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Distribution of Cycle Lengths in Graphs

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Bondy and Vince proved that every graph with minimum degree at least three contains two cycles whose lengths differ by one or two, which answers a question raised by Erdős. By a different approach, we show in this paper that if G is a graph with minimum degree $\delta(G) \geqslant 3k$ for any positive integer k, then G contains $k+1$ cycles $C_0, C_1, \ldots, C_k$ such that $k+1 < |E(C_0)| < |E(C_1)| < \cdots < |E(C_k)|$, $|E(C_i)| - |E(C_{i-1})| = 2$, $i \leqslant i \leqslant k-1$, and $1 \leqslant |E(C_k)| - |E(C_{k-1})| \leqslant 2$, and furthermore, if [illegible] minimum degree at least $3k$ for any positive integer k, then G contains $2k$ cycles of consecutive lengths $m, m+1, \ldots, m+2k-1$ for some integer $m \geqslant k+2$.

1. INTRODUCTION

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. A *cycle* is a connected 2-regular graph. Erdős [2] asked whether every graph with minimum degree at least three contains two cycles whose lengths differ by one or two. Using non-separating induced cycles, Bondy and Vince [1] answered the question affirmatively. By a different approach, we prove in this paper the following more general result.

THEOREM 1.1. *Let xy be an edge in a 2-connected graph G. For any positive integer k, if every vertex other than x and y has degree at least $3k$, then xy is contained in $k+1$ cycles $C_0, C_1, \ldots, C_k$ such that $k+1 < |E(C_0)| < |E(C_1)| < \cdots < |E(C_k)|$, $|E(C_i)| - |E(C_{i-1})| = 2$, $1 \leqslant i \leqslant k-1$, and $1 \leqslant |E(C_k)| - |E(C_{k-1})| \leqslant 2$.*

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Evidently a bipartite graph cannot have cycles whose lengths differ by one. It is natural to ask what graphs contain cycles whose lengths differ by two. Häggkvist and Scott [4] have proved that every connected cubic graph other than the complete graph on four vertices contains two cycles whose lengths differ by two. A stronger result, together with other results in graphs with minimum degree at least three, has been obtained in [3]. In this paper, we prove that

THEOREM 1.2. *Let xy be an edge in a 2-connected graph G. For any positive integer k, if every vertex other than x and y has degree at least $3k+1$, then xy is contained in $k+1$ cycles of consecutive even lengths or consecutive odd lengths $m, m+2, m+4, \ldots, m+2k$ for some integer $m \geqslant k+2$.*

An immediate consequence of Theorem 1.1 (Theorem 1.2) is that each edge of a 2-connected graph with minimum degree at least $3k$ $(3k+1)$ is contained in $k+1$ cycles with the described property. If we apply the theorems to a 2-connected component or an endblock of a graph with minimum degree at least $3k$ (in the latter case, xy is an edge incident with the only cut vertex), then we obtain the following corollary.

COROLLARY 1.3. *If G is a graph with minimum degree $\delta(G) \geqslant 3k$ for any positive integer k, then G contains $k+1$ cycles $C_0, C_1, \ldots, C_k$ such that $k+1 < |E(C_0)| < |E(C_1)| < \cdots < |E(C_k)|$, $|E(C_i)| - |E(C_{i-1})| = 2$, $1 \leqslant i \leqslant k-1$, and $1 \leqslant |E(C_k)| - |E(C_{k-1})| \leqslant 2$, and furthermore, if $\delta(G) \geqslant 3k+1$, then $|E(C_k)| - |E(C_{k-1})| = 2$.*

To generalize Bondy and Vince's result [1] mentioned above, Wang [6], along the same lines of Bondy and Vince [1], proved that if the graph has minimum degree at least d, then one can require the two cycles whose lengths differ by one or two to have lengths at least d. We note that in Corollary 1.3, the last two cycles C_{k-1} and C_k have lengths at least $3k$. The corollary and the theorems are best possible in the following sense. If G is the complete graph on $3k+1$ or $3k+2$ vertices, then we cannot have more than $k+1$ cycles with the described property. In [1], Bondy and Vince gave an infinite family of nonbipartite 2-connected graphs with arbitrarily large minimum degree, but containing no cycles whose lengths differ by one. In the same paper, they asked whether there exists a function $f(k)$ such that every nonbipartite 3-connected graph with minimum degree at least $f(k)$ contains cycles of k consecutive lengths. Theorem 1.4 below, to be proved in the last section, answers the question in the affirmative.

THEOREM 1.4. *If G is a nonbipartite 3-connected graph with minimum degree at least $3k$ for any positive integer k, then G contains $2k$ cycles of consecutive lengths $m, m+1, m+2, \ldots, m+2k-1$ for some integer $m \geqslant k+2$.*

Let G be a graph. For $S \subseteq V(G)$, $G-S$ denotes the graph obtained from G by deleting all the vertices of S together with all the edges with at least one end in S. When $S=\{x\}$, we simplify this notation to $G-x$. A vertex $x \in V(G)$ is a *cut vertex* of G if $G-x$ has more components than G. A connected graph is *nonseparable* if it has no cut vertex. A *block* of G is a maximal nonseparable subgraph of G, and an *endblock* of G is a block that contains at most one cut vertex of G. If $xy \in E(G)$, we say that xy is *incident* with x and that y is a *neighbor* of x. For a subgraph H of G, $N_H(x)$ is the set of the neighbors of x which are in H, and $d_H(x)=|N_H(x)|$ is the *degree* of x in H. When no confusion can occur, we shall write $N(x)$ and $d(x)$, instead of $N_G(x)$ and $d_G(x)$. For A, $B \subseteq V(G)$, $e(A, B)$ is the number of edges with one end in A and the other end in B. When $A=\{a\}$, we simplify the notation to $e(a, B)$. For x, $y \in V(G)$, an (x, y)-*path* is a path from x to y; an (x, y)-path is *trivial* if $x=y$, in which the path consists of a single vertex.

An edge is *contracted* if it is removed and its two ends are identified. Contraction might create multiple edges. By *removing multiple edges* of a graph, we mean the removal of $m-1$ edges between every two vertices joined by m edges. Sometimes we identify a graph with its edge set.

2. STRINGS OF CYCLES

For an edge $uv \in E(G)$, by *replacing uv with a cycle*, we mean the operation of deleting the edge uv and adding a new cycle C such that $V(C) \cap V(G)=\{u, v\}$. An (x, y)-*string* (of k cycles) is the graph obtained from an (x, y)-path by replacing k edges of the path with k cycles, one edge with one cycle. Figure 1 is an (x, y)-string of 4 cycles obtained from the path $xa_1a_2\cdots a_m y$ by replacing a_1a_2, $a_{i-1}a_i$, a_ia_{i+1}, and $a_m y$ with cycles.

When no need to specify the ends, we simply use *strings*, instead of (x, y)-strings. In a string, if C is the cycle replacing uv, then u and v are called the *connection vertices* of C. C is t-*defective* if the two segments of C divided by u and v differ in length by t. A string is t-*defective* if each of its cycles is t-defective. We note that in a string of cycles distinct cycles can intersect only at connection vertices. Figure 2 is a string of three cycles in which the first and the last cycles are 1-defective and the second one is 2-defective.

An (x, y)-string S of k cycles can be represented by $S=P_0C_1P_1C_2\cdots C_kP_k$, where C_i is a cycle with connection vertices y_i and x_i, $1 \leqslant i \leqslant k$, P_j is a path

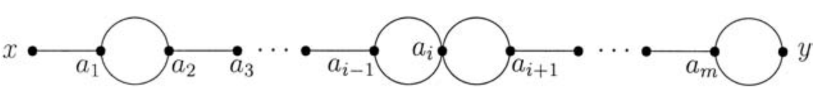

FIGURE 1

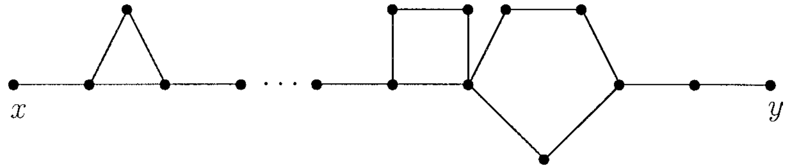

FIGURE 2

from x_j to y_{j+1}, $0 \leqslant j \leqslant k$, $x_0 = x$ and $y_{k+1} = y$. (P_j may be the trivial path consisting of a single vertex in which $x_j = y_j + 1$.) For each i, $1 \leqslant i \leqslant k$, let C'_i and C''_i be the two segments of C_i divided by its connection vertices such that $|E(C''_i)| \geqslant |E(C'_i)|$. The *length* of S is defined by

$$\ell(S) = \sum_{i=1}^{k} |E(C'_i)| + \sum_{i=0}^{k} |E(P_i)|,$$

which is the minimum length of a path from x to y in S. For any s, $1 \leqslant s \leqslant k$, let P be a path from x_s to y in $P_s C_{s+1} \cdots C_k P_k$, for instance, let $P = P_s C'_{s+1} \cdots C'_k P_k$, then $P_0 C_1 P_1 \cdots C_s P$ is an (x, y)-string of s cycles. This gives the following easy observation.

Observation 2.1. If $P_0 C_1 P_1 \cdots C_k P_k$ is an (x, y)-string of k cycles, then for any s, $1 \leqslant s \leqslant k$, there is a path P such that $\bigcup_{i=s}^{k} E(P_i) \subseteq E(P) \subseteq E(P_s C_{s+1} \cdots C_k P_k)$ and $P_0 C_1 P_1 \cdots C_s P$ is an (x, y)-string of s cycles.

LEMMA 2.2. *Let S be a t-defective (x, y)-string of k cycles. Then S contains (x, y)-paths of lengths $m, m+t, m+2t, \ldots, m+kt$, where $m = \ell(S)$.*

Proof. Let $S = P_0 C_1 P_1 \cdots C_k P_k$. For each i, $1 \leqslant i \leqslant k$, let C'_i and C''_i be the two segments of C_i divided by its connection vertices such that $|E(C''_i)| - |E(C'_i)| = t$. Taking the smaller segment of each C_i, $1 \leqslant i \leqslant k$, together with all P_j, $0 \leqslant j \leqslant k$, we have an (x, y)-path $Q_0 = P_0 C'_1 P_1 C'_2 \cdots C'_k P_k$ with $|E(Q_0)| = \ell(S)$. For each s, $1 \leqslant s \leqslant k$, let

$$Q_s = P_0 C''_1 P_1 \cdots C''_s P_s C'_{s+1} \cdots C'_k P_k,$$

where we use the larger segments of C_i for all $i \leqslant s$ and the smaller segments of C_j for all $j \geqslant s+1$. Then Q_s is an (x, y)-path of length $|E(Q_0)| + st$, $1 \leqslant s \leqslant k$. This proves the lemma. ▌

LEMMA 2.3. *Let S be an (x, y)-string of k cycles in which s cycles are 1-defective and the rest $k-s$ cycles are 2-defective. If $s \geqslant 1$, then S contains (x, y)-paths of consecutive lengths $m, m+1, m+2, \ldots, m+2k-s$, where $m = \ell(S)$.*

Proof. As before, let $S = P_0 C_1 P_1 \cdots C_k P_k$, and for each i, $1 \leqslant i \leqslant k$, let C'_i and C''_i be the two segments of C_i divided by its connection vertices such that $|E(C''_i)| - |E(C'_i)| = 1$ or 2, depending on C_i being 1- or 2-defective. We use induction on k to prove the lemma. If $k = 1$, then $s = k = 1$ and C_1 is 1-defective. ($S = P_0 C_1 P_1$.) Then $Q_0 = P_0 C'_1 P_1$ and $Q_1 = P_0 C''_1 P_1$ are two (x, y)-paths of lengths $\ell(S)$, $\ell(S) + 1$. Suppose that $k \geqslant 2$ and the lemma is true for smaller values of k. Let y_k and x_k be the connection vertices of C_k, where y_k is an end of P_{k-1} and x_k is an end of P_k. Consider the (x, y_k)-string $S' = P_0 C_1 P_1 \cdots C_{k-1} P_{k-1}$.

If C_k is 2-defective, then the number of 1-defective cycles in S' is the same as that in S. By the induction hypothesis, S' contains (x, y_k)-paths $R_0, R_1, \ldots, R_{2(k-1)-s}$ of lengths $|E(R_i)| = m' + i$, $0 \leqslant i \leqslant 2(k-1) - s$, where $m' = \ell(S')$. Let

$$Q_i = R_i \cup C'_k \cup P_k \qquad \text{for each } i, \qquad 0 \leqslant i \leqslant 2(k-1) - s,$$

and

$$Q_{2k-s-1} = R_{2k-s-3} \cup C''_k \cup P_k \qquad \text{and} \qquad Q_{2k-s} = R_{2(k-1)-s} \cup C''_k \cup P_k.$$

Since $|E(C''_k)| = |E(C'_k)| + 2$, we have that Q_i is an (x, y)-path in S of length $m + i$ for each i, $0 \leqslant i \leqslant 2k - s$, where $m = m' + |E(C'_k)| + |E(P_k)| = \ell(S)$.

If C_k is 1-defective and $s \geqslant 2$, then S' has $(s-1)$ 1-defective cycles, and by the induction hypothesis, S' contains (x, y_k)-paths $R_0, R_1, \ldots, R_{2k-s-1}$ of lengths $|E(R_i)| = m' + i$, $0 \leqslant i \leqslant 2k - s - 1$, where $m' = \ell(S')$. Let $Q_i = R_i \cup C'_k \cup P_k$, $0 \leqslant i \leqslant 2k - s - 1$, and $Q_{2k-s} = R_{2k-s-1} \cup C''_k \cup P_k$. Since $|E(C''_k)| = |E(C'_k)| + 1$, we see that $Q_0, Q_1, \ldots, Q_{2k-s}$ are (x, y)-paths in S with the required property.

If C_k is 1-defective and $s = 1$, then S' is a 2-defective (x, y_k)-string of $k - 1$ cycles. By Lemma 2.2 (with $t = 2$), S' contains (x, y_k)-paths $R_0, R_1, \ldots, R_{k-1}$ of lengths $|E(R_i)| = m' + 2i$, $0 \leqslant i \leqslant k - 1$, where $m' = \ell(S')$. Let

$$Q_{2i} = R_i \cup C'_k \cup P_k \qquad \text{and} \qquad Q_{2i+1} = R_i \cup C''_k \cup P_k, \qquad 0 \leqslant i \leqslant k-1.$$

Then $Q_0, Q_1, \ldots, Q_{2k-1}$ are (x, y)-paths in S with the required property. This completes the proof of Lemma 2.3. ▮

DEFINITION 2.4. Let $S = P_0 C_1 P_1 \cdots C_k P_k$ be an (x, y)-string of k cycles in a graph G. S is *feasible* (with respect to k and G) if all the following three statements hold.

(i) $\sum_{i=0}^{k} |E(P_i)| \neq 0$.

(ii) C_i is 2-defective for every i, $1 \leqslant i \leqslant k$, with at most one exception.

(iii) If C_j is the exceptional cycle in (ii), then C_j is 1-defective, and moreover, there is $uv \in E(C_j)$ such that $\{u, v\} \cap \{x, y\} = \emptyset$ and $d_G(u) = d_G(v) = 3k$.

In the definition, if the exceptional cycle does not exist (so (iii) does not occur), then S is called a *feasible* 2-*defective* (x, y)-string of k cycles in G.

THEOREM 2.5. *Let x and y be two distinct vertices in a* 2-*connected graph G. For any positive integer k, if every vertex other than x and y has degree at least* 3*k, then G contains a feasible* (x, y)-*string of k cycles.*

Proof. We use induction on $|V(G)|$. By the given condition, $|V(G)| \geqslant 4$. If $|V(G)| = 4$, then $k = 1$, and G is either K_4 or K_4 minus xy. Suppose that $V(G) = \{x, y, z, w\}$. Let $P_0 = x$, C_1 be the triangle on $\{x, w, z\}$, and $P_1 = zy$. Then $P_0 C_1 P_1$ is a feasible (x, y)-string in G. (The only 1-defective cycle C_1 contains wz and $d_G(w) = d_G(z) = 3$.) Suppose that $|V(G)| \geqslant 5$ and the theorem holds for every 2-connected graph G' with $|V(G')| < |V(G)|$. By symmetry, we may assume that $d(x) \leqslant d(y)$. The proof is divided into two parts.

Part I. There is no vertex $a \in V(G) \setminus \{x, y\}$ such that $N(a) \cap (N(x) \setminus \{y\})| \geqslant 2$.

For simplicity, Let $X = N(x) \setminus \{y\}$. ($X = N(x)$ if $xy \notin E(G)$.) Then we have that

$$e(v, X) \leqslant 1 \qquad \text{for each} \quad v \in V(G) \setminus \{x, y\}. \tag{1}$$

Let G^* be the simple graph obtained from G by contracting the subgraph induced by $X \cup \{x\}$ into a single vertex x^* and then removing multiple edges. By (1), the contraction only results in multiple edges between x^* and y, and thus

$$d_{G^*}(v) = d_G(v) \qquad \text{for every} \quad v \in V(G^*) \setminus \{x^*, y\}. \tag{2}$$

We note that if G^* is not 2-connected, then x^* is the only cut vertex of G^* and each block of G^* is an endblock containing x^*. Let B be a block of G^* which contains y. ($B = G^*$ if G^* is 2-connected.)

Case 1. $|V(B)| \geqslant 3$. So B is 2-connected. By the induction hypothesis and using (2), B contains a feasible (x^*, y)-string $S^* = P_0 C_1 P_1 \cdots C_k P_k$ of k cycles, with respect to k and B. (Note that since $\sum_{i=0}^{k} |E(P_i)| \neq 0$, we have that $x^* y \notin E(S^*)$.) If $|E(P_0)| \geqslant 1$, say that the edge incident with x^* in P_0 corresponds to an edge incident with some $u \in X$ in G, let $P_0' = P_0 \cup \{ux\}$. Then $P_0' C_1 P_1 \cdots C_k P_k$ is a feasible (x, y)-string of k cycles in G. If $E(P_0) = \emptyset$, that is, $P_0 = x^*$ and x^* is a connection vertex of C_1, let e and f

be the two edges incident with x^* in C_1. If e and f correspond to two edges incident with the same $u \in X$ in G, let $P'_0 = xu$ and $C'_1 = C_1$; if e and f correspond to two edges incident with different u, $w \in X$ in G, let $P'_0 = x$ and $C'_1 = C_1 \cup \{xu, xw\}$. In either case, set $S = P'_0 C'_1 P_1 \cdots C_k P_k$. By the construction, the defectiveness of C'_1 in S is the same as that of C_1 in S^*. If C'_1 is 1-defective in S, then C_1 is 1-defective in S^*, and by the induction, there is $uv \in E(C_1)$ such that $\{u, v\} \cap \{x^*, y\} = \emptyset$ and $d_B(u) = d_B(v) = 3k$. So, $uv \in E(C'_1)$, $\{u, v\} \cap \{x, y\} = \emptyset$, and by (2), $d_G(u) = d_G(v) = 3k$. (In fact, $\{u, v\} \cap (\{x, y\} \cup X) = \emptyset$.) Therefore, S is a feasible (x, y)-string of k cycles in G.

Case 2. $|V(B)| = 2$. Then $B = x^*y$ and $N(y) \setminus \{x\} \subseteq X$ in G. But $d(x) \leqslant d(y)$, and so $N(y) \setminus \{x\} = X$.

Case 2a. $G^* \neq B$. Then there exists a block D of G^* other than B. By (1) and (2), $|V(D)| \geqslant 4$. Split x^* (in D) into two new vertices u_1 and u_2 such that (i) each u_i is incident with at least one edge of D, $i = 1, 2$; (ii) if two edges incident with the same $u \in X$ in G, then they are incident with the same u_1 or u_2. Since G is 2-connected, such a splitting of x^* exists. Add a new edge joining u_1 and u_2, and denote the resulting graph by U. ($V(U) = (V(D) \setminus \{x^*\}) \cup \{u_1, u_2\}$.) Evidently U is 2-connected, $5 \leqslant |V(U)| < |V(G)|$, and

$$d_U = d_G(v) \qquad \text{for every} \quad v \in V(U) \setminus \{u_1, u_2\}.$$

By the induction hypothesis, U contains a feasible (u_1, u_2)-string $P_0 C_1 P_1 \cdots C_k P_k$. By the construction of U, (u_1, u_2) corresponds to a partition (X_1, X_2) of X such that for any $v \in V(U) \setminus \{u_1, u_2\}$, $vu_i \in E(U)$ if and only if $vb \in E(G)$ for some $b \in X_i$, $i = 1, 2$. If $|E(P_0)| \geqslant 1$, let wu_1 be the edge incident with u_1 in P_0, which corresponds to an edge wb in G for some $b \in X_1$, and define $P'_0 = (P_0 - u_1) \cup \{wb, bx\}$ and $C'_1 = C_1$. If $E(P_0) = \emptyset$, that is, $P_0 = u_1$ and u_1 is a connection vertex of C_1, then consider the two edges e and f incident with u_1 in C_1. Let e' and f' be the edges in G corresponding to e and f, respectively. If e' and f' are incident with the same $b \in X_1$ in G, let $P'_0 = xb$ and $C'_1 = (C_1 - u_1) \cup \{e', f'\}$; if e' and f' are incident with different b_1, $b_2 \in X_1$ in G, let $P'_0 = x$ and $C'_1 = (C_1 - u_1) \cup \{e', f', xb_1, xb_2\}$. Since $N(y) \setminus \{x\} = X$, using X_2 and y, similarly we transfer P_k into P'_k, and C_k into C'_k, so that $P'_0 C'_1 P_1 \cdots C'_k P'_k$ is a feasible (x, y)-string of k cycles in G.

Case 2b. $G^* = B (= x^*y)$. Then $N(y) \setminus \{x\} = X = V(G) \setminus \{x, y\}$. It follows from (1) that $k = 1$ and $d_G(v) = 3$ for every $v \in X$. So there is $uw \in E(G)$ for u, $w \in X$. Let $P_0 = x$, C_1 be the triangle on $\{x, u, w\}$, and $P_1 = wy$. Then $P_0 C_1 P_1$ is a feasible (x, y)-string in G, as required. This ends Part I.

Remark. In the arguments above, except for Case 2b, we do not need degree requirement on the vertices of $N(x) \cap N(y)$. (Note that we need to switch x and y if $d(y) < d(x)$.) As long as (1) holds for x and $d_G(v) \geqslant 3k$ for every $v \in V(G) \setminus (\{x, y\} \cup (N(x) \cap N(y)))$, we have a feasible (x, y)-string S of k cycles in G, unless $|V(G^*)| = 2$ which is Case 2b, and moreover, if S has a 1-defective cycle C such that $uv \in E(C)$ and $d_G(u) = d_G(v) = 3k$, then $\{u, v\} \cap (\{x, y\} \cup (N(x) \cap N(y))) = \varnothing$.

Part II. There is a vertex $a \in V(G) \setminus \{x, y\}$ such that $|N(a) \cap (N(x) \setminus \{y\})| \geqslant 2$.

Let $\{x_1, x_2\} \subseteq N(a) \cap (N(x) \setminus \{y\})$ and denote by Q_a the cycle xx_1ax_2x of length 4. Suppose that a has been chosen so that the component in $G - V(Q_a)$ containing y is as large as possible. For simplicity, we use Q for Q_a. Let H be the component of $G - V(Q)$ which contains y.

If $e(\{x_1, x_2\}, H) = 0$, then, since G is 2-connected, we have $e(a, H) \geqslant 1$, and moreover, if we let $G' = G - V(H)$, then G' is 2-connected and $d_{G'}(v) = d_G(v)$ for every $v \in V(G') \setminus \{x, a\}$. By the induction hypothesis, G' contains a feasible (x, a)-string of k cycles, which can be extended to a feasible (x, y)-string of k cycles in G by adding an (a, y)-path with all internal vertices in H. Suppose therefore that $e(\{x_1, x_2\}, H) \geqslant 1$, say $e(x_1, H) \geqslant 1$.

If $k = 1$, let $P_0 = x$, $C_1 = Q$, and P_1 be an (x_1, y)-path with all internal vertices in H. Then $P_0 C_1 P_1$ is a feasible (x, y)-string in G, as required. (Note that $|E(P_1)| \geqslant 1$ and C_1 is 2-defective in the string.) In the rest of the proof, assume that $k \geqslant 2$. We claim that either we are done or

$$\text{if } e(a, H) \geqslant 1, \text{ then } H \text{ is the only component of } G - V(Q). \tag{3}$$

If (3) is not true, let $R = G - V(H)$. Then R is 2-connected and $|V(R)| \geqslant 5$. (Note that $V(Q) \subseteq V(R)$.) If there is $a' \in V(R) \setminus \{x, a\}$ such that $|N_R(a') \cap (N_R(x) \setminus \{a\})| \geqslant 2$, let Q' be a cycle formed by a', x and two vertices from $N_R(a') \cap (N_R(x) \setminus \{a\})$. Then, since $e(a, H) \geqslant 1$, there is a component in $G - V(Q')$ which contains $V(H) \cup \{a\}$, contrary to the choice of Q. Suppose therefore that $|N_R(v) \cap (N_R(x) \setminus \{a\}| \leqslant 1$ for every $v \in V(R) \setminus \{x, a\}$. That is, $e(v, N_R(x) \setminus \{a\}) \leqslant 1$ for every $v \in V(R) \setminus \{x, a\}$. Note that $d_R(v) = d_G(v)$ for every $v \in V(R) \setminus V(Q)$. Let $z \in V(R) \setminus V(Q)$. Since $k \geqslant 2$, it follows that

$$e(z, V(R) \setminus (N_R(x) \cup \{x, a\})) \geqslant 3k - 3 \geqslant 3.$$

Thus, if R^* denotes the simple graph obtained from R by contracting the subgraph induced by $(N_R(x) \setminus \{a\}) \cup \{x\}$ into a single vertex x^* and then removing multiple edges between x^* and a, then $|V(R^*)| \geqslant 5$. Now, we are in situations similar to Case 1 and Case 2a of Part I, with R and (x, a) in place of G and (x, y), respectively. Note that Case 2b of Part I cannot occur since $|V(R^*)| \geqslant 5$. Since $\{x_1, x_2\} \subseteq N_R(x) \cap N_R(a)$ and $d_R(v) = d_G(v)$ for every $v \in V(R) \backslash V(Q)$, and by the remark at the end of Part I, we have a feasible (x, a)-string of k cycles in R, which can be extended to a feasible (x, y)-string of k cycles in G by adding an (a, y)-path with all internal vertices in H. (Such an (a, y)-path exists since $e(a, H) \geqslant 1$.) This proves (3).

If $|V(H)| = 1$, that is, $H = y$, let $R = G - y$. Since $k \geqslant 2$, we have that $|V(R)| \geqslant 7$, and by (3), $e(a, H) = 0$. So $N(y) \backslash \{x\} \subseteq \{x_1, x_2\}$. But $d(x) \leqslant d(y)$, and thus $N(y) \backslash \{x\} = N(x) \backslash \{y\} = \{x_1, x_2\}$. Let R' be the graph obtained from $G - \{x, y\}$ by adding x_1x_2 (if $x_1x_2 \notin E(G)$). Then R' is 2-connected and $d_{R'}(v) = d_G(v)$ for every $v \in V(R') \backslash \{x_1, x_2\}$. By the induction hypothesis, R' contains a feasible (x_1, x_2)-string of k cycles, which can be extended to a feasible (x, y)-string of k cycles in G by adding xx_1 and x_2y. Suppose therefore that $|V(H)| \geqslant 2$.

If $e(\{x_1, x_2\}, H - y) = 0$, then H is not the only component of $G - V(Q)$, and by (3), we have that $e(a, H) = 0$. It follows, since G is 2-connected, that $e(x, H - y) \neq 0$. Let H' be the graph induced by $V(H) \cup \{x\}$ plus xy if $xy \notin E(G)$. Then H' is 2-connected and $d_{H'}(v) = d_G(v)$ for every $v \in V(H') \backslash \{x, y\}$. By the induction hypothesis, H' contains a feasible (x, y)-string S of k cycles. Since S is feasible, the possibly added xy is not in S, and hence S is a feasible (x, y)-string of k cycles in G, as required. Suppose therefore that $e(\{x_1, x_2\}, H - y) \geqslant 1$. Without loss of generality, suppose that

$$\text{there is} \quad z \in V(H - y) \quad \text{such that} \quad x_1 z \in E(G). \tag{4}$$

Let B be an endblock of H and b the unique cut vertex of H contained in B (if H is 2-connected, $B = H$ and $b = y$) such that $y \notin V(B - b)$. Since $k \geqslant 2$ and $y \notin V(B - b)$, we have that $|V(B)| \geqslant 3$ and so B is 2-connected. Let P be a path from b to y in H. (Note that $V(P) \cap V(B - b) = \varnothing$ and it is possible that $P = b = y$.)

If $z \in V(B - b)$ (z is defined in (4)), let F be the graph induced by $V(B) \cup \{x_1\}$ plus x_1b if $x_1b \notin E(G)$. Then, F is 2-connected and for every $v \in V(F) \backslash \{x_1, b\}$

$$d_F(v) \geqslant d_G(v) - 3 \geqslant 3k - 3 = 3(k - 1). \tag{5}$$

By the induction hypothesis, F contains a feasible (x_1, b)-string $S' = P_1C_2 \cdots C_kP_k$ of $k - 1$ cycles. (Note that the subscripts of C_i start from 2.) Let $P_0 = x$, $C_1 = Q$, $P'_k = P_k \cup P$, and $S = P_0C_1P_1 \cdots C_kP'_k$. Clearly, C_1 is 2-defective in S. If some C_i, $i \geqslant 2$, is 1-defective in S, then it is also 1-defective in S', and by the induction, there is $uv \in E(C_i)$ such that

$\{u, v\} \cap \{x_1, b\} = \emptyset$ (so $\{u, v\} \cap \{x, y\} = \emptyset$) and $d_F(u) = d_F(v) = 3(k-1)$, which implies, by (5), $d_G(u) = d_G(v) = 3k$. This shows that S is a feasible (x, y)-string of k cycles in G. Thus, by the arbitrariness of z, we may assume that $e(x_1, B-b) = 0$, and similarly, $e(x_2, B-b) = 0$. Therefore,

$$e(\{x_1, x_2\}, B-b) = 0, \tag{6}$$

which implies that for every $v \in V(B-b)$,

$$d_B(v) \geqslant d_G(v) - 2 > 3(k-1). \tag{7}$$

By the induction hypothesis, for any $u \in V(B-b)$, B contains a feasible (u, b)-string S' of $k-1$ cycles, and by (7), each cycle in S' is 2-defective. Hence,

$$B \text{ contains a feasible 2-defective } (u, b)\text{-string of } k-1 \text{ cycles.} \tag{8}$$

If $e(a, B-b) = 0$, then, since G is 2-connected, $e(x, B-b) \neq 0$. Let F_x be the graph induced by $V(B) \cup \{x\}$ plus xb if $xb \notin E(G)$. Then F_x is 2-connected, and using (6), $d_{F_x}(v) = d_G(v)$ for every $v \in V(F_x) \setminus \{x, b\}$. By the induction hypothesis, F_x contains a feasible (x, b)-string of k cycles, which can be extended to a feasible (x, y)-string of k cycles in G by adding P. Suppose therefore that

$$N_G(a) \cap V(B-b) \neq \emptyset. \tag{9}$$

If $e(x, B-b) = 0$, then let F_a be the graph induced by $V(B) \cup \{a\}$ plus ab if $ab \notin E(G)$. As above, with a in place of x, we see that F_a contains a feasible (a, b)-string of k cycles, which can be extended to a feasible (x, y)-string of k cycles in G by adding xx_1a and P. Suppose therefore that

$$N_G(x) \cap V(B-b) \neq \emptyset. \tag{10}$$

Let $h \in V(H)$ which is not a cut vertex of H. Then there is a path P' from b to y in $H - V(B-b)$ such that $h \notin V(P')$. If $e(h, Q) = 4$, then by (6) $h \notin V(B-b)$. Let C_1 be the cycle ax_1hx_2a and $P_0 = xx_2$. By (9) and (8), B contains a feasible 2-defective (u, b)-string S' of $k-1$ cycles, where $au \in E(G)$. Then $P_0C_1auS'bP'y$ is a feasible 2-defective (x, y)-string of k cycles in G. Suppose thus that

$$e(h, Q) \leqslant 3. \tag{11}$$

If H is 2-connected, then $H-y=B-b$, and by (6), H is not the only component of $G-V(Q)$. It follows from (3) that $e(a, H)=0$, contradicting (9). Suppose therefore that H is not 2-connected. Let $B_1, B_2, \ldots, B_m$ be all endblocks of H and b_i the unique cut vertex of H contained in B_i, $1 \leqslant i \leqslant m$, where $m \geqslant 2$.

If $y \notin V(B_i-b_i)$ for each B_i, $1 \leqslant i \leqslant m$, let $D=H-(\bigcup_{i=1}^{m} V(B_i-b_i))$. Then $y \in V(D)$. Now, (6) holds for every B_i, $1 \leqslant i \leqslant m$, and hence $z \in V(D)$. (z is defined in (4).) It is not difficult to verify that there are distinct j and t such that D contains a (z, b_j)-path P_j and a (y, b_t)-path P_t such that $V(P_j) \cap V(P_t)=\varnothing$. By (9) and (8), B_j (B_t) contains a feasible 2-defective (u_j, b_j)-string S_j((u_t, b_t)-string S_t) of $k-1$ cycles, where $u_j \in V(B_j-b_j)$ ($u_t \in V(B_t-b_t)$), and au_j, $au_t \in E(G)$. Then $xx_1zP_jb_jS_ju_jau_tS_tb_tP_ty$ is a feasible 2-defective (x, y)-string of $2(k-1)$ cycles in G, and the theorem follows from Observation 2.1. Suppose therefore that $y \in V(B_i-b_i)$ for some i, and by relabeling if necessary, we suppose that $y \in V(B_1-b_1)$. Then, by (10) and (8), there is $u_2 \in V(B_2-b_2)$ with $xu_2 \in E(G)$ and B_2 contains a feasible 2-defective (u_2, b_2)-string S_2 of $k-1$ cycles.

By (11), for every $v \in V(B_1) \setminus \{y, b_1\}$,

$$d_{B_1}(v) \geqslant d_G(v)-3 \geqslant 3(k-1). \tag{12}$$

If $|V(B_1)| \geqslant 3$, then by the induction hypothesis, B_1 contains a feasible (b_1, y)-string S_1 of $k-1$ cycles. By Observation 2.1, we may reduce S_2 to a 2-defective (u_2, b_2)-string S_2' of one cycle. Let P_{12} be a path from b_1 to b_2 in H and let $S=xu_2S_2'b_2P_{12}b_1S_1y$. As before, using (12), S is a feasible (x, y)-string of k cycles in G. Suppose thus that $|V(B_1)|=2$. So $B_1=b_1y$.

If $e(y, \{x_1, x_2\})=2$, let C_k be the cycle $x_1ax_2yx_1$, $P_k=y$, and P_z be a (b_2, z)-path in H. Then $xu_2S_2b_2P_zzx_1C_kP_k$ is a feasible 2-defective (x, y)-string of k cycles in G. Suppose thus that $e(y, \{x_1, x_2\}) \leqslant 1$. Since $B_1=b_1y$, we see that $|N(y) \setminus \{x\}| \leqslant 3$. On the other hand, by (10), $e(x, B_i) \geqslant 1$ for every $i \geqslant 2$, and we have that $|N(x) \setminus \{y\}| \geqslant 2+(m-1)=m+1$. But $d(x) \leqslant d(y)$, and so $m=2$. This means that H is a chain of blocks which can be represented by $B_1A_1A_2 \cdots A_tB_2$, where each A_i is a block containing exactly two cut vertices of H, $1 \leqslant i \leqslant t$.

If there is A_j with $|V(A_j)| \geqslant 3$, let b and b' be the two cut vertices contained in A_j such that $V(A_{j-1}) \cap V(A_j)=\{b\}$ and $V(A_j) \cap V(A_{j+1})=\{b'\}$, where $A_0=B_1$ and $A_{t+1}=B_2$. By (11), for every $v \in V(A_j) \setminus \{b, b'\}$,

$$d_{A_j}(v) \geqslant d_G(v)-3 \geqslant 3(k-1). \tag{13}$$

It follows from the induction hypothesis that A_j contains a feasible (b, b')-string S_j of $k-1$ cycles. Let P_1 be a (b_1, b)-path in H and P_2 a (b_2, b')-path in H. As before, use Observation 2.1 to reduce S_2 to a 2-defective (u_2, b_2)-string S'_2 of one cycle, and then, using (13), $xu_2S'_2b_2P_2b'S_jbP_1b_1y$ is a feasible (x, y)-string of k cycles in G.

Suppose therefore that $|V(A_j)| = 2$ for each i, that is, each A_i is a single edge, $1 \leqslant i \leqslant t$. By (9) and (3), H is the only component of $G - V(Q)$, and hence $e(x_1, H) \geqslant 3k - 3 \geqslant 3$, which implies, since $|V(B_1)| = 2$ and by (6), that $t \geqslant 1$. So we have that $d_G(b_1) \leqslant 6$, which implies that $k = 2$ and

$$e(v, Q) = 4 \qquad \text{for each} \quad v \in V(A_1A_2 \cdots A_t) \setminus \{b_2\}. \tag{14}$$

If $t = 1$, then $A_1 = b_1b_2$ and $e(x_1, A_1) = 2$, which gives a triangle T on $\{x_1, b_2, b_1\}$, and moreover, $d_G(x_1) = d_G(b_1) = 6$. Thus xQx_1Tb_1y is a feasible (x, y)-string of k $(=2)$ cycles in G. If $t \geqslant 2$, let $A_1 = b_1b$. We have that $d_G(b_1) = d_G(b) = 6$, and by (14), we have a triangle T' on $\{x_1, b, b_1\}$. Then $xQx_1T'b_1y$ is a feasible (x, y)-string of k $(=2)$ cycles in G. This completes the proof of Theorem 2.5. ∎

As we have already seen in the proof of Theorem 2.5, if there is no edge $uv \in E(G - \{x, y\})$ such that $d_G(u) = d_G(v) = 3k$, then statement (iii) in Definition 2.4 cannot occur and so the string is feasible 2-defective. As an immediate consequence of Theorem 2.5, we have that

THEOREM 2.6. *Let x and y be two distinct vertices in a 2-connected graph G. For any positive integer k, if $d_G(v) \geqslant 3k$ for every $v \in V(G) \setminus \{x, y\}$, and in addition, if there is no edge $uv \in E(G - \{x, y\})$ such that $d_G(u) = d_G(v) = 3k$, then G contains a feasible 2-defective (x, y)-string of k cycles.*

3. PATHS AND CYCLES OF CONSECUTIVE LENGTHS

Consider a feasible (x, y)-string $S = P_0C_1P_1 \cdots C_kP_k$. We have that

$$\ell(S) = \sum_{i=1}^{k} |E(C'_i)| + \sum_{i=0}^{k} |E(P_i)| \geqslant k + 1.$$

COROLLARY 3.1. *Let x and y be two distinct vertices in a 2-connected graph G. For any positive integer k, if $d_G(v) \geqslant 3k$ for every $v \in V(G) \setminus \{x, y\}$, then G contains $k+1$ (x, y)-paths $R_0, R_1, \ldots, R_k$ such that $k < |E(R_0)| < |E(R_1)| < \cdots < |E(R_k)|$, $|E(R_i)| - |E(R_{i-1})| = 2$, $1 \leqslant i \leqslant k-1$, and $1 \leqslant |E(R_k)| - |E(R_{k-1})| \leqslant 2$.*

Proof. By Theorem 2.5, G contains a feasible (x, y)-string S of k cycles. If S is 2-defective, then by Lemma 2.2, S contains $k+1$ (x, y)-paths of lengths $m, m+2, \ldots, m+2k$, where $m = \ell(S) > k$, and we are done.

Otherwise, one and only one cycle in S is 1-defective, and by Lemma 2.3, S contains (x, y)-paths P_i of lengths $m+i$, $0 \leqslant i \leqslant 2k-1$, where $m = \ell(S) > k$. Let $R_i = P_{2i}$, $0 \leqslant i \leqslant k-1$, and $R_k = P_{2k-1}$. Then $R_0, R_1, \ldots, R_k$ are $k+1$ (x, y)-paths with the required property. This proves the corollary. ∎

COROLLARY 3.2. *Let x and y be two distinct vertices in a 2-connected graph G. For any positive integer k, if $d_G(v) \geqslant 3k+1$ for every $v \in V(G) \setminus \{x, y\}$, then G contains $k+1$ (x, y)-paths of consecutive even lengths or consecutive odd lengths m, $m+2$, $m+4$, $\ldots$, $m+2k$ for some integer $m \geqslant k+1$.*

Proof. This follows from Theorem 2.6 and Lemma 2.2. ∎

In Corollaries 3.1 and 3.2 above, all the (x, y)-paths have lengths at least $k+1 \geqslant 2$, and hence the edge xy is not contained in any of the (x, y)-paths. So it is clear that Theorems 1.1 and 1.2 follows from Corollaries 3.1 and 3.2, respectively, by adding xy to those (x, y)-paths.

Instead of proving Theorem 1.4 directly, we shall prove Theorem 3.5 in this section. We need some additional notation and two more lemmas. Let C be an odd cycle. The *diameter-graph* of C is the graph with vertex set $V(C)$ in which two vertices u and v are joined by an edge if and only if the two segments of C divided by u and v differ in length by one.

LEMMA 3.3. *The diameter-graph of any odd cycle is connected.*

Proof. Let C be an odd cycle with vertices $x_1, x_2, \ldots, x_{2t+1}$, around C in that order. Let D be the diameter-graph of C. By definition, $\{x_i x_{i+t}\colon 1 \leqslant i \leqslant t\}$ and $\{x_i x_{i+t+1}\colon 1 \leqslant i \leqslant t\}$ are two matchings (independent sets of edges) in D, which gives a hamiltonian path (of D) from x_{t+1} to x_{2t+1}, and so D is connected. (In fact, $x_{t+1} x_{2t+1} \in E(D)$ and hence D is a hamiltonian cycle, so isomorphic to C.) ∎

A cycle C is *non-separating* in G if $G - V(C)$ is connected. Non-separating cycles were studied intensively in [5] by Thomassen and Toft. In [1], Bondy and Vince proved that every nonbipartite 3-connected graph contains a non-separating induced odd cycle.

LEMMA 3.4. *Let G be a graph with minimum degree at least four. If G contains a nonseparating induced odd cycle, then G contains a non-separating induced odd cycle C such that either C is a triangle or $e(v, C) \leqslant 2$ for every $v \in V(G) \setminus V(C)$ which is not a cut vertex of $G - V(C)$.*

Proof. Choose C to be a non-separating induced odd cycle with $|V(C)|$ minimum. Let $H = G - V(C)$. If C is a triangle, we are done. Suppose that

$|V(C)| \geqslant 5$. Let $u \in V(H)$ which is not a cut vertex of H. Suppose, to the contrary, that $e(u, C) \geqslant 3$. Let $N_C(u) = \{x_1, x_2, ..., x_t\}$ and let P_i be the segment of C with $V(P_i) \cap N_C(u) = \{x_i, x_{i+1}\}$, $1 \leqslant i \leqslant t$ ($x_{t+1} = x_1$). Since $\sum_{i=1}^{t} |E(P_i)| = |V(C)|$ is odd, we have that $|E(P_s)|$ is odd for some s, $1 \leqslant s \leqslant t$. Choose $P \in \{P_1, P_2, ..., P_t\}$ such that $|E(P)|$ is odd, and subject to this, $|E(P)|$ is minimum. By relabeling if necessary, we may assume that $P = P_1$. Let $C' = P_1 \cup \{ux_1, ux_2\}$. Then C' is an induced odd cycle. Since $t \geqslant 3$ and $d_G(x_3) \geqslant 4$, there is $z \in V(H-u)$ such that $x_3 z \in E(G)$, which implies, since u is not a cut vertex of H, that C' is non-separating in G. By the minimality of C, $|V(C')| \geqslant |V(C)|$, which implies that $t = 3$ and $|E(P_2)| = |E(P_3)| = 1$, and hence, by the minimality of P_1, $|E(P_1)| = 1$. It follows that $|V(C)| = 3$, a contradiction. This proves Lemma 3.4. ▮

THEOREM 3.5. *If G is a 2-connected graph with minimum degree at least 3k for any positive integer k, and in addition, if G contains a non-separating induced odd cycle, then G contains 2k cycles of consecutive lengths m, m+1, m+2, ..., m+2k−1 for some integer m ⩾ k+2.*

Proof. If $k = 1$, as in [1], let u and v be two vertices dividing the non-separating induced odd cycle into two segments P_1 and P_2 such that $|E(P_2)| - |E(P_1)| = 1$, and let P be a (u, v)-path that has only u and v in common with the cycle. Then $P_1 \cup P$ and $P_2 \cup P$ are two cycles with the required property. In what follows, we assume that $k \geqslant 2$.

By Lemma 3.4, G contains a non-separating induced odd cycle C such that either C is a triangle or $e(v, C) \leqslant 2$ for every $v \in V(H)$ which is not a cut vertex of H, where $H = G - V(C)$.

Case 1. C is a triangle. Suppose that $V(C) = \{x_1, x_2, x_3\}$. Let B_1, B_2, ..., B_t be all endblocks of H and b_i the unique cut vertex of H contained in B_i, $1 \leqslant i \leqslant t$. (If H is 2-connected, then $t = 1$, $B_1 = H$, and b_1 is an arbitrary vertex in H.) Since G is 2-connected, we have that $e(C, B_i - b_i) \geqslant 1$ for each i, $1 \leqslant i \leqslant t$.

If there is $x \in V(C)$ such that $e(x, B_i - b_i) \geqslant 1$ for every i, $1 \leqslant i \leqslant t$, say $x = x_1$, let R be the graph induced by $V(H) \cup \{x_1\}$. Then R is 2-connected and for all $v \in V(R)$

$$d_R(v) \geqslant d_G(v) - 2 \geqslant 3k - 2 > 3(k-1).$$

Since $d_G(x_2) \geqslant 3k \geqslant 6$, there is $y \in V(H)$ such that $x_2 y \in E(G)$. Applying Theorem 2.6 to R with x_1 and y as the two specified vertices, we have a feasible 2-defective (x_1, y)-string S' of $k-1$ cycles in R. Since S' is feasible, we have that $\ell(S') \geqslant (k-1)+1 = k$. Let $S = x_2 C x_1 S' y$. Then S is an (x_2, y)-string of k cycles in G, in which C is the only 1-defective cycle and all the other $k-1$ cycles are 2-defective. By Lemma 2.3, S contains (x_2, y)-paths of lengths p, $p+1$, ..., $p+2k-1$, where $p=\ell(S)=\ell(S')+1\geqslant k+1$. These (x_2, y)-paths plus the edge x_2y give $2k$ cycles of lengths $m, m+1, \ldots,$ $m+2k-1$, where $m=p+1\geqslant k+2$, as required by the theorem. Suppose that this is not the case.

Without loss of generality, assume that there are $y_1\in V(B_1-b_1)$ and $y_2\in V(B_2-b_2)$ such that $x_1y_1, x_2y_2\in E(G)$. Let R be the graph induced by $V(B_1)\cup\{x_1\}$ plus x_1b_1 if $x_1b_1\notin E(G)$. Then R is 2-connected and for all $v\in V(R)\setminus\{x_1, b_1\}$, $d_R(v)\geqslant 3k-2>3(k-1)$. As above, with b_1 in place of y, we have a feasible 2-defective (x_1, b_1)-string S' of $k-1$ cycles in R. Let P be a path from b_1 to y_2 in H. Then $x_2Cx_1S'b_1Py_2$ is an (x_2, y_2)-string of k cycles in G, in which C is the only 1-defective cycle and all the other $k-1$ cycles are 2-defective. As above, by Lemma 2.3, we have $2k$ (x_2, y_2)-paths of consecutive lengths, which plus x_2y_2 give the required $2k$ cycles.

Case 2. $|V(C)|\geqslant 5$ and $e(v, C)\leqslant 2$, and so

$$d_H(v)\geqslant d_G(v)-2\geqslant 3k-2>3(k-1)$$

for every $v\in V(H)$ which is not a cut vertex of H. Let B be an endblock of H and b the unique cut vertex of H contained in B. (If H is 2-connected, then $B=H$ and b is an arbitrary vertex in H.) Let $x_1\in V(C)$ such that $x_1y_1\in E(G)$ for some $y_1\in V(B-b)$. (x_1 exists since G is 2-connected.) Let $x_2\in V(C)$ such that the two segments of C divided by x_1 and x_2 differ in length by one. Since $d_G(x_2)\geqslant 3k\geqslant 6$, there is $y_2\in V(H)\setminus\{y_1\}$ such that $x_2y_2\in E(G)$.

If H is 2-connected, applying Theorem 2.6 to H with y_1 and y_2 as the two specified vertices, we have a feasible 2-defective (y_1, y_2)-string S' of $k-1$ cycles in H. Then $x_2Cx_1y_1S'y_2$ is an (x_2, y_2)-string of k cycles in G, in which C is the only 1-defective cycle and all the other $k-1$ cycles are 2-defective, and as before, we are done.

Suppose thus that H is not 2-connected. If $y_2\notin V(B-b)$, we apply Theorem 2.6 to B with y_1 and b as the two specified vertices and obtain a feasible 2-defective (y_1, b)-string S' of $k-1$ cycles in B. Let P be a path from b to y_2 in H. Then $x_2Cx_1y_1S'bPy_2$ is an (x_2, y_2)-string of k cycles in G, in which C is the only 1-defective cycle and all the other $k-1$ cycles are 2-defective, and the theorem follows as before. Suppose therefore that $y_2\in(B-b)$, which means, by the arbitrariness of y_2, that $N_H(x_2)\subseteq V(B-b)$. Switching x_1 and x_2, and by the same argument, we have that $N_H(x_1)\subseteq V(B-b)$. Note that x_1x_2 is an edge of the diameter-graph of C, which is connected by Lemma 3.3. Applying this argument to every edge of the diameter-graph of C, either we are done, or $N_H(x)\subseteq V(B-b)$ for every $x\in V(C)$ which implies that b is a cut vertex of G, contradicting that G is 2-connected. This ends Case 2, and completes the proof of Theorem 3.5. ▮

Proof of Theorem 1.4. By [1, Lemma 2] (or from the proof of [1, Theorem 2]), G contains a non-separating induced odd cycle, and the theorem follows from Theorem 3.5. (The case that $k = 1$ is a result of Bondy and Vince [1].) ∎

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