7 to show that the corollary is true.

(3). If conclusion (iii) of Lemma 4.5 holds, then we have that $B \cap S' = D \cap B = \{d\} \subset \{x_2, x_3, y_2\}$.

We may employ an argument analogous to that used in (2) to show that l_{i+2} is not an end-vertex of P. We may also employ an argument similar to that used in (3) of Theorem 4.7 to conclude that the corollary is true.

(4). If conclusion (iv) of Lemma 4.5 holds, then the corollary is true. $\square$

5. The number of removable edges in a 4-connected graph

After we have been well prepared with the results in the above section, we are arriving at the point to show our main results of this paper in this section.

Let M be a 5-wheel such that $V(M) = \{a, x, y, z, v\}$ and a is its center. Let T_1, T_2, T_3, T_4 be four trees such that for each $i \in \{1, 2, 3, 4\}$, T_i has k vertices of degree one and $|T_i| - k$ vertices of degree four. Let the vertices of degree four be $u_i^{(1)}, u_i^{(2)}, \ldots, u_i^{(|T_i|-k)}$, and the vertices of degree one be $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(k)}$. Let $M_1, M_2, \ldots, M_k$ be k copies of M and $a^{(j)}, x^{(j)}, y^{(j)}, z^{(j)}, v^{(j)}$ be the vertices of M_j corresponding to the vertices a, x, y, z, v of M, respectively, where $j = 1, 2, \ldots, k$. For each $j \in \{1, \ldots, k\}$, identify $x_1^{(j)}, x_2^{(j)}, x_3^{(j)}, x_4^{(j)}$ with $x^{(j)}, y^{(j)}, z^{(j)}, v^{(j)}$ such that each of $x_1^{(j)}, x_2^{(j)}, x_3^{(j)}, x_4^{(j)}$ identifies with one and only one of $x^{(j)}, y^{(j)}, z^{(j)}, v^{(j)}$. Denote the resulting graph by G. It is easy to see that G is 4-connected. Next we will show that for each 4-cycle $C = x^{(j)}y^{(j)}z^{(j)}\ v^{(j)}x^{(j)}$ of G, we have that $E(C) \subset E_R(G)$, and the other edges in G are unremovable, where $j = 1, 2, \ldots, k$. For $y^{(j)}u_i^{(l)} \in E(G)$, let $S = \{x^{(j)}, v^{(j)}, z^{(j)}\}, A = \{a^{(j)}, y^{(j)}\}, B = G - y^{(j)}u_i^{(l)} - S - A$, then $(y^{(j)}u_i^{(l)}, S; A, B)$ is a separating group of G, and hence $y^{(j)}u_i^{(l)} \in E_N(G)$. Symmetrically, we can show that $x^{(j)}u_i^{(l)}, z^{(j)}u_i^{(l)}, v^{(j)}u_i^{(l)} \in E_N(G)$, where $j = 1, 2, \ldots, k; i = 1, 2, 3, 4; l = 1, 2, \ldots, |T| - k$. For each edge $a^{(j)}x^{(j)}$, it is easy to see that $(a^{(j)}x^{(j)}, T)$ is a separating pair of G such that $T = \{y^{(j)}, v^{(j)}, u_i^{(j)}\}$ and $u_i^{(l)}z^{(j)} \in E(G)$. By symmetry, we have that $a^{(j)}y^{(j)}, a^{(j)}z^{(j)}, a^{(j)}v^{(j)} \in E_N(G)$. From Corollary 2.3 it is easy to see that for each 4-cycle $C = x^{(j)}y^{(j)}z^{(j)}v^{(j)}x^{(j)}$, we have that $E(C) \subset E_R(G)$. For each edge e of T_i, for example, $e = u_1^{(l)}u_1^{(l+1)}$, it is easy to see that (e, S) is a separating pair of G such that $S = \{u_2^{(l)}, u_3^{(l)}, u_4^{(l)}\}$. Therefore, for each edge e of T_i, where $i = 1, 2, 3, 4$, we have that

$e \in E_N(G)$, and so $e_R(G) = 4k, |T_i| = (3k-2)/2, (i = 1,2,3,4), |G| = 7k-4$, $e_R(G) = (4|G| + 16)/7$. We denote the set of all the above constructed graphs by $\mathfrak{I}$.

Theorem 5.1. *Let G be a 4-connected graph of order at least 5. If G is neither C_5^2 nor C_6^2, then $e_R(G) \geqslant (4|G| + 16)/7$ and the equality holds if and only if $G \in \mathfrak{I}$.*

Proof. Let $|G| = n, |E(G)| = m$. We proceed by induction on $(n+m)$. Since G is not C_5^2, we have that $n \geqslant 6$. If $n = 6$, since G is not C_6^2, we have that $m \geqslant 13, (n+m) \geqslant 19$. It is easy to see that $e_R(G) \geqslant 9 > (4n+16)/7$. If $n = 7$, then it is easy to that $e_R(G) \geqslant 9 > (4n+16)/7$. Therefore, we may assume that $n \geqslant 8$.

Case 1: If G does not have property (★), i.e., there exists an edge $e = xy \in E_R(G)$ such that $d(x) \geqslant 5$ and $d(y) \geqslant 5$ in G, then consider $G \ominus e = G - xy$. It is easy to see that removable edges in $G - xy$ are also removable edges in G, and hence $e_R(G) \geqslant e_R(G \ominus e) + 1$. Then, $|G| = |G \ominus e|, |E(G \ominus e)| = m - 1$, and therefore $|G \ominus e| + |E(G \ominus e)| < n + m$. If $G \ominus e$ is C_5^2 or C_6^2, then $e_R(G) \geqslant 9 > (4n+16)/7$. If $G \ominus e$ is neither C_5^2 nor C_6^2, by the induction hypothesis we know that $e_R(G) \geqslant e_R(G \ominus e) + 1 \geqslant (4n+16)/7 + 1 > (4n+16)/7$.

Next we suppose that G has property (★).

Case 2: If G contains a 2-bi-fan as its subgraph, from Theorem 4.1 we know that there exists an edge $e \in E(G)$ such that $e_R(G) \geqslant e_R(G \ominus e) + 1$. Here, $|G \ominus e| = n - 1$, $|E(G \ominus e)| = m - 3$. Then, $|G \ominus e| + |E(G \ominus e)| < n + m$. If $G \ominus e$ is C_5^2 or C_6^2, then $e_R(G) \geqslant 10 > (4n+16)/7$. If $G \ominus e$ is neither C_5^2 nor C_6^2, by the induction hypothesis we know that $e_R(G) \geqslant e_R(G \ominus e) + 1 \geqslant [4(n-1) + 16]/7 + 1 > (4n+16)/7$.

Case 3: If G contains an l-belt as its subgraph where $l \geqslant 3$. Then, from Theorem 4.2 we have that there exists an edge $e \in E(G)$ such that $e_R(G) \geqslant e_R(G \ominus e) + 2$. If $G \ominus e$ is either C_5^2 or C_6^2, then $e_R(G) \geqslant 12 > (4n+16)/7$. If $G \ominus e$ is neither C_5^2 nor C_6^2, by the induction hypothesis we know that $e_R(G) \geqslant e_R(G \ominus e) + 2 \geqslant [4(n-2) + 16]/7 + 2 > (4n+16)/7$.

Case 4: If for any edge $e \in E_R(G)$, when $|G \ominus e| = n$, we have that $e_R(G) < e_R(G \ominus e)$; when $|G \ominus e| = n - 1$, we have that $e_R(G) < e_R(G \ominus e) + 1$; when $|G \ominus e| = n - 2$, we have that $e_R(G) < e_R(G \ominus e) + 2$, then we discuss the following subcases.

Subcase 4.1: If $[E_N(G)]$ is a forest, then $e_N(G) = n - t$ such that t is the number of components in $[E_N(G)]$. Therefore, $e_R(G) \geqslant 2n - n + t = n + t > (4n+16)/7$.

Subcase 4.2: If $[E_N(G)]$ contains a cycle, from Theorem 4.4 and the above argument in Cases 2 and 3 we can get that G contains some structures in $\mathfrak{R}$ as its subgraphs. Let G contain k_1 maximal 1-belts, k_2 maximal 1-bi-fans, k_3 maximal 1-co-belts, k_4 W-frameworks, k_5 W'-frameworks, k_6 maximal 2-belts, k_7 maximal 2-co-belts, and h helms. Let E_1 be the set of inner edges of the above-mentioned subgraphs. Then,

$$|E_1| = 2k_1 + k_2 + 3k_3 + 2k_4 + 3k_5 + 4k_6 + 5k_7 + 4h. \tag{1}$$

Let $E_0 = E_N(G) - E_1$, then we have the following results:

(1). $[E_0]$ is a forest. This follows from Theorem 4.4, Lemma 3.7, and the definitions of the above-mentioned subgraphs.

(2). Let $r = \sum_{x \in G}(d(x) - 4) = \sum_{x \in G} d(x) - 4n$, then $e(G) = 2n + r/2$. Let $n_1 = n - h - |[E_0]|$, then $n_1 \geqslant 0$, and $n_1 = 0$ if and only if $V(G) = V([E_0]) \bigcup \{a_1, a_2, \ldots, a_h\}$ such that a_i is the center of a helm, where $i = 1, 2, \ldots, h$.

(3). $e_R(G) = e(G) - e_N(G), e_N(G) = |E_0| + |E_1| = |[E_0]| - t + |E_1| = n - n_1 - h - t + |E_1|$, where t is the number of components in $[E_0]$.

By noticing the number of removable edges in the above-mentioned subgraphs, we have the following result

$$\begin{aligned} e_R(G) &= e(G) - e_N(G) = 2n + r/2 - n + h + n_1 + t - |E_1| \\ &\geqslant 3k_1 + 4k_2 + 4k_3 + 5k_4 + 5k_5 + 5k_6 + 6k_7 + 4h. \end{aligned} \tag{2}$$

From the formulas $\langle 1 \rangle$ and $\langle 2 \rangle$, we have the following result

$$n + r/2 - 7h + n_1 + t - 5k_1 - 5k_2 - 7k_3 - 7k_4 - 8k_5 - 9k_6 - 11k_7 \geqslant 0.$$

Then,

$$6n + 3r - 42h + 6n_1 + 6t - 30k_1 - 30k_2 - 42k_3 - 42k_4 - 48k_5 - 54k_6 - 66k_7 \geqslant 0$$

and so

$$\begin{aligned} e_R(G) &= n + r/2 + n_1 + t + h - |E_1| = 4n/7 + (6n + 7r + 14n_1 + 14t - 42h - 28k_1 \\ &\quad - 14k_2 - 42k_3 - 28k_4 - 42k_5 - 56k_6 - 70k_7)/14 \\ &\geqslant 4n/7 + (6n + 3r + 6n_1 + 6t - 42h - 30k_1 - 30k_2 - 42k_3 \\ &\quad - 42k_4 - 48k_5 - 54k_6 - 66k_7)/14 \\ &\quad + (4r + 8n_1 + 8t + 2k_1 + 16k_2 + 14k_4 + 6k_5 - 2k_6 - 4k_7)/14 \\ &\geqslant 4n/7 + (4r + 8n_1 + 8t + 2k_1 + 16k_2 + 14k_4 + 6k_5 - 2k_6 - 4k_7)/14. \end{aligned} \tag{3}$$

Therefore, $e_R(G) \geqslant (4n + 16)/7$ holds only if the following formula holds

$$\Delta = 2r + 4n_1 + 4t + k_1 + 8k_2 + 7k_4 + 3k_5 - k_6 - 2k_7 \geqslant 16. \tag{4}$$

Let L_1' be a maximal 1-co-belt. It is easy to see that $x_2 \in G - \{a_1, a_2, \ldots, a_h\} - V([E_0])$, and so L_1' will contribute 1 to n_1. Since G contains k_3 maximal 1-belts, and so they will contribute k_3 to n_1. Analogously, for each maximal 2-belt, it will contribute 2 to n_1, and so k_6 maximal 2-belts will contribute $2k_6$ to n_1. For W'-frameworks, maximal 2-co-belts and W-frameworks, we analyze them analogously. Then, we can get the following formula

$$n_1 \geqslant k_3 + k_4 + k_5 + 2k_6 + 3k_7. \tag{5}$$

From the formulas $\langle 5 \rangle$ and $\langle 4 \rangle$, we can get the following formula

$$\Delta \geqslant 2r + 4t + k_1 + 8k_2 + 4k_3 + 11k_4 + 7k_5 + 7k_6 + 10k_7. \tag{6}$$

We will discuss the following cases.

(4). $\mathrm{h} = 0$, $\mathrm{k} = \mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3 + \mathrm{k}_4 + \mathrm{k}_5 + \mathrm{k}_6 + \mathrm{k}_7 \leqslant 2$.

First, we claim that $[E_N(G)]$ contains at most two cycles. Otherwise, suppose that there are at least three cycles in $[E_N(G)]$. Then, we take a cycle C_1. From Theorem 4.6 and the assumption, we have that G contains some structure $H_1 \in \mathfrak{R}$ as its subgraph such that H_1 has an inner edge e_1 on C_1. We take another cycle C_2 in $[E_N(G)] - C_1$. Analogously, we have that G contains some structure $H_2 \in \mathfrak{R}$ as its subgraph such that H_2 has an inner edge e_2 on C_2. Last, we take a cycle C_3 in $[E_N(G)] - C_1 - C_2$. Then, G contains some structure $H_3 \in \mathfrak{R}$ as its subgraph such that H_3 has an inner edge e_3 on C_3. Since e_1 is an inner edge of H_1, but not any of H_2, we have that $H_1 \neq H_2$. Analogously, we have that $H_1 \neq H_3, H_2 \neq H_3$. From Lemma 3.7 we know that any two of H_1, H_2 and H_3 do not have common inner edge, and so $k \geqslant 3$, a contradiction. Therefore, there are at most two cycles in $[E_N(G)]$. So, $e_N(G) \leqslant n+1$, and hence $e_R(G) \geqslant 2n - n - 1 > (4n+16)/7$.

(5). $h = 0, k = k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 \geqslant 3$.

(5.1). $k_1 + k_3 = 0$, and so $k_2 + k_4 + k_5 + k_6 + k_7 \geqslant 3$. Noticing that $t \geqslant 1$, from the formula $\langle 6 \rangle$ we have that

$$\begin{aligned} \varDelta &\geqslant 2r + 4 + 7(k_2 + k_4 + k_5 + k_6 + k_7) + k_2 + 4k_4 + 3k_7 \\ &\geqslant 4 + 7(k_2 + k_4 + k_5 + k_6 + k_7) \geqslant 25, \end{aligned}$$

here the inequality $\langle 4 \rangle$ rigidly holds.

(5.2). $k_1 + k_3 \geqslant 1$. We may assume that G contains a maximal 1-belt L_1 such that $V(L_1) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. From Theorem 4.7 we know that if $x_3, y_1 \in [E_0]$, then $n_1 \geqslant 2, t \geqslant 2$. From the formulas $\langle 4 \rangle$ and $\langle 5 \rangle$ we have that

$$\begin{aligned} \varDelta \geqslant\ & 2r + 3n_1 + 4t + (k_1 + k_3) + 8k_2 + 8k_4 + 4k_5 + k_6 + k_7 \geqslant 3n_1 + 4t + (k_1 + k_2 \\ &+ k_3 + k_4 + k_5 + k_6 + k_7) \geqslant 6 + 8 + 3 = 17. \end{aligned}$$

If $x_3 \in [E_0], y_1 \notin [E_0]$, then $n_1 \geqslant 1, t \geqslant 3$. Similarly, we can get that $\varDelta \geqslant 18$.

If $x_3, y_1 \in [E_0]$, then $t \geqslant 4$, and so $\varDelta \geqslant 19$, here the inequality $\langle 4 \rangle$ rigidly holds.

(6). $h \geqslant 1$. We take a helm H such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$. From Theorem 4.6 we have that any two of the edges $x_1v_1, x_2v_2, x_3v_3, x_4v_4$ are in different components, and so $t \geqslant 4$. From the formula $\langle 6 \rangle$ we know that $\varDelta \geqslant 16$, i.e., $e_R(G) \geqslant (4n+16)/7$, and the equality holds only if $k_i = 0$, where $i = 1, 2, \ldots, 7, r = 0, t = 4, n_1 = 0$, i.e., $[E_0]$ has only four components T_1, T_2, T_3, T_4, and $V(G) = V([E_0]) \cup \{a_1, a_2, \ldots, a_h\}$. Then, from $r = 0$ we know that G is a 4-connected and 4-regular graph. From $e_R(G) = 4h, e_N(G) = 10h - 8$, we can get that $n = 7h - 4$. Moreover, all the edges but $x_1^{(p)}x_2^{(p)}, x_2^{(p)}x_3^{(p)}, x_3^{(p)}x_4^{(p)}, x_4^{(p)}x_1^{(p)}$ of each helm H_p in G are unremovable, whereas different edges of $x_i^{(p)}v_i^{(p)}$ of H_p are in different components T_i, and every vertex $v_i^{(p)}$ is of degree 1 in T_i. Based on the above arguments, we can conclude that T_i has h vertices with degree 1 and $|T_i| - h$ vertices with degree 4. Therefore, $G \in \mathfrak{I}$. The proof is now complete. $\square$

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