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NEW CONDITIONS FOR THE ADDITIVE INVERSE EIGENVALUE PROBLEM FOR MATRICES*

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Abstract—This paper deals with the following inverse eigenvalue problem: Given an n by n real symmetric matrix A and a set of real numbers $\{\lambda_i\}_1^n$, find a diagonal matrix D such that $A+D$ has eigenvalues λ_i. For the solvability of this problem a number of necessary conditions and sufficient conditions are known. In this work, new necessary conditions are derived, while some sufficient conditions are optimized. In particular, one sufficient condition due to Morel is obtained by optimizing a sufficient condition discovered by Laborde.

1. INTRODUCTION

Consider the following inverse eigenvalue problem for matrices: Given an n by n real symmetric matrix A and a set of real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, find a real diagonal matrix D such that $A+D$ has as its eigenvalues the numbers λ_i. This problem is called the Additive Inverse Eigenvalue Problem (AIEP).

The AIEP was posed by Downing and Householder [1] in 1956. To compute a solution to this problem, they proposed a numerical algorithm based on the Newton's method, in which they assume the existence of a solution, although they did not discuss whether a solution is possible. We quote them [1, p. 203]: "Criteria for the existence of solutions do not appear to be known. In practical applications, however, circumstantial evidence may be sufficient to justify the assumption that at least one solution exists."

There have been important advances in the question of the existence of a solution to the AIEP problem in the last 20 years. Necessary conditions and sufficient conditions have been obtained by Hadeler [2], Laborde [3], De Oliveira [4,5], Friedland [6], Morel [7–9] and Biegler-König [10].

In this paper, new necessary conditions for the solvability of the AIEP are derived. Concerning the sufficient conditions, some of them are optimized. The paper is organized as follows. In Section 2, we include a summary of necessary conditions which are known, and we derived new necessary conditions. In Section 3, we present the most well known sufficient conditions and we optimize some of them. In particular, one sufficient condition due to Morel is obtained by optimizing a sufficient condition discovered by Laborde.

2. NECESSARY CONDITIONS

Consider the AIEP as defined in Section 1. We may assume without loss of generality, that the diagonal entries of A are all zeros, since we can incorporate them into the diagonal matrix D. We may also suppose that the desired eigenvalues λ_i sum to zero, since we can always translate the spectrum by an amount h by adding the scalar matrix hI, where $h = -\frac{1}{n}\sum \lambda_i$. Thus, if $A+D$ has the modified spectrum $\lambda_i + h$, then $A+D-hI$ will have the original spectrum.

In this section, we assume that the AIEP problem has a solution $D=\text{diag}\{d_1, d_2, \dots, d_n\}$. That is, we suppose that there exist D such that the matrix $A+D$ has eigenvalues λ_i. Since all diagonal

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX entries of A are zeros and $\sum_{i=1}^{n}\lambda_i = 0$, it follows that

$$\sum_{i=1}^{n} d_i = 0, \tag{2.1}$$

and

$$\|A+D\|_F^2 = \|A\|_F^2 + \|D\|_F^2, \tag{2.2}$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm. From (2.2) and the symmetry of A we have

$$\sum_{i=1}^{n}\mu_i^2 \le \sum_{i=1}^{n}\lambda_i^2, \tag{NC 1}$$

where the μ_i are the eigenvalues of A. This is the simplest necessary condition, and it tells us that the given eigenvalues must be more spread out than the eigenvalues of A.

We shall assume, for the rest of this paper, that the given eigenvalues λ_i and the eigenvalues μ_i of A satisfy

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n, \tag{2.3}$$
$$\mu_1 \le \mu_2 \le \cdots \le \mu_n. \tag{2.4}$$

A second necessary condition can be obtained by using properties of the Rayleigh quotient. In fact, for all vectors $x \neq 0$, we have

$$\lambda_1 \le \frac{x^T(A+D)x}{x^T x} \le \lambda_n. \tag{2.5}$$

In particular, for $x = (1,1,\dots,1)^T$ we obtain, recalling Condition (2.1),

$$\lambda_1 \le \frac{1}{n}\sum_{i\neq j} a_{ij} \le \lambda_n. \tag{NC 2}$$

Now, for $x = e_i$, the i^{th} unit vector, we have from (2.5)

$$\lambda_1 \le d_i \le \lambda_n, \qquad i = 1,2,\dots,n. \tag{2.6}$$

The following lemma together with the monotonicity theorem (Wilkinson [11, p. 102]) will give us another necessary condition.

LEMMA 1. *Suppose that $d_1 + d_2 + \cdots + d_n = 0$ with d_k all real. Then,*

$$d_k^2 \le \frac{n-1}{n}\sum_{i=1}^{n} d_i^2, \quad \text{for all } k. \tag{2.7}$$

PROOF. If $d_k = 0$, for all k, there is nothing to prove. Let $d_k \neq 0$ and $x_i = -\frac{d_i}{d_k}$, for $i \neq k$. Define

$$f(x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = 1 + \sum_{i\neq k} x_i^2.$$

It is clear that f attains its minimum value, under the constraint $\sum_{i\neq k} x_i = 1$, at $(1/n-1, 1/n-1, \dots, 1/n-1)$. Hence,

$$f\left(\frac{1}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1}\right) = \frac{n}{n-1} \le 1 + \sum_{i\neq k} x_i^2 = \frac{\sum d_i^2}{d_k^2},$$

and (2.7) follows. In particular, for $d_k = 1$ and $d_i = -1/n-1$, $i \neq k$, we have an equality in (2.7), which shows that the inequality is sharp. ∎

We now consider another necessary condition. From (2.1), we have $\max_k d_k > 0$ and $\min_k d_k < 0$. We apply the monotonicity theorem [11, p. 102] and we obtain

$$\mu_i + \min_k d_k \leq \lambda_i \leq \mu_i + \max_k d_k, \qquad i = 1, 2, \ldots, n,$$

whence

$$\max_i(\lambda_i - \mu_i) \leq \max_k d_k, \tag{2.8}$$

$$\max_i(\mu_i - \lambda_i) \leq -\min_k d_k. \tag{2.9}$$

Using the result of the previous lemma, (2.8) and (2.9), we obtain

$$\max_i(\lambda_i - \mu_i) \leq \sqrt{\frac{n-1}{n}} \left(\sum d_i^2\right)^{1/2}, \tag{2.10}$$

$$\max_i(\mu_i - \lambda_i) \leq \sqrt{\frac{n-1}{n}} \left(\sum d_i^2\right)^{1/2}. \tag{2.11}$$

From (2.2), we have

$$\sum \lambda_i^2 = \sum \mu_i^2 + \sum d_i^2. \tag{2.12}$$

Now, from (2.10), (2.11) and (2.12), follow the necessary conditions

$$\max_i(\lambda_i - \mu_i) \leq \sqrt{\frac{n-1}{n}} \left(\sum \lambda_i^2 - \sum \mu_i^2\right)^{1/2}. \tag{NC 3}$$

$$\max_i(\mu_i - \lambda_i) \leq \sqrt{\frac{n-1}{n}} \left(\sum \lambda_i^2 - \sum \mu_i^2\right)^{1/2}. \tag{NC 4}$$

Let $M = \max_k d_k$ and $m = \min_k d_k$. From the monotonicity theorem, we have

$$m + \mu_{i+1} \leq \lambda_{i+1} \leq M + \mu_{i+1},$$
$$-M - \mu_i \leq -\lambda_i \leq -m - \mu_i.$$

Then,

$$-(M-m) + (\mu_{i+1} - \mu_i) \leq \lambda_{i+1} - \lambda_i \leq M - m + (\mu_{i+1} - \mu_i),$$

whence,

$$\max_i |(\lambda_{i+1} - \lambda_i) - (\mu_{i+1} - \mu_i)| \leq M - m.$$

From this inequality and the previous lemma, it follows that

$$\max_i |(\lambda_{i+1} - \lambda_i) - (\mu_{i+1} - \mu_i)| \leq 2\sqrt{\frac{n-1}{n}} \left(\sum \lambda_i^2 - \sum \mu_i^2\right)^{1/2}. \tag{NC 5}$$

In order to derive the next necessary condition, we recall the Wielandt-Hoffman inequality [12]. If B and C are Hermitian matrices with eigenvalues β_i ($\beta_i \leq \beta_{i+1}$) and γ_i ($\gamma_i \leq \gamma_{i+1}$), respectively, then

$$\sum (\beta_i - \gamma_i)^2 \leq \|B - C\|_F^2. \tag{2.13}$$

Applying this inequality to $B = A + D$ and $C = A$, we obtain

$$\sum (\lambda_i - \mu_i)^2 \leq \|D\|_F^2 = \sum \lambda_i^2 - \sum \mu_i^2, \tag{2.14}$$

whence

$$\sum \mu_i^2 \leq \sum \lambda_i \, \mu_i. \tag{2.15}$$

For $B = -(A + D)$ and $C = -A$, the inequality (2.13) gives us

$$\sum \mu_i^2 \geq \sum \lambda_i \, \mu_{n+1-i}. \tag{2.16}$$

From (2.15) and (2.16), we have the necessary condition

$$\sum \lambda_i \, \mu_{n+1-i} \leq \sum \mu_i^2 \leq \sum \lambda_i \, \mu_i. \tag{NC 6}$$

We hasten to point out that the necessary condition (2.15) was first obtained by Morel [8] and later extended by Friedland to NC 6.

Lemma 2. *If the AIEP has a solution $D = \operatorname{diag}\{d_1, d_2, \dots, d_n\}$, then*

$$\min_i |\lambda_j - \mu_i| \leq \max_k |d_k| \quad \text{for all } j, \tag{2.17}$$

$$\min_i |\lambda_i - \mu_j| \leq \max_k |d_k| \quad \text{for all } j. \tag{2.18}$$

Proof. Let $\delta_j = \min_i |\lambda_j - \mu_i|$. If $\delta_j = 0$, then inequality (2.17) is immediate. We assume $\delta_j > 0$. Then $\lambda_j I - A$ is a nonsingular matrix, while $I - (\lambda_j I - A)^{-1} D$ is singular. In fact, if this matrix were invertible, the product $(\lambda_j I - A)\,(I - (\lambda_j I - A)^{-1} D) = \lambda_j I - (A + D)$ would also be invertible, contradicting the fact that λ_j is an eigenvalue of $A + D$. Hence, 1 is an eigenvalue of $(\lambda_j I - A)^{-1} D$ and

$$1 \leq \|(\lambda_j I - A)^{-1} D\|_2 \leq \|(\lambda_j I - A)^{-1}\|_2 \, \|D\|_2 \leq \frac{1}{\delta_j} \, \|D\|_2,$$

where $\|\cdot\|_2$ denotes the spectral norm. Thus, (2.17) is obtained. The inequality (2.18) can be obtained by taking $\delta_j = \min_i |\lambda_i - \mu_j|$, and considering the matrix $\mu_j I - (A + D)$. ∎

From inequalities (2.17) and (2.18) and Lemma 1, we have

$$\max_j \min_i |\lambda_j - \mu_i| \leq \sqrt{\frac{n-1}{n}} \left(\sum \lambda_i^2 - \sum \mu_i^2\right)^{1/2} \tag{2.19}$$

$$\max_j \min_i |\mu_j - \lambda_i| \leq \sqrt{\frac{n-1}{n}} \left(\sum \lambda_i^2 - \sum \mu_i^2\right)^{1/2}. \tag{2.20}$$

We observe that, in general,

$$\max_j \min_i |\mu_j - \lambda_i| \neq \max_j \min_i |\lambda_j - \mu_i|,$$

and, therefore, Conditions (2.19) and (2.20) are different. From these conditions, we obtain

$$\max \left\{ \max_j \min_i |\lambda_j - \mu_i|, \ \max_j \min_i |\mu_j - \lambda_i| \right\} \leq \sqrt{\frac{n-1}{n}} \left(\sum \lambda_i^2 - \sum \mu_i^2\right)^{1/2}. \tag{NC 7}$$

3. SUFFICIENT CONDITIONS

The cornerstone of all sufficient conditions that are known is the Brouwer fixed point theorem. A very important role in the application of this theorem is played by the mapping

$$T : \mathcal{B} \longrightarrow \mathbf{R}^n$$
$$T(v) = v + \lambda - \lambda^*(A + V),$$

where $\mathcal{B}$ is a convex, compact and non-empty subset of $\mathbf{R}^n$, λ is the vector of the desired eigenvalues and $\lambda^*(A+V)$ is the vector of the eigenvalues of $A+V$, with $V = \text{diag}\{v_1, v_2, v_3, \ldots, v_n\}$ and $v = (v_1, v_2, \ldots, v_n)^T$. Since T is a continuous mapping, if $T(\mathcal{B}) \subseteq \mathcal{B}$, then there exists $d \in \mathcal{B}$ such that $T(d) = d$ and so $\lambda^*(A+D) = \lambda$. Thus, D is a solution for the AIEP.

All the known sufficient conditions tell us that, if the given eigenvalues are sufficiently separated, then the problem has a solution. The earliest result of this type is due to Hadeler [2] who established

$$\text{If } m(\lambda) = \min_i(\lambda_{i+1} - \lambda_i) \geq 2\sqrt{3} \max_i \left(\sum_j a_{ij}^2\right)^{1/2}, \tag{SC 1}$$

then the AIEP has a solution.

Among the most well known sufficient conditions for the AIEP, we can mention:

(Laborde [3]): If $m(\lambda) \geq 2\rho(A)$, where A is an Hermitian matrix and $\rho(A)$ denotes the spectral radius of A, then the AIEP has a solution. (SC 2)

(Morel [8,9]): If $m(\lambda) \geq 2^{1-(1/p)} \left(\sum |\mu_i|^p\right)^{(1/p)}$ for at least one $p, p \geq 1$ (SC 3)

or if $m(\lambda) \geq \mu_n - \mu_1$, then the AIEP has a solution. (SC4)

In SC 3 and SC 4, A is an Hermitian matrix with eigenvalues μ_i listed in increasing order.

Since $\rho(A) \leq \|A\|$, for any matrix norm and $|\mu_i - \mu_j| \leq \sqrt{2}\|A\|_F$ (Mirsky [13]), Conditions SC 2 and SC 4 can be replaced by

$$m(\lambda) \geq 2\|A\|, \tag{SC 5}$$

$$m(\lambda) \geq \sqrt{2}\,\|A\|_F, \tag{SC 6}$$

which are easier to handle, although less precise.

The following sufficient condition is for a real matrix A, not necessarily symmetric

$$\text{(De Oliveira [5]): if } |\lambda_{\sigma(i)} - \lambda_{\sigma(j)}| \geq 2(S_i + S_j), \quad i,j = 1,2,\ldots,n \tag{SC 7}$$

with $i \neq j$, where $S_i = \sum_{j \neq i} |a_{ij}|$, and σ is a permutation, then the AIEP has a solution.

Let $\lambda - \alpha$ be the vector

$$\lambda - \alpha = (\lambda_1 - \alpha,\ \lambda_2 - \alpha, \ldots, \lambda_n - \alpha).$$

Clearly, $m(\lambda) = \min(\lambda_{i+1} - \lambda_i) = m(\lambda - \alpha)$. Using this invariance, we try to improve the sufficient conditions in which $m(\lambda)$ appears, by choosing α such that the respective right side is a minimum. We start with the condition of Laborde. For the matrix $A - \alpha I$, this condition becomes $m(\lambda) \geq 2\rho(A - \alpha I)$. We now look for the value of α that minimizes $\rho(A - \alpha I)$. Clearly,

$$\rho(A - \alpha I) = \max\{|\mu_n - \alpha|, |\mu_1 - \alpha|\}$$

attains its minimum value for α satisfying the equation

$$\alpha - \mu_1 = \mu_n - \alpha,$$

that is, for $\alpha_{\text{op}} = (\mu_1 + \mu_n)/2$. Hence,

$$\min_\alpha \rho(A - \alpha I) = \frac{\mu_n - \mu_1}{2},$$

and the best α leads us to the sufficient condition of Morel SC 4.

We now apply Morel's condition SC 3 to the matrix $A - \alpha I$. We seek an α such that

$$2^{1-\frac{1}{p}} \left(\sum |\mu_i - \alpha|^p\right)^{\frac{1}{p}}, \qquad p \geq 1$$

be a minimum. We shall consider the cases $p = 1$ and $p = 2$. Let $p = 1$ and let $f(\alpha) = \sum |\mu_i - \alpha|$. It is not hard to see that if n is an odd number, say $n = 2m+1$, then the best α is $\alpha_{\text{op}} = \mu_{m+1}$, while if n is even, $n = 2m$, then $\alpha_{\text{op}} = \mu_m$. In both events, we have

$$\min_{\alpha} f(\alpha) = \sum_{k=1}^{m} (\mu_{m+k+1} - \mu_k).$$

Then, we have

Let A be a Hermitian matrix in the AIEP. If $m(\lambda) \geq \sum_{k=1}^{m}(\mu_{m+k+1} - \mu_k)$, (SC 8)

where A is of order $n = 2m+1$ or $n = 2m$, then the AIEP has a solution.

Let now $p = 2$ and $f(\alpha) = \sum(\mu_i - \alpha)^2$. In this case, $\alpha_{\text{op}} = (\sum \mu_i)/n$ is the optimum. However, we have assumed that the diagonal entries of A are all zero. Therefore, we cannot improve upon Morel's condition for $p = 2$.

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