The existence, uniqueness and global attractivity of periodic solution for a type of neutral functional differential system with delays

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Abstract

In this article, by applying the coincidence degree theory and constructing a suitable Lyapunov functional, the authors study a kind of neutral functional differential system with delays. Some sufficient conditions, which guarantee the existence, uniqueness and global attractivity of periodic solution for this system, are obtained.

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1. Introduction

In recent years, by using Mawhin’s coincidence degree theory, many authors studied the existence of periodic solutions for ordinary or functional differential equations. In papers [1–5], the authors studied the existence of periodic solutions for the following several types of equations and systems:

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\[ x''(t) + f(x'(t)) + g(x(t - \tau(t))) = p(t), \]
\[ \frac{d}{dt}(u(t) - ku(t - \tau)) = g_1(u(t)) + g_2(u(t - \tau)) + p(t), \]
\[ \frac{d}{dt}(x(t) - bx(t - s)) = -ax(t - r + \nu h(t, x(t + \cdot))) + f(t), \]
\[ x' = Ax(t) + f(t), \]
\[ x' = A(t)x(t) + F(x(t)) + p(t). \]

But they did not investigate the global attractivity of these equations and systems. We only found Ma and Yasuhiro’s paper [6] which had studied the global attractivity of the nonnegative equilibria of the equation as follows
\[ \frac{d}{dt}(u(t) - g(u_t)) = Hu(t) - f(u_t)u(t). \]

But it did not study the existence, uniqueness and global attractivity of periodic solution for the above equation. As far as we known, the corresponding problem to neutral functional differential systems with delays were studied far less. One of the main reasons is that the methods to estimate \textit{a priori bounds} of all periodic solutions in the case of ordinary differential equation cannot be directly adapted to the case of functional differential system, especially neutral functional differential system with delays, and the methods to construct a suitable Lyapunov functional (or function) are different from those ones of ordinary differential equation. It is well known that the globally attractive periodic solution play an important role in some fields, such as medicine, ecology, mechanics, physics and so on.

Stimulated by these reasons, in this paper we study the existence, uniqueness and global attractivity of periodic solution for a kind of neutral functional differential system in the form
\[ (x(t) + cx(t - \sigma))^\prime = A(t, x(t))x(t) + f(t, x(t), x(t - \tau)), \]
\[ (x(t) + cx(t - \sigma))^\prime = A(t)x(t) + f(t, x(t), x(t - \tau)), \]
\[ x(s) = \varphi(s), \quad \varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s))^\top, \]
\[ \varphi_i(s) \in C([-\gamma, 0], R), \quad i = 1, 2, \ldots, n, \]

where \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^\top, f(t, x, y) \in C(R \times R^n \times R^n, R^n), f(t + T, x, y) \equiv f(t, x, y), A(t, x) \in C(R \times R^n, R^n), A(t + T, x) \equiv A(t, x), \gamma = \max\{\sigma, \tau\}, c, \sigma > 0 \) and \( \tau > 0 \) are constants.

By employing Mawhin’s coincidence degree theory and constructing a suitable Lyapunov functional, we study system (1.1) and (1.2)–(1.3), and obtain some new results. The methods to estimate \textit{a priori bounds} of all periodic solutions to system (1.1) and (1.2) are different from those ones in papers [1–5], and the methods to construct the Lyapunov functional also differ from the one in [6]. The significance of this article is that our conditions imposed on operator \( B \) and function \( f \) are very weak, also they are easy to be verified.

The organization of this paper is that: In Section 2, some necessary lemmas are given. In Section 3, by using Mawhin’s coincidence degree theorem, some sufficient conditions are obtained for the existence of periodic solutions of system (1.1). By using Theorem 3.1 and constructing a Lyapunov functional in Section 4 we verify the existence, uniqueness and global attractivity
of periodic solution for system (1.2)–(1.3). In the last section, an example is given to show the feasibility of the main results of this paper.

2. Main lemmas

Let $X := \{ x \mid x \in C(R^n, R^n), \ x(t + T) \equiv x(t) \}$ with the norm $\|x\| = \max_{t \in [0,T]}|x(t)| = \left( \sum_{i=1}^{n} x_i^2(t) \right)^{1/2}$, $\forall x \in X$. Clearly, $X$ is a Banach space. We define operator $B$ in the following form:

$$B : X \rightarrow X, \quad (Bx)(t) = x(t) + cx(t - \sigma).$$

Obviously, $B$ is a continuous linear operator with $\|B\| \leq 1 + |c|$. We also define an operator $L$ as follows

$$L : \text{Dom} \ L \subset X \rightarrow X, \quad [Lx] = (Bx)' ,$$

where $\text{Dom} \ L = \{ x \mid x \in C^{1}(R^n, R^n), \ x(t + T) \equiv x(t) \}$, and

$$N : X \rightarrow X, \quad (Nx)(t) = A(t, x(t))x(t) + f(t, x(t), x(t - \tau)) . \quad (2.1)$$

If $|c| \neq 1$, we know that $\text{Ker} \ L = R^n$, $\text{Im} \ L = \{ x \mid x \in X, \ \int_{0}^{T} x(s) \, ds = 0 \}$. Hence, $L$ is a Fredholm operator with index zero [7]. Let us define the projection operators $P$ and $Q$ as follows, respectively

$$P : X \cap \text{Dom} \ L \rightarrow \text{Ker} \ L, \quad Px = (Bx)(0);$$

$$Q : X \rightarrow X/\text{Im} \ L , \quad Qy = \frac{1}{T} \int_{0}^{T} y(s) \, ds .$$

Then $P = \text{Ker} \ L = R^n$, $Q = \text{Im} \ L$. Set operator $L_p = L|_{\text{Dom} \ L \cap \text{Ker} \ P} : \text{Dom} \ L \cap \text{Ker} \ P \rightarrow \text{Im} \ L$ and denote by $L_p^{-1} : \text{Im} \ L \rightarrow \text{Dom} \ L \cap \text{Ker} \ P$ the unique inverse of $L_p$, then

$$\left[ L_p^{-1} y \right](t) = B^{-1} \left( \int_{0}^{t} y(s) \, ds \right) . \quad (2.2)$$

From (2.1) and (2.2), it is easy to see that $N$ is $L$-compact on $\Omega$, where $\Omega$ is an any open subset of $X$.

For the sake of convenience, we denote by $A^\top$ the transposed matrix of $A$, $\lambda_M(t, x)$ the maximum eigenvalue of matrix $\frac{A(t, x) + A^\top(t, x)}{2}$, and $\lambda_m(t, x)$ the minimum eigenvalue of matrix $\frac{A(t, x) + A^\top(t, x)}{2}$. In view of

$$\left| A(t, x)u(t) \right|^2 = \left( A(t, x)u(t) \right)^\top A(t, x)u(t)$$

$$= u^\top(t)A^\top(t, x)A(t, x)u(t) = \sum_{i=1}^{n} \lambda_i(t, x)|u(t)|^2 , \quad (2.3)$$

where $\lambda_i(t, x), \ i = 1, 2, \ldots, n$, are the eigenvalues of matrix $A^\top(t, x)A(t, x)$. So we can let $\lambda_A := \max_{1 \leq i \leq n} \max_{t \in [0,T]}{\sqrt{\lambda_i(t, x)}} \ | x \in R^n \}.$
Lemma 2.1. For any square matrix $A_{n \times n}$, if the rank of $A$ is $r$ ($r \leq n$), then we have

$$\lambda_B(A^\top A) \geq \lambda_i \left( \frac{A + A^\top}{2} \right), \quad i = 1, 2, \ldots, n,$$

(2.4)

where $\lambda_B(A^\top A) := \max_{1 \leq i \leq r} \{ \lambda_i(A^\top A) | \lambda_i(A^\top A) > 0 \}$, $\lambda_i(A^\top A)$ are the nonzero eigenvalues of $A^\top A$, and $\lambda_i(\frac{A + A^\top}{2})$, $i = 1, 2, \ldots, n$, are the eigenvalues of $\frac{A + A^\top}{2}$.

Proof. Firstly, we denote by $\lambda_i(X)$, $i = 1, 2, \ldots, n$, the eigenvalues of matrix $X$, and assume that $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$. By the singular value decomposition theory, there must be two orthogonal square matrices $P_{n \times n}$ and $Q_{n \times n}$ such that

$$A = P A_r Q^\top \quad \text{and} \quad A^\top = Q A_r P^\top,$$

(2.5)

where $A_r := (\bigwedge_r 0)\bigwedge_r = \text{diag}(\lambda_1(A^\top A), \lambda_2(A^\top A), \ldots, \lambda_r(A^\top A))$.

Notice that

$$PA_r P^\top + QA_r Q^\top - PA_r Q^\top - QA_r P^\top = PA_r (P^\top - Q^\top) + QA_r (Q^\top - P^\top)$$

$$= (P - Q) A_r (P^\top - Q^\top)$$

$$= (P - Q) A_r (P - Q)^\top \geq 0.$$

So we have

$$\frac{PA_r Q^\top + QA_r P^\top}{2} \leq \frac{PA_r P^\top + QA_r Q^\top}{2}.$$

(2.6)

Taking (2.5) into (2.6) we get

$$\frac{A + A^\top}{2} \leq \frac{PA_r P^\top + QA_r Q^\top}{2}.$$

(2.7)

Now, we let

$$B_1 = \frac{A + A^\top}{2}; \quad B_2 = \frac{PA_r P^\top + QA_r Q^\top}{2}.$$

By using Weyl inequality, following (2.7) we get

$$\lambda_i(B_1) = \lambda_i(B_1 - B_2 + B_2) \leq \lambda_i(B_1 - B_2) + \lambda_i(B_2), \quad i = 1, 2, \ldots, n.$$

(2.8)

As $B_1 - B_2 \leq 0$, which yields $\lambda_i(B_1 - B_2) \leq 0$, $i = 1, 2, \ldots, n$, from (2.8) we have

$$\lambda_i(B_1) \leq \lambda_i(B_2), \quad i = 1, 2, \ldots, n.$$

(2.9)

On the other hand, by applying Weyl inequality again we obtain

$$\lambda_i(B_2) = \lambda_i \left( \frac{PA_r P^\top + QA_r Q^\top}{2} \right)$$

$$\leq \lambda_i \left( \frac{PA_r P^\top}{2} + \frac{QA_r Q^\top}{2} \right)$$

$$\leq \frac{\lambda_i(P A_r P^\top) + \lambda_i(Q A_r Q^\top)}{2}$$

$$= \frac{\lambda_i(A_r) + \lambda_i(A_r)}{2}$$

$$\leq \lambda_i(A_r), \quad i = 1, 2, \ldots, n.$$

(2.10)

Substituting (2.10) into (2.9), we get
\[ \lambda_i \left( \frac{A + A^T}{2} \right) \leq \lambda_B(A^T A), \quad i = 1, 2, \ldots, n. \]

So inequality (2.4) holds. \( \square \)

**Lemma 2.2.** (See [7].) Let \( X \) and \( Y \) be two Banach spaces, \( L: \text{Dom} \subset Y \rightarrow X \) be a Fredholm operator with index zero, \( \Omega \subset Y \) be an open bounded set, and \( N: \Omega \rightarrow X \) be \( L \)-compact on \( \Omega \).

If all the following conditions hold:

- [B1] \( Lx \neq \lambda Nx, \ \forall x \in \partial \Omega \cap \text{Dom} L, \ \forall \lambda \in (0, 1); \)
- [B2] \( Nx \notin \text{Im} L, \ \forall x \in \partial \Omega \cap \text{Ker} L; \)
- [B3] \( \text{deg}\{JQN, \Omega \cap \text{Ker} L, 0\} \neq 0, \) where \( J: \text{Im} Q \rightarrow \text{Ker} L \) is an isomorphism map;

then equation \( Lx = Nx \) has at least one solution on \( \Omega \cap \text{Dom} L. \)

**Lemma 2.3.** (See [8].) If function \( f \) is nonnegative, integrable and uniformly continuous on \([0, +\infty)\), then we have \( \lim_{t \to +\infty} f(t) = 0. \)

### 3. Existence of periodic solutions

**Theorem 3.1.** Suppose there are constants \( a \geq 0, b \geq 0, d > 0 \) and \( w > 0 \), such that the following conditions hold:

- [C1] \( |f(t, x, y)| \leq a|x| + b|y| + d, \ \forall (t, x, y) \in R \times R^n \times R^n; \)
- [C2] \( |c| < 1, \ \lambda_m(t, x) \geq w \) (or \( |c| \neq 1, \ \lambda_M(t, x) \leq -w), \ \forall (t, x) \in R \times R^n. \)

Then system (1.1) has at least one \( T \)-periodic solution, if \( w > |c| \lambda_A + (1 + |c|)(a + b). \)

**Remark 1.** The assumption \( |c| < 1 \) is necessary. In fact, from Lemma 2.1 and condition \( \lambda_m(t, x) \geq w \) we know that \( \lambda_A \geq w, \) if \( |c| > 1 \), then condition \( w > |c| \lambda_A + (1 + |c|)(a + b), \) which is crucial in estimating \textit{a priori bounds} of all periodic solutions, does not hold any more. And in the critical case \( |c| = 1 \), there is an interesting but challenging problem associated with the existence of periodic solutions of system (1.1), we leave this problem for future work.

**Proof.** We only give the proof for \( |c| < 1, \ \lambda_m(t, x) \geq w \), it is easy to see that system (1.1) has \( T \)-periodic solutions if and only if the following operator equation:

\[ Lx = Nx, \]

has \( T \)-periodic solutions.

Take \( \Omega_1 = \{x \mid x \in \text{Dom} L, \ Lx = \lambda Nx, \ \lambda \in (0, 1)\}. \ \forall x \in \Omega_1, \) then \( x \) must satisfy the following system:

\[ (x(t) + cx(t - \sigma))' = \lambda A(t, x(t))x(t) + \lambda f(t, x(t), x(t - \tau)). \]

Multiplying both sides of system (3.2) with \( (Bx)^\top(t) \), and integrating them over \([0, T]\), we have

\[ \int_0^T (Bx)^\top(t)(A(t, x(t))x(t) + f(t, x(t), x(t - \tau))) \ dt = 0. \]
that is
\[
\int_0^T \left( x(t)^\top A(t,x(t)) + A(t,x(t))^\top x(t) \right) dt
= \int_0^T x(t)^\top A(t,x(t)) x(t) dt
\]
\[
= -c \int_0^T x(t)^\top (t-\sigma) A(t,x(t)) x(t) dt
- c \int_0^T x(t)^\top (t-\sigma) f(t,x(t),x(t-\tau)) dt
- \int_0^T x(t)^\top f(t,x(t),x(t-\tau)) dt.
\]  
(3.3)

From assumption [C2], we get
\[
\int_0^T \left( A(t,x(t)) x(t) + A(t,x(t))^\top x(t) \right) dt \geq w \int_0^T |x(t)|^2 dt.
\]  
(3.4)

Following (2.3), we have
\[
|A(t,x(t)) x(t)| \leq \lambda_A |x(t)|.
\]  
(3.5)

So substituting (3.4) and (3.5) into (3.3), we obtain
\[
w \int_0^T |x(t)|^2 dt \leq -c \int_0^T x(t)^\top (t-\sigma) A(t,x(t)) x(t) dt
- \int_0^T x(t)^\top f(t,x(t),x(t-\tau)) dt
\]
\[
\leq |c| \int_0^T |x(t)^\top (t-\sigma) A(t,x(t)) x(t)| dt + \int_0^T |x(t)^\top f(t,x(t),x(t-\tau))| dt
\]
\[
+ |c| \int_0^T |x(t)^\top (t-\sigma) f(t,x(t),x(t-\tau))| dt
\]
\[
\leq \lambda_A |c| \int_0^T |x(t-\sigma)| |x(t)| dt + a \int_0^T |x(t)|^2 dt + b \int_0^T |x(t)| |x(t-\tau)| dt
\]
\[
+ d \int_0^T |x(t)| dt + a |c| \int_0^T |x(t-\sigma)| |x(t)| dt
\]
\[ + b |c| \int_0^T |x(t - \sigma)| |x(t - \tau)| \, dt + d |c| \int_0^T |x(t - \sigma)| \, dt \]
\[ \leq \left( |c| \lambda_A + (1 + |c|)(a + b) \right) \int_0^T |x(t)|^2 \, dt \]
\[ + d(1 + |c|) \sqrt{T} \left( \int_0^T |x(t)|^2 \, dt \right)^{1/2}. \]  
\[
(3.6)
\]

As \( w > |c| \lambda_A + (1 + |c|)(a + b) \), following (3.6) we get
\[
\left( \int_0^T |x(t)|^2 \, dt \right)^{1/2} \leq \frac{d(1 + |c|) \sqrt{T}}{w - (|c| \lambda_A + (1 + |c|)(a + b))},
\]
which implies that there must be a constant \( \xi \in (0, T) \) such that
\[
|x(\xi)| \leq \frac{d(1 + |c|)}{w - (|c| \lambda_A + (1 + |c|)(a + b))} := M_1.
\]

Meanwhile we get
\[
|x(t)| \leq |x(\xi)| + \int_\xi^T |x'(t)| \, dt \leq M_1 + \int_0^T |x'(t)| \, dt. \]  
\[
(3.8)
\]

On the other hand, in respect of \((Bx')(t) = (Bx)'(t), \forall x \in \text{Dom } L\), we obtain, from (3.2)
\[
|x'(t)| \leq |A(t, x(t)) x(t)| + |f(t, x(t), x(t - \tau))| + |c||x'(t - \sigma)|.
\]

Integrating both sides of the above inequality, we have
\[
\int_0^T |x'(t)| \, dt \leq \int_0^T |A(t, x(t)) x(t)| \, dt + \int_0^T |f(t, x(t), x(t - \tau))| \, dt + \int_0^T |c||x'(t - \sigma)| \, dt \]
\[ \leq (\lambda_A + a + b) \int_0^T |x(t)| \, dt + |c| \int_0^T |x'(t)| \, dt + dT. \]  
\[
(3.9)
\]

From (3.7) and (3.9) we get
\[
(1 - |c|) \int_0^T |x'(t)| \, dt \leq (\lambda_A + a + b) \sqrt{T} \left( \int_0^T |x(t)|^2 \, dt \right)^{1/2} + dT \]
\[ \leq (\lambda_A + a + b) M_1 T + dT, \]

note \(|c| < 1\), then we get
\[
\int_0^T |x'(t)| \, dt \leq \frac{(\lambda_A + a + b) M_1 T + dT}{1 - |c|} := M_2. \]  
\[
(3.10)
\]
Substituting (3.10) into (3.8), we obtain
\[ \|x\| \leq M_1 + M_2 =: M. \]

Let \( M_0 = M + \varepsilon \), where \( \varepsilon > 0 \) is a small constant, \( \Omega = \{ x \mid x \in \Omega_1, \|x\| < M_0 \} \) and \( \Omega_2 = \{ x \mid x \in \partial \Omega \cap \text{Ker} L \} \), then
\[ QN x = \frac{1}{T} \int_0^T (A(t, x(t)) x(t) + f(t, x(t), x(t - \tau))) \, dt \neq 0, \ \forall x \in \partial \Omega. \]

Thus conditions [B1] and [B2] of Lemma 2.2 are both satisfied. Next define the isomorphism map \( J : \text{Im} \, Q \to \text{Ker} L, \, J(x) \equiv x \). And let
\[ H(x, \mu) = \mu x + (1 - \mu) J Q N x, \ \forall (x, \mu) \in \Omega \times [0, 1]. \]

Then we have
\[ H(x, \mu) = \mu x + \frac{1 - \mu}{T} \int_0^T (A(t, x(t)) x(t) + f(t, x(t), x(t - \tau))) \, dt, \]
\[ \forall (x, \mu) \in \Omega_2 \times [0, 1]. \]

We can get that \( x H(x, \mu) > 0 \). Hence
\[ \text{deg} \{ J Q N, \Omega \cap \text{Ker} L, 0 \} = \text{deg} \{ H(x, 0), \Omega \cap \text{Ker} L, 0 \} = \text{deg} \{ H(x, 1), \Omega \cap \text{Ker} L, 0 \} \neq 0. \]

That is condition [B3] of Lemma 2.2 is satisfied. So system (3.1) has at least one \( T \)-periodic solution \( x(t) \) on \( \overline{\Omega} \cap \text{Dom} L \), i.e., system (1.1) has at least one \( T \)-periodic solution \( x(t) \).

4. Uniqueness and global attractivity

**Definition 4.1.** Suppose \( \bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))^\top \) is the \( T \)-periodic solution of system (1.2), if each solution \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^\top \) of system (1.2)–(1.3) satisfies
\[ \lim_{t \to +\infty} |x(t) - \bar{x}(t)| = 0. \]

Then \( \bar{x}(t) \) is globally attractive.

**Remark 2.** If \( \bar{x}(t) \) is a periodic solution and globally attractive, then \( \bar{x}(t) \) must be unique.

**Theorem 4.1.** Suppose \( |c| < 1 \), and there are constants \( L \geq 0, \, H \geq 0 \) and \( w > 0 \), such that the following conditions hold:
\[ \begin{align*}
[D_1] & \quad |f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L|x_1 - x_2| + H|y_1 - y_2|, \ \forall (t, x_i, y_i) \in R \times R^n \times R^n, \ i = 1, 2; \\
[D_2] & \quad \lambda_M(t) \leq -w, \ \forall t \in R.
\end{align*} \]

Then system (1.2)–(1.3) has only one globally attractive \( T \)-periodic solution, if \( w > |c|\lambda_A + (1 + |c|)(L + H) \).
**Proof.** Firstly, we prove the existence of periodic solutions of system (1.2)–(1.3). Let \( x_2 = 0, y_2 = 0 \), from assumption [D2] we can get
\[
|f(t, x_1, y_1)| \leq L|x_1| + H|y_1| + e,
\]
where \( e = \max_{t \in [0, T]} |f(t, 0, 0)| \). Thus condition [C1] of Theorem 3.1 holds, then we know that system (1.2) has at least one \( T \)-periodic solution \( \bar{x}(t) \).

Next, we show the global attractivity of \( \bar{x}(t) \). Define operator \( D \) as follows
\[
D : Y \rightarrow Y, \quad (Du)(t) = u(t) + cu(t - \sigma),
\]
where \( Y := \{ u : R^n \rightarrow R^n \text{ is uniformly continuous and bounded on } R^n \} \). So similar to (4.4) in paper [3], we can obtain
\[
(D^{-1}g)(t) = \sum_{i \geq 0} (-c)^i g(t - i\sigma), \quad \forall g \in Y. \tag{4.1}
\]
Assume that \( x(t) \) is a solution of system (1.2)–(1.3), and let \( u(t) = x(t) - \bar{x}(t) \), then
\[
(Du)'(t) = A(t)u(t) + f(t, x(t), x(t - \tau)) - f(t, \bar{x}(t), \bar{x}(t - \tau)).
\]
Now, we construct a Lyapunov functional as follows
\[
V(u) = \frac{|(Du)(t)|^2}{2} + \frac{(\lambda_A + L + H)|c|}{2} \int_{-\sigma}^{0} |u(t + s)|^2 ds + \frac{H(1 + |c|)}{2} \int_{-\tau}^{0} |u(t + s)|^2 ds.
\]
Obviously, we can get
\[
\frac{\|Du\|^2}{2} \leq V(u) \leq \left( \frac{(\lambda_A + L + H)|c|\sigma}{2} + \frac{H(1 + |c|)\tau}{2} \right) \|u\|^2
\]
and
\[
V(u(0)) = \frac{|(Du)(0)|^2}{2} + \frac{(\lambda_A + L + H)|c|}{2} \int_{-\sigma}^{0} |\phi(s) - \bar{x}(s)|^2 ds
\]
\[
+ \frac{H(1 + |c|)}{2} \int_{-\tau}^{0} |\phi(s) - \bar{x}(s)|^2 ds.
\]
Also we obtain
\[
V'(u) = (Du)^\top(t)(Du)'(t) + \frac{(\lambda_A + L + H)|c|}{2} (|u(t)|^2 - |u(t - \sigma)|^2)
\]
\[
+ \frac{H(1 + |c|)}{2} (|u(t)|^2 - |u(t - \tau)|^2). \tag{4.2}
\]
As
\[
(Du)^\top(t)(Du)'(t)
\]
\[
= \left( u(t) + cu(t - \sigma) \right)^\top A(t)u(t)
\]
\[
+ \left( u(t) + cu(t - \sigma) \right)^\top \left[ f(t, x(t), x(t - \tau)) - f(t, \bar{x}(t), \bar{x}(t - \tau)) \right]
\]
easy to get |tion of system (1.2)–(1.3), we can easily get that
\( \dot{x} \mid_{\text{DM}} \)

where
\[
\lambda A + |c| (L + H) > 0.
\]

Substituting (4.3) into (4.2), we get
\[
V'(u) \leq (|c| \lambda A + (1 + |c|) (L + H - w)|u(t)|^2 = -K|u(t)|^2,
\]

where \( K = w - (|c| \lambda A + (1 + |c|) (L + H)) > 0. \) Integrating both sides of (4.4) on \([0, t]\), we have
\[
V(u) + K \int_0^t |u(s)|^2 ds \leq V(u(0)) < +\infty, \quad t \geq 0.
\]

Thus \( \int_0^t |u(s)|^2 ds \leq V(u(0))/K < +\infty, \) which yields that \( |u(t)|^2 \) is integrable on \([0, +\infty)\), also we get \( |(Du)(t)| < +\infty. \) Meanwhile, one know that \( |u(t)|^2 \) is continuous on \([0, +\infty)\). So it is easy to get \( |u(t)|^2 < +\infty, t \geq 0, \) i.e., \( |x(t) - \bar{x}(t)| < +\infty, t \geq 0. \) From the bounded of \( \bar{x}(t) \), we know that \( x(t) \) is bounded on \([0, +\infty)\). By system (1.2) we can get that \( (\dot{Dx})(t) \) is uniformly continuous and bounded on interval \([0, +\infty)\), so following (4.1) we have
\[
|x'(t)| \leq \frac{D_M}{1 - |c|} < +\infty,
\]

where \( D_M = \max_{t \geq 0} |(Dx)'(t)|, \) i.e., \( x'(t) \) is bounded. On the other hand, \( \bar{x}(t) \) is a periodic solution of system (1.2)–(1.3), we can easily get that \( \bar{x}'(t) \) is bounded too. Thus \( |u(t)|^2 \) is uniformly continuous on \([0, +\infty)\). By Lemma 2.3, we obtain
\[
\lim_{t \to +\infty} |x(t) - \bar{x}(t)| = 0.
\]

Therefore, \( \bar{x}(t) \) is globally attractive. From Remark 2 one know that system (1.2)–(1.3) has only one globally attractive \( T \)-periodic solution. \( \square \)

5. Example

As an application, we consider the following system,
\[
\begin{align*}
\begin{cases}
\left( x_1(t) - \frac{1}{6}x_1(t - 3) \right)' &= (\cos t - 2)x_1 + x_2 \cos t + \frac{1}{20} x_1 \cos 2t + \frac{1}{30} x_2(t - 1), \\
\left( x_2(t) - \frac{1}{6}x_2(t - 3) \right)' &= -x_1 \cos t - 4x_2 + \frac{1}{20} x_2 \sin t + \frac{1}{30} \sin(t - x_1(t - 1)).
\end{cases}
\end{align*}
\] (5.1)
Let

\[ A(t) = \begin{pmatrix} \cos t - 2 & \cos t \\ -\cos t & -4 \end{pmatrix} \]

\[ f(t, x, x(t - \tau)) = \begin{pmatrix} \frac{1}{20} x_1 \cos 2t + \frac{1}{30} x_2 (t - 1) \\ \frac{1}{20} x_2 \sin t + \frac{1}{30} \sin(t - x_1 (t - 1)) \end{pmatrix} \]

Then

\[ A^\top(t) A(t) = \begin{pmatrix} 2 \cos^2 t - 4 \cos t + 4 & -(2 + \cos t) \cos t \\ -(2 + \cos t) \cos t & 16 + \cos^2 t \end{pmatrix} \]

So we have

\[ \lambda_M(t) \leq -1, \quad \lambda_A = \sqrt{\frac{27 + \sqrt{560}}{2}} < \frac{51}{10}, \]

\[ |f(t, x, y) - f(t, z, s)| \leq \frac{1}{20} |x - z| + \frac{1}{30} |y - s|. \]

We can easily get

\[ K = w - \left( |c| \lambda_A + (1 + |c|) (L + H) \right) > 1 - \left( \frac{51}{60} + \frac{35}{360} \right) = \frac{19}{360} > 0. \]

So by applying Theorem 4.1, we know that system (5.1) has only one globally attractive $2\pi$-periodic solution.

References