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A Directed Graph Version of Strongly Regular Graphs

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We study a directed graph version of strongly regular graphs whose adjacency matrices satisfy $A^2 + (\mu - \lambda)A - (t - \mu)I = \mu J$, and $AJ = JA = kJ$. We prove existence (by construction), nonexistence, and necessary conditions, and construct homomorphisms for several families of parameter sets.

1. INTRODUCTION

Let G be a directed **graph** consisting of a set of n **vertices**, together with a set of **edges** from one vertex to another. The graphs we use will not have more than one edge from one vertex to another, nor will they have any edges from a vertex to itself. We will say there is an **undirected** edge joining vertices a and b if there are edges both from a to b and b to a. An **undirected graph** has undirected edges only.

Notation. If a and b are vertices of a graph G, we will write $a \leftrightarrow b$ to denote that there is an undirected edge joining a and b in G, and $a \rightarrow b$ to denote that there is directed edge from a to b, but not *vice versa.* We will also string these expressions together, like so, $a \leftrightarrow b \rightarrow c$, meaning $a \leftrightarrow b$ and $b \rightarrow c$, for instance.

G may be characterized by its **adjacency matrix**, an $n \times n$ (0, 1)-matrix (matrix of 0's and 1's) A defined by

$$A_{ij} = 1 \qquad \text{if and only if} \qquad i \rightarrow j \text{ or } i \leftrightarrow j.$$

A graph and its adjacency matrix will be used interchangeably. The adjacency matrix of an undirected graph is symmetric. Since the graphs we are looking at have no edges from a vertex to itself, the diagonal of A will always be all 0's.

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A special kind of graph that has been researched extensively (see [3; 15], for expository articles) is a **strongly regular graph** with parameters (n, k, μ, λ), which is defined to be an undirected graph of n vertices whose adjacency matrix A satisfies

$$A^2 + (\mu - \lambda)A - (k - \mu)I = \mu J, \tag{1.1}$$

$$AJ = JA = kJ, \tag{1.2}$$

where I is the identity matrix of order n, and J is the $n \times n$ matrix of all 1's. Equation (1.1) implies (1.2) for undirected graphs. We see the significance of these graphs by rewriting Eq. (1.1) as

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Then we may see that the number of paths of length 2 from a vertex a to a vertex b in G is λ if there is an edge from a to b, μ if there is not, and k if $b = a$ (i.e., each vertex has degree k).

A generalization of strongly regular graphs that does not seem to have been studied before is to allow G to be directed, while restricting it to the same conditions on the number of paths of length 2. We also require that G still have constant in- and out-degree (number of edges coming in and out, respectively). We therefore define a **(n, k, μ, λ, t)-graph** G to be a directed graph on n vertices whose adjacency matrix A satisfies

$$A^2 + (\mu - \lambda)A - (t - \mu)I = \mu J, \tag{1.3}$$

$$AJ = JA = kJ. \tag{1.4}$$

As with (undirected) strongly regular graphs, it is often useful to rewrite Eq. (1.3) as

$$A^2 = tI + \lambda A + \mu(J - I - A). \tag{1.5}$$

Note that each vertex of G still has in- and out-degree k, but now with only t edges being undirected, leaving $k - t$ edges coming in only and $k - t$ coming out only. The interpretation of μ and λ remains the same as with undirected strongly regular graphs. As A is (0, 1), Eq. (1.5) immediately implies $0 \leqslant \mu, \lambda$. Of course, $0 \leqslant t \leqslant k \leqslant n - 1$.

An example of such a graph that is not a strongly regular graph is the graph G, in Fig. 1, with the adjacency matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

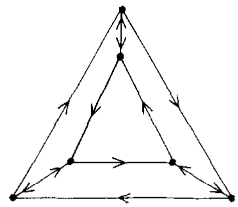

FIGURE 1

We see that $A^2 + A = J$, so that G is an (n, k, μ, λ, t)-graph with parameters $(6, 2, 1, 0, 1)$. We will demonstrate several other interesting examples that have been discovered, in Sections 4 through 9.

2. NECESSARY CONDITIONS ON PARAMETERS

We now develop necessary conditions on the parameters of (n, k, μ, λ, t)-graphs. We start with a useful lemma.

LEMMA 2.1. *If G is an (n, k, μ, λ, t)-graph with adjacency matrix A, then the complementary graph G' is an $(n, k', \mu', \lambda', t')$-graph with adjacency matrix $A' = (J - I - A)$, where*

$$\begin{aligned} k' &= (n-2k) + (k-1), \\ \lambda' &= (n-2k) + (\mu - 2), \\ t' &= (n-2k) + (t-1), \\ \mu' &= (n-2k) + \lambda. \end{aligned}$$

Proof. That A' and k' satisfy (1.4) is immediate. It thus remains to show that A', μ', λ', and t' satisfy (1.3). We do this by simply evaluating

$$\begin{aligned} A'^2 = (J-I-A)^2 &= (n-k-1)J - (J-I-A) - (kJ - A - A^2) \\ &= (n-2k-2)J + I + 2A + (\lambda A + t + \mu(J-I-A)) \\ &= (n-2k+t-1)I + (n-2k+\mu-2)(J-I-A) + (n-2k+\lambda)A \\ &= t'I + \lambda' A' + \mu'(J-I-A'). \quad \blacksquare \end{aligned}$$

To find necessary conditions, we start with arguments that are similar to those used for strongly regular graphs (as in [3]), by calculating the multiplicities of the eigenvalues of A, and using the fact that they are nonnegative integers.

THEOREM 2.2. *If G is an (n, k, μ, λ, t)-graph with adjacency matrix A, and G is not*

(i) *an (undirected) strongly regular graph ($t=k$) or*

(ii) *a complete graph ($A=J-I$),*

then A is equivalent to a Hadamard matrix ($A+A^T=J-I$, $AA^T=(\mu-1)J+\mu I$), or, for some positive integer d,

$$k(k+(\mu-\lambda))=t+(n-1)\mu, \tag{2.1}$$

$$(\mu-\lambda)^2+4(t-\mu)=d^2, \tag{2.2}$$

$$d \mid 2k-(\mu-\lambda)(n-1), \tag{2.3}$$

$$\frac{2k-(\mu-\lambda)(n-1)}{d} \equiv n-1 \pmod 2, \tag{2.4}$$

$$\left|\frac{2k-(\mu-\lambda)(n-1)}{d}\right| \leqslant n-1. \tag{2.5}$$

Proof. Recall that the eigenvalues of J are n with multiplicity 1, corresponding to the eigenvector j of all 1's, and 0 with multiplicity $n-1$, corresponding to the space of eigenvectors the sum of whose elements is 0. Equation (1.4) gives that j is also an eigenvector of A, corresponding to the eigenvalue k. Let the other eigenvalues of A be θ_i $(i=1, ..., n-1)$. By (1.3), we have

$$\begin{aligned} k^2+(\mu-\lambda)k-(t-\mu)&=\mu n, \\ \theta_i^2+(\mu-\lambda)\theta_i-(t-\mu)&=0, \end{aligned} \tag{2.6}$$

so

$$\theta_i=\tfrac{1}{2}\left(-(\mu-\lambda)\pm\sqrt{(\mu-\lambda)^2+4(t-\mu)}\right). \tag{2.7}$$

Equation (2.6) proves (2.1). Let ρ and σ be the θ_i, $\rho>\sigma$, occurring with multiplicity r and s, respectively. Then

$$r+s=n-1, \tag{2.8}$$

$$0=\operatorname{tr} A=k+r\rho+s\sigma, \tag{2.9}$$

so

$$0=k-\tfrac{1}{2}(n-1)(\mu-\lambda)+(r-s)\tfrac{1}{2}\sqrt{(\mu-\lambda)^2+4(t-\mu)}. \tag{2.10}$$

First suppose that $(\mu-\lambda)^2+4(t-\mu)\neq d^2$, for any positive integer d. Then

$$0=k-\tfrac{1}{2}(n-1)(\mu-\lambda), \qquad k=\tfrac{1}{2}(n-1)(\mu-\lambda).$$

So $\mu-\lambda$ is 1 or 2, as $0<k\leqslant n-1$.

Case 1. $\mu-\lambda=2$, $k=n-1$. Because $k=n-1$, we have $A=J-I$, which contradicts hypothesis (ii).

Case 2. $\mu-\lambda=1$, $k=\frac{1}{2}(n-1)$, $n=2k+1$. Substituting into (2.6) gives us

$$k^2+k-(t-\mu)=\mu(2k+1),$$
$$k(k+1-2\mu)=t.$$

So $t=k+1-2\mu=0$ as $0\leqslant t<k$.

Because $t=0$, there are no undirected edges. This and $k=\frac{1}{2}(n-1)$ together imply that

$$A+A^{\mathrm{T}}=J-I.$$

Substituting $\mu-\lambda=1$ and $t=0$ into (1.3), we get

$$A^2+A=\mu(J-I).$$

Thus

$$AA^{\mathrm{T}}=A(J-I-A)=kJ-A-A^2=kJ-\mu(J-I),$$
$$AA^{\mathrm{T}}=(k-\mu)J+\mu I=(\mu-1)J+\mu I.$$

So A is equivalent to a Hadamard matrix of order 4μ (see, for instance, [12]). Thus, we may assume

$$(\mu-\lambda)^2+4(t-\mu)=d^2,$$

for some positive integer d. Substituting this into (2.7), we get

$$\rho=\tfrac{1}{2}(-(\mu-\lambda)+d), \qquad \sigma=\tfrac{1}{2}(-(\mu-\lambda)-d).$$

Solving the system of Eq. (2.8) and (2.9) for r and s, we have

$$s=\frac{k+\rho(n-1)}{\rho-\sigma}, \qquad r=-\frac{k+\sigma(n-1)}{\rho-\sigma},$$

so

$$s-r=\frac{2k+(\rho+\sigma)(n-1)}{\rho-\sigma}=\frac{2k-(\mu-\lambda)(n-1)}{d}.$$

As s and r are eigenvalue multiplicities, they must be integers. This will occur if and only if $s+r$ and $s-r$ are integers and have the same parity. Therefore

$$d \mid 2k-(\mu-\lambda)(n-1),$$
$$\frac{2k-(\mu-\lambda)(n-1)}{d}\equiv n-1 \pmod 2.$$

We also require that s and r be nonnegative. As $s+r>0$, this occurs if and only if

$$|s-r| \leqslant s+r=n-1. \tag{2.11}$$

i.e.,

$$\left|\frac{2k-(\mu-\lambda)(n-1)}{d}\right| \leqslant n-1. \quad \blacksquare$$

In the remainder of the paper, we will assume $t \neq k$, $A \neq J-I$, and A is not equivalent to a Hadamard matrix as described above. Note that if G satisfies these conditions, then so does its complement G'.

We may see that Lemma 2.1 does not strengthen this result; it is easily verified that

$$d'=d,$$

$$2k'-(\mu'-\lambda')(n-1)=-(2k-(\mu-\lambda)(n-1)),$$

so that (2.2), (2.3), (2.4), and (2.5) are satisfied by $J-I-A$. Equation (2.1) is also satisfied by the parameters of $J-I-A$. However, Lemma 2.1 will be used in the proof of the next two theorems.

THEOREM 2.3. *If G is an (n, k, μ, λ, t)-graph, then*

$$0 \leqslant \lambda < t < k,$$

$$0 < \mu \leqslant t < k.$$

Proof. First we show that $t \geqslant \mu$. Assume not: $t-\mu<0$. Recall the definition of d, s, and r from the proof of the previous theorem. By (2.2),

$$(\mu-\lambda)^2-d^2=-4(t-\mu)>0,$$

$$|\mu-\lambda|>|d|=d.$$

Also note that

$$s-r \leqslant n-1.$$

Case 1. $\mu-\lambda<0$, $d<-(\mu-\lambda)$. In this case,

$$\begin{aligned} k &= \tfrac{1}{2}(n-1)(\mu-\lambda)+\tfrac{1}{2}(s-r)\,d \\ &< \tfrac{1}{2}(n-1)(\mu-\lambda)-\tfrac{1}{2}(n-1)(\mu-\lambda)=0. \qquad \Rightarrow\Leftarrow \end{aligned}$$

Case 2. $\mu-\lambda>0$, $d<\mu-\lambda$. By (2.2), $\mu-\lambda$ and d have the same parity, so, in this case, we may state

$$(\mu-\lambda)-d \geqslant 2.$$

Then by (2.10) and (2.11),

$$k \geqslant \tfrac{1}{2}(n-1)(\mu-\lambda) - \tfrac{1}{2}(n-1)\,d \geqslant n-1,$$

so $k = n-1$, $A = J - I$, a case we are not considering. Thus, either case produces a contradiction, and $\mu \leqslant t$.

Then by Lemma 2.1, $\mu' \leqslant t'$, i.e., $\lambda < t$.

Finally, we prove that $\mu > 0$. Assume not: $\mu = 0$. Substituting into (2.1), we get

$$t = k(k-\lambda).$$

But $t < k$, by hypothesis, and $0 \leqslant \lambda < t$, by above, so $0 < t < k$, $\Rightarrow\Leftarrow$. ∎

There is one necessary condition that results from a graph-theoretic argument, not an eigenvalue argument.

THEOREM 2.4. *If G is an (n, k, μ, λ, t)-graph, then*

$$-2(k-t-1) \leqslant \mu - \lambda \leqslant 2(k-t).$$

Proof. Suppose $a \rightarrow b$ in G. Then μ counts the number of vertices c in G such that $b \rightarrow c \rightarrow a$, $b \rightarrow c \leftrightarrow a$, $b \leftrightarrow c \rightarrow a$, or $b \leftrightarrow c \leftrightarrow a$. Since there are $k-t$ directed edges coming into or out of one vertex,

$$|\{c\colon b \rightarrow c \rightarrow a \text{ or } b \rightarrow c \leftrightarrow a\}| \leqslant |\{c\colon b \rightarrow c\}| \leqslant k-t,$$
$$|\{c\colon b \leftrightarrow c \rightarrow a\}| \leqslant |\{c\colon c \rightarrow a\}| \leqslant k-t.$$

Also, as $a \rightarrow b$, there are λ paths of length 2 from a to b, and hence,

$$|\{c\colon b \leftrightarrow c \leftrightarrow a\}| \leqslant \lambda.$$

Thus,

$$\begin{aligned} \mu &\leqslant 2(k-t) + \lambda, \\ \mu - \lambda &\leqslant 2(k-t). \end{aligned} \tag{2.12}$$

By Lemma 2.1,

$$0 \leqslant 2(k'-t') - (\mu' - \lambda') = 2(k-t) - (\lambda - \mu + 2),$$
$$(\lambda - \mu) \leqslant 2(k-t-1). \quad ∎ \tag{2.13}$$

Equation (2.13) may also be obtained directly by an argument similar to the one used to obtain (2.12), considering paths of length 2 from a to b rather than from b to a.

This result is not a consequence of the other conditions; the parameter set (25, 11, 6, 3, 10) satisfies all the other conditions, but not this one.

A list of the first few possible parameter sets ($n \leq 20$) and the status of their existence follows; this list was generated by a computer program from the conditions of Theorems 2.2, 2.3, and 2.4. (Only the cases $k < n/2$ are listed, so that of a complementary pair of graphs, exactly one is listed.)

n	k	μ	λ	t	Existence/Construction
6	2	1	0	1	Sections 4, 8
8	3	1	1	2	exists, see [2]
10	4	2	1	2	Sections 5, 6
12	3	1	0	1	Section 8
12	4	2	0	2	Theorem 7.1 and (6, 2, 1, 0, 1)
12	5	2	2	3	Theorem 7.2 and (6, 2, 1, 0, 1)
14	5	2	1	4	?
14	6	3	2	3	Section 6
15	4	1	1	2	Section 9
15	5	2	1	2	?
16	6	3	1	3	?
16	7	2	4	5	Theorem 7.2 and (8, 3, 1, 1, 2)
16	7	3	3	4	?
18	4	1	0	3	Section 4
18	5	1	2	3	?
18	6	3	0	3	Theorem 7.1 and (6, 2, 1, 0, 1)
18	7	3	2	5	?
18	8	3	4	5	Theorem 7.2 and (6, 2, 1, 0, 1)
18	8	4	3	4	Sections 5, 6
20	4	1	0	1	Section 8
20	7	2	3	4	?
20	8	4	2	4	Theorem 7.1 and (10, 4, 2, 1, 2)
20	9	4	4	5	Theorem 7.2 and (10, 4, 2, 1, 2)

3. Nonexistence of Prime Orders

One of the first things one notices from the list of possible parameters at the end of the previous section is that there are no graphs listed with a prime number of vertices. Just from the necessary conditions on the parameters proved in the previous section, we may prove this to be true in general.

Theorem 3.1. *If G is an (n, k, μ, λ, t)-graph, then n is not prime.*

Proof. Assume n is prime. The case $n = 2$ is trivial, so let n be odd, say

$n = 2p + 1$. Without loss of generality, assume $k \leqslant p$; if not, consider the complement matrix $J - I - A$ of degree $n - k - 1$.

Theorem 2.3 now gives

$$|\mu - \lambda| \leqslant k \leqslant p, \tag{3.1}$$

$$0 \leqslant t - \mu \leqslant k \leqslant p. \tag{3.2}$$

Substituting this into (2.2), we get

$$\begin{aligned} d^2 &= (\mu - \lambda)^2 + 4(t - \mu) \leqslant p^2 + 4p < p^2 + 4p + 4 \\ &= (p + 2)^2, \qquad d < p + 2. \end{aligned} \tag{3.3}$$

Combining (2.1) and (2.2), we get

$$\mu n = k(k + (\mu - \lambda)) - \tfrac{1}{4}(d^2 - (\mu - \lambda)^2),$$

$$4\mu n = (2k + (\mu - \lambda) + d)(2k + (\mu - \lambda) - d),$$

$$n \mid (2k + (\mu - \lambda) + d)(2k + (\mu - \lambda) - d).$$

We are assuming n is prime, so

$$n \mid (2k + (\mu - \lambda) \pm d).$$

Further, $\mu - \lambda$ and d have the same parity, by (2.2), so

$$2 \mid (2k + (\mu - \lambda) \pm d),$$

and since 2 and n are relatively prime, we finally have

$$2n \mid (2k + (\mu - \lambda) \pm d). \tag{3.4}$$

We use (3.1), (3.2), and (3.3) to turn the divisibility relation of (3.4) into a useful equality:

$$-2n < 2k - k - (p + 2) < 2k + (\mu - \lambda) \pm d,$$

$$2k + (\mu - \lambda) \pm d < 2p + p + (p + 2) = 4p + 2 = 2n,$$

$$-2n < 2k + (\mu - \lambda) \pm d < 2n.$$

Recalling (3.4), we get

$$4\mu n = (2k + (\mu - \lambda) + d)(2k + (\mu - \lambda) - d) = 0,$$

which implies $\mu = 0$, contradicting Theorem 2.3. Thus n is not prime. ∎

4. THE FAMILY $\mu=1$, $\lambda=0$, $t=k-1$

In this case the parameter conditions reduce to a simple condition, by arguments that are similar to those used to find possible parameter sets of strongly regular graphs with $\mu=1$, $\lambda=0$ (and which may be found in [6]). We are then left with only three cases to consider.

LEMMA 4.1. *If G is an (n, k, μ, λ, t)-graph, with $\mu=1$, $\lambda=0$, $t=k-1$, then $d \mid 9$.*

Proof. Theorem 2.2, in this case, simplifies to

$$k^2=n-2, \tag{4.1}$$

$$d \mid 2k-n+1=2k-(k^2+2)+1=-(k-1)^2, \tag{4.2}$$

$$4k-7=d^2. \tag{4.3}$$

Equations (4.2) and (4.3) lead to

$$(k-1)^2 \equiv 0 \pmod{d},$$

$$4k \equiv 7 \pmod{d}.$$

Then

$$0 \equiv 16(k-1)^2 \equiv (4k)^2-8(4k)+16 \equiv 9 \pmod{d},$$

$$d \mid 9. \quad \blacksquare$$

Thus $d=1$, 3, or 9. Substituting into (4.1) and (4.3), we get that the other parameters in each of these cases are

d	n	k
1	6	2
3	18	4
9	486	22

To construct these graphs, we show that there is a homomorphism from such a graph to an (undirected) strongly regular graph with $\frac{1}{3}$ as many vertices.

LEMMA 4.2. *Let G be an (n, k, μ, λ, t)-graph with $\mu=1$, $\lambda=0$, $t=k-1$. If $a \to b$ in G, then there is $c \in G$ such that $a \to b \to c \to a$.*

Proof. Because $\mu=1$ and $\lambda=0$, there is a path of length 2 from x to y in G iff there is no path of length 1 from x to y. Thus there is a path of length 2 from b to a, say via $c \in G$ (see Fig. 2). If $b \leftrightarrow c \leftrightarrow a$, then there is a

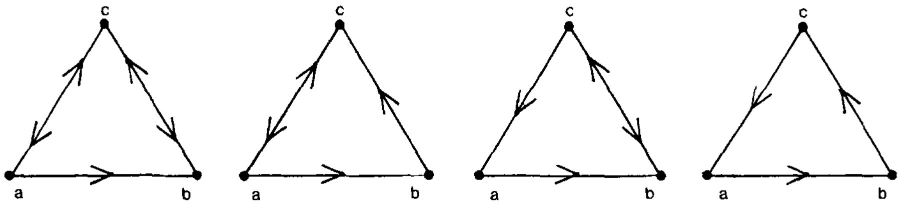

FIGURE 2

path of length 2 from a to b, $\Rightarrow\Leftarrow$. If $b \to c \leftrightarrow a$, then there are paths of length 1 and 2 from a to c, $\Rightarrow\Leftarrow$. If $b \leftrightarrow c \to a$, then there are paths of length 1 and 2 from c to b, $\Rightarrow\Leftarrow$. Thus $b \to c \to a$. ∎

Since $t = k - 1$, there is only one directed edge going into and coming out of each vertex, and thus the directed edges form disjoint polygons in G. By Lemma 4.2, these polygons are triangles. The homomorphism will map these directed triangles into vertices. We now show how these triangles are connected in G.

LEMMA 4.3. *Let G be an (n, k, μ, λ, t)-graph with $\mu = 1$, $\lambda = 0$, $t = k - 1$. Suppose $a \to b \to c \to a$ and $x \to y \to z \to x$ in G, and $a \leftrightarrow x$. Then $b \leftrightarrow z$ and $c \leftrightarrow y$, and there are no other edges among these vertices.*

Proof. Assume there is no edge joining b and z. There must be a path of length 2 from b to x, otherwise there is an edge from b to x, and then a path of length 2 from a to x, and an edge from a to x. It cannot be via c (otherwise $x \leftrightarrow c \to a$ and $x \leftrightarrow a$), and it cannot be via z (by assumption), so there is a new vertex $w \in G$ such that $b \leftrightarrow w \leftrightarrow x$ (see Fig. 3). But then $a \to b \leftrightarrow w$ and $a \leftrightarrow x \leftrightarrow w$, $\Rightarrow\Leftarrow$. So $b \leftrightarrow z$, and similarly, $c \leftrightarrow y$. By similar arguments, it is also easy to see that there are no other edges among these vertices. ∎

Let G' be an undirected graph with vertex set $= \{\{a, b, c\} : a \to b \to c \to a$ in $G\}$. Lemmas 4.2 and 4.3 allow us to define the following homomorphism:

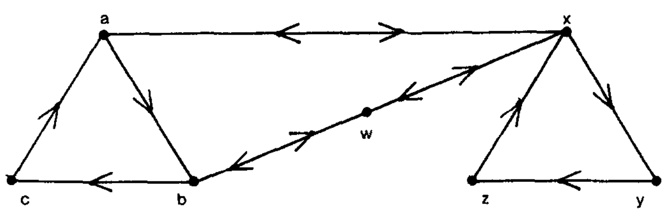

FIGURE 3

α: $G \to G'$: $a\alpha = \{a, b, c\}$ such that $a \to b \to c \to a$ in G;

$a\alpha \leftrightarrow x\alpha$ in G' if and only if $a\alpha = \{a, b, c\}$ and $x\alpha = \{x, y, z\}$ are connected in G as described in Lemma 4.3.

THEOREM 4.4. *Let G be an (n, k, μ, λ, t)-graph, with $\mu = 1$, $\lambda = 0$, and $t = k - 1$. Then there is a strongly regular graph, G', with parameters $n' = n/3$, $k' = t$, $\mu' = 3$, $\lambda' = 0$, and a homomorphism α: $G \to G'$, as defined above.*

Proof. It is easy to see that $n' = n/3$ and that $k' = t$. To show $\lambda' = 0$, we must show that G' has no triangles. Assume $a\alpha \leftrightarrow x\alpha \leftrightarrow y\alpha \leftrightarrow a\alpha$ in G'. Without loss of generality, assume $a \leftrightarrow x \leftrightarrow y$ in G, and also let $a \to b \to c \to a$ in G (see Fig. 4). If $y\alpha \leftrightarrow a\alpha$ in G', then $y \leftrightarrow a$, $y \leftrightarrow b$, or $y \leftrightarrow c$ in G. If $y \leftrightarrow x \leftrightarrow a$ in G then there is no edge from y to a in G, so $y \leftrightarrow b$ or $y \leftrightarrow c$. If $y \leftrightarrow b$, then $a \to b \leftrightarrow y$ and $a \leftrightarrow x \leftrightarrow y$ in G, $\Rightarrow\Leftarrow$. If $y \leftrightarrow c$, then $y \leftrightarrow c \to a$ and $y \leftrightarrow x \leftrightarrow a$ in G, $\Rightarrow\Leftarrow$. So G' has no triangles, and $\lambda' = 0$.

To show that $\mu' = 3$, assume that there is no edge joining $a\alpha = \{a, b, c\}$ and $x\alpha = \{x, y, z\}$ in G' (say $a \to b \to c \to a$ and $x \to y \to z \to x$ in G). We show that there are exactly three paths of length 2 joining $a\alpha$ and $x\alpha$ in G'. Since there is no edge joining $a\alpha$ and $x\alpha$, there are no edges in G joining the members of $a\alpha$ and the members of $x\alpha$. Thus, there is a path of length 2 from a to each of x, y, and z in G. Because there are no edges joining $a\alpha$ and $x\alpha$, each of these paths of length 2 must be via some new vertex in G. Each path must be via a different vertex in G, otherwise the same vertex is connected to two vertices of the same directed triangle, $x\alpha = \{x, y, z\}$, violating Lemma 4.3. Also, these 3 different vertices must each be from a different directed triangle of G, otherwise a has two edges going to the same directed triangle. Thus, each of the three paths of length 2 from a to a member of $x\alpha$ in G corresponds to a different path of length 2 from $a\alpha$ to

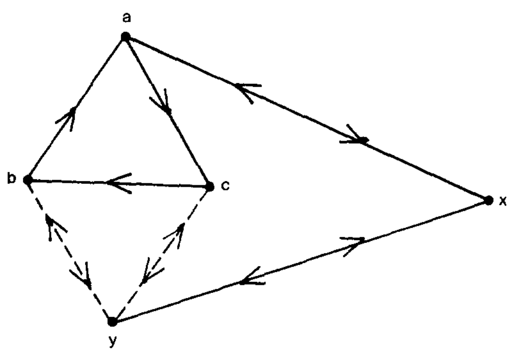

FIGURE 4

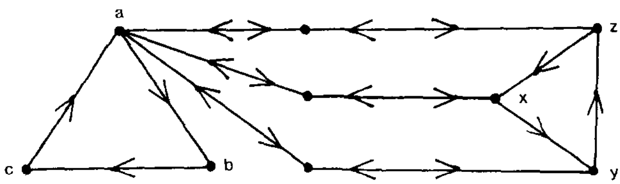

FIGURE 5

$x\alpha$ in G' (see Fig. 5). If there are more than three paths of length 2 from $a\alpha$ to $x\alpha$ in G', then 2 of them correspond to distinct paths of length 2 from a to some member of $x\alpha$. Therefore $\mu' = 3$. ∎

This shows that there is a unique solution for $d = 1$, namely two directed triangles connected as described in Lemma 4.3. There is a solution to $d = 3$, whose homomorphic graph is the complete bipartite graph on 6 vertices. Its adjacency matrix is A. The existence of a solution to $d = 9$ is unknown; if it exists, then the homomorphic graph is a strongly regular graph with parameters (162, 21, 3, 0).

$$A = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0\\
0&1&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1\\
\hline
0&0&0&0&0&1&0&0&0&1&0&0&0&0&1&1&0&0\\
0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&1\\
\hline
0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0\\
\hline
0&1&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0\\
\hline
1&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0\\
0&0&1&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\
\hline
1&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&1&0&0
\end{array}\right).$$

5. A CONSTRUCTION USING QUADRATIC RESIDUE MATRICES

We look for (n, k, μ, λ, t)-graphs with parameters $(2q, q-1, \frac{1}{2}(q-1), \frac{1}{2}(q-1)-1, \frac{1}{2}(q-1))$, where q is a prime power and $q = 4m+1$, whose adjacency matrices are of the form

$$A = \begin{pmatrix} Q & C_1 \\ C_2 & Q \end{pmatrix}, \tag{5.1}$$

where $Q=(Q_{ij})$ is a **quadratic residue matrix** of order q, and C_1 and C_2 are **σ_1- and σ_2-circulants**, respectively. Recall that

> $Q_{ij}=1$ if and only if $i-j$ is a quadratic residue in $GF(q)$, the finite field of q elements,

and that a σ-circulant $C=(C_{ij})$ satisfies

$$C_{ij}=C_{i-k,\,j-\sigma k},$$

where σ is in $GF(q)^{\#}$, and the rows and columns of C are labelled symmetrically by the elements of $GF(q)$. Notice that this labelling means that the top (left) row (column) of C is numbered 0, not 1. An elementary result of circulant matrices is that the product of an a-circulant and a b-circulant is an ab-circulant.

We recall some elementary properties of Q (see [5], for instance). Let R be the set of all quadratic residues in $GF(q)$, and let N be the set of all non-residues in $GF(q)$. It is immediate that Q is a 1-circulant. As R contains $2m$ elements, the line sums of Q are $2m$, and thus the line sums of both C_1 and C_2 are $2m$. By definition, 0 is not in R, so $\operatorname{tr} Q=0$. Q is also symmetric, and

$$Q^2+Q=m(J+I). \tag{5.2}$$

LEMMA 5.1. *We may assume, up to simultaneous permutations of rows and columns, that*

$$(C_1)_{0,0}=(C_2)_{0,0}=0,$$
$$(C_1)_{i,0}\neq(C_2)_{0,i}, \qquad \textit{for} \quad i\neq 0.$$

Proof. Consider all the pairs $(C_1)_{i,0}$, $(C_2)_{0,i}$; there are $4m+1$ such pairs. Let r be the first row (and hence column) of Q. Then

$$2m=(A^2+A)_{0,0}=\sum_i r_i r_i+\sum_i (C_1)_{i,0}\,(C_2)_{0,i}$$
$$=2m+\sum_i (C_1)_{i,0}\,(C_2)_{0,i},$$

so

$$\sum_i (C_1)_{0,i}\,(C_2)_{i,0}=0.$$

Thus, of the $4m+1$ pairs, none are $(1,1)$. Since the line sums of C_1 and C_2 are each $2m$, there are $2m$ $(1,0)$'s, $2m$ $(0,1)$'s, and 1 $(0,0)$. Suppose

$(C_1)_{0,j} = (C_2)_{j,0} = 0$. Simultaneously rotating the columns of C_1 and the rows of C_2 j times corresponding to a simultaneous permutation of rows and columns of A, moves the 0's to the proper positions of C_1 and C_2, and preserves all the assumed properties of A. ∎

LEMMA 5.2. *Suppose $\sigma \in GF(q)^{\#}$. Then $R\sigma = R$ if $\sigma \in R$, and $R\sigma = N$ if $\sigma \in N$.*

Proof. Suppose $\sigma \in R$. Residues are closed under multiplication, so $R\sigma \subseteq R$. But $\sigma \neq 0$, so multiplication by σ is a 1–1 map in $GF(q)$, and $|R\sigma| = |R|$. Hence $R\sigma = R$.

Alternatively, suppose $\sigma \in N$. $R\sigma \subseteq N$, otherwise there are $r_1, r_2 \in R$ such that

$$r_1\sigma = r_2,$$
$$\sigma = r_2 r_1^{-1} \in R, \qquad \Rightarrow\Leftarrow.$$

Again, $|R\sigma| = |R| = 2m = |N|$, so $R\sigma = N$. ∎

We now return to our original matrix A, and Eq. (5.1), which expands as

$$2mJ = A^2 + A = \begin{pmatrix} Q^2 + C_1C_2 + Q & QC_1 + C_1Q + C_1 \\ C_2Q + QC_2 + C_2 & Q^2 + C_2C_1 + Q \end{pmatrix}.$$

We now find necessary and sufficient conditions for each of the four $q \times q$ blocks of $A^2 + A$ to be $2mJ$.

LEMMA 5.3. *$QC_i + C_iQ + C_i = 2mJ$ if and only if $\sigma_i \in N$ $(i = 1, 2)$.*

Proof. Let $C = C_i$, $\sigma = \sigma_i$. $M = QC + CQ + C$ is a σ-circulant, so we only need show that all the elements of the first row of M are $2m$ if and only if $\sigma \in N$. To simplify matters, we may think of C as the sum of σ-circulant permutation matrices,

$$C = \sum_{i=1}^{2m} P_i,$$
$$M = \sum_{i=1}^{2m} QP_i + P_iQ + P_i.$$

Suppose row 0 of P_i has a 1 in column x_i only. Note that the x_i are distinct. Then row y of P_i has a 1 in column $x_i + \sigma y$ only. The first row of QP_i is rP_i, where r is the first row of Q. By definition, r has 1's only in columns s where $s \in R$, so rP_i has 1's only in columns $x_i + \sigma s$, $s \in R$. The first row of P_iQ is row x_i of Q which has 1's only in columns $s + 1x_i$, $s \in R$. The first row of P_i has a 1 only in column x_i.

Case 1. $\sigma \in N$. Then the three sets of 1's are in the columns $x_i + R\sigma = x_i + N$, $x_i + R$, and x_i, so there is exactly one 1 in each column of the first row of $QP_i + P_iQ + P_i$, and hence,

$$\text{first row of } M = \sum_{i=1}^{2m} (1, ..., 1) = (2m, ..., 2m).$$

Case 2. $\sigma \in R$. Then the three sets of 1's are in the columns $x_i + R\sigma = x_i + R$, $x_i + R$, and x_i, so the first row of $QP_i + P_iQ + P_i$ has a 1 in column x_i, and 2's and 0's in all the other columns, depending on whether or not they are in $x_i + R$. The x_i are distinct, so column x_i of the first row of M is the sum of only one 1 (from $QP_1 + P_1Q + P_1$) and a series of 0's and 2's, and is thus an odd number. Therefore, the first row of M cannot be $(2m, ..., 2m)$. ∎

LEMMA 5.4. *$Q^2 + C_1C_2 + Q = 2mJ$ if and only if $\sigma_1\sigma_2 = 1$, and the first row of C_1C_2 is $(0, m, ..., m)$ (and similarly for $Q^2 + C_2C_1 + Q$).*

Proof. Assume $Q^2 + C_1C_2 + Q = 2mJ$. By (5.2) we have

$$C_1C_2 = 2mJ - (Q^2 + Q) = m(J - I),$$

which is a 1-circulant. The result follows from multiplication of circulants. Conversely, if the first row of C_1C_2 is $(0, m, ..., m)$, then $Q^2 + C_1C_2 + Q = 2mJ$, by (5.2) and multiplication of circulants. ∎

We now give necessary and sufficient conditions for the first row of C_1C_2 to be $(0, m, ..., m)$. Partition the $4m$ elements of $GF(q)^\#$ into two parts, each containing $2m$ elements, by the following partition:

$$S = \{x \in GF(q)^\#: (C_2)_{0,x} = 1\}, \quad T = \{x \in GF(q)^\#: (C_2)_{0,x} = 0\}.$$

LEMMA 5.5. *The first row of C_1C_2 is $(0, m, ..., m)$ if and only if the partition described above satisfies the following "difference-partition" property:*

> *Among the $(2m)^2$ differences $s - t$ such that $s \in S$ and $t \in T$, each of the $4m$ elements of $GF(q)^\#$ occurs exactly m times.*

Proof. The first row of C_1C_2 is the vector sum of those rows r in C_2 such that $(C_1)_{0,r} = 1$, which is the vector sum of rows $r \neq 0$ in C_2 such that $(C_2)_{r,0} = 0$. Thus, column c of the first row of C_1C_2 is the number of pairs (r, c) such that

$$(C_2)_{r,0} = 0, \qquad r \neq 0 \tag{5.3}$$

$$(C_2)_{r,c} = 1. \tag{5.4}$$

We show that there is a 1–1 correspondence between (r, c) pairs satisfying (5.3) and (5.4) and differences $s-t=c$ $(s \in S, t \in T)$. Then column c is the number of occurrences of c among the differences $s-t$ $(s \in S, t \in T)$, proving the result. Because C_2 is a σ_2-circulant, we see that

$$(C_2)_{r,0} = (C_2)_{0,-\sigma_2 r},$$

$$(C_2)_{r,c} = (C_2)_{0,c-\sigma_2 r}.$$

So (r, c) satisfies (5.3) and (5.4) if and only if $-\sigma_2 r \in T$ and $c - \sigma_2 r \in S$.

If (r, c) satisfies (5.3) and (5.4), then $c = s - (-\sigma_2 r)$, where $s \in S$ and $-\sigma_2 r \in T$. Conversely, if $c = s - t$, where $s \in S$ and $t \in T$, let $r = -t\sigma_2^{-1}$; then $-\sigma_2 r = t \in T$ and $c - \sigma_2 r = s \in S$, so (r, c) satisfies (5.3) and (5.4). This proves the 1–1 correspondence. ∎

Combining Lemmas 5.3, 5.4, and 5.5, we get

THEOREM 5.6. *If*

$$A = \begin{pmatrix} Q & C_1 \\ C_2 & Q \end{pmatrix},$$

where Q is a quadratic residue matrix of prime power order $q = 4m+1$, and C_1 and C_2 are σ_1- and σ_2-circulants, respectively, then $A^2 + A = 2mJ$ (and thus A is the adjacency matrix of an (n, k, μ, λ, t)-graph, with parameters $(2q, q-1, \frac{1}{2}(q-1), \frac{1}{2}(q-1)-1, \frac{1}{2}(q-1))$ if and only if

(i) $\sigma_1 \sigma_2 = 1$ *in* $GF(q)$;

(ii) $\sigma_1, \sigma_2 \in N$; *and*

(iii) *the partition described by the first row satisfies the "difference-partition" property described in Lemma* 5.5.

Further, such a partition always exists, one such partition being R and N, the quadratic residues and the nonresidues.

Proof. The first part of the theorem is proven by Lemmas 5.3, 5.4, and 5.5. To show that the partition of R and N has the "difference-partition" property, we show that every nonzero element x can be expressed in the form $r-n$ (r will denote an element of R, and n will denote an element of N) in the same number of ways that 1 can.

Case 1. $x \in R$ (then also $x^{-1} \in R$). Suppose $r - n = 1$. Then $xr - xn = x$, and $xr \in R$ and $xn \in N$. Conversely, suppose $r - n = x$. Then $x^{-1}r - x^{-1}n = 1$, and $x^{-1}r \in R$, and $x^{-1}n \in N$.

Case 2. $x \in N$ (then also $-x, \pm x^{-1} \in N$). Suppose $r - n = 1$. Then $x = xr - xn = (-x)n - (-x)r$, and $(-x)n \in R$ and $(-x)r \in N$. Conversely, suppose $r-n=x$. Then $1=x^{-1}r-x^{-1}n=(-x^{-1})n-(-x^{-1})r$, and $(-x^{-1})n\in R$ and $(-x^{-1})r\in N$.

Also note that 0 cannot be represented at all. Since all nonzero elements can be represented in the same number of ways, and 0 not at all, this partition has the "difference-partition" property. ∎

This partition is not always unique; for instance, with $GF(9)^{\#}=Z_3[x]/(x^2-x-1)$, the partition of $GF(9)^{\#}$ into $\{1, 2, x, 2x\}$ and $\{x+1, x+2, 2x+1, 2x+2\}$ has the "difference-partition" property, but the quadratic residues are $\{1, 2, x+1, 2x+2\}$.

An example of a matrix satisfying Theorem 5.6 is A. Here, the parameters are (10, 4, 2, 1, 2); $A^2+A=2J$; $\sigma_1=2$, $\sigma_2=3$; and the partition is $R=\{1,4\}$, and $N=\{2,3\}$.

$$A=\left(\begin{array}{ccccc|ccccc}
0&1&0&0&1&0&1&0&0&1\\
1&0&1&0&0&0&1&0&1&0\\
0&1&0&1&0&1&0&0&1&0\\
0&0&1&0&1&1&0&1&0&0\\
1&0&0&1&0&0&0&1&0&1\\
\hline
0&1&0&0&1&0&1&0&0&1\\
0&0&1&0&1&1&0&1&0&0\\
1&0&1&0&0&0&1&0&1&0\\
1&0&0&1&0&0&0&1&0&1\\
0&1&0&1&0&1&0&0&1&0
\end{array}\right).$$

6. Another Block Construction

We look for (n, k, μ, λ, t)-graphs with parameter sets $(2(2\mu+1), 2\mu, \mu, \mu-1, \mu)$ whose adjacency matrices are of the form

$$A=\begin{pmatrix} Q & PQ \\ (PQ)^{\mathrm{T}} & Q\end{pmatrix},$$

where

$$Q+Q^{\mathrm{T}}=J-I, \tag{6.1}$$

$$QJ=JQ=\mu J, \tag{6.2}$$

and P is a permutation matrix of rank 2. This implies

$$PJ=JP=J,$$

$$P=P^{\mathrm{T}}=P^{-1}.$$

Note that

$$(PQ)^{\mathrm{T}}=Q^{\mathrm{T}}P=(J-I-Q)P=J-P-QP.$$

We therefore search for A such that

$$\mu J = A^2 + A$$
$$= \begin{pmatrix} Q^2 + PQ(J-P-QP) + Q & QPQ + PQ^2 + PQ \\ (J-P-QP)Q + Q(J-P-QP) + (J-P-QP) & (J-P-QP)PQ + Q^2 + Q \end{pmatrix}. \tag{6.3}$$

Note that the set of parameters that satisfy these conditions contains the set of parameters studied in the last section, but that this construction is not a generalization of the previous construction; here Q is completely asymmetric ($Q^{\mathrm{T}} = J - I - Q$), and in the previous section, Q was symmetric ($Q^{\mathrm{T}} = Q$).

THEOREM 6.1. *If A is of the form described above, then $A^2 + A = \mu J$ if and only if $PQ = (PQ)^{\mathrm{T}}$.*

Proof. First we show that $A^2 + A = \mu J$ if and only if

$$Q^2 + Q = P(Q^2 + Q)P, \tag{6.4}$$

$$\mu J = QPQ + P(Q^2 + Q), \tag{6.5}$$

$$PQ = (PQ)^{\mathrm{T}}, \tag{6.6}$$

by finding necessary and sufficient conditions for each of the $(2\mu + 1) \times (2\mu + 1)$ blocks to be μJ. For the upper-left block, this is equivalent to (6.4). For the upper-right block, this is equivalent to (6.5). The lower-right block is identically μJ. Now assume that (6.4) and (6.5) hold. Note that (6.4) implies

$$(Q^2 + Q)P = P(Q^2 + Q).$$

Then for the lower-left block, (6.3) is equivalent to

$$\mu J = (2\mu + 1)J - (QPQ + QP + Q^2P) - (PQ + QP + P)$$
$$= (\mu + 1)J - (PQ + QP + P),$$

i.e.,

$$PQ = J - QP - P = (PQ)^{\mathrm{T}}.$$

Thus, it suffices to show that Eq. (6.6) implies (6.4) and (6.5). So assume that (6.6) holds. Then

$$Q(PQ) + (Q^2 + Q)P = Q(J - P - QP) + (Q^2 + Q)P = \mu J.$$

Also

$$PQP=(PQ)^{\mathrm{T}}P=(Q^{\mathrm{T}}P)P=Q^{\mathrm{T}}=(J-I-Q),$$

so

$$\begin{aligned}P(Q^2+Q)P&=(PQP)^2+(PQP)=(J-I-Q)^2+(J-I-Q)\\&=(n-1-2\mu)J+Q^2+Q=Q^2+Q. \quad \blacksquare\end{aligned}$$

It is easy to construct P and Q satisfying (6.1) and (6.2), but (6.6) is a little harder to achieve. One such construction is this: let Q be a circulant, and let

$$P=\begin{pmatrix}1 & 0 & \cdot & \cdot & \cdot & 0\\ 0 & \cdot & \cdot & \cdot & 0 & 1\\ 0 & \cdot & \cdot & 0 & 1 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 1 & 0 & \cdot & \cdot & 0\end{pmatrix};$$

Then PQ is a back-circulant ((-1)-circulant), which is symmetric. To make circulant matrices satisfying (6.1) and (6.2), put 1's in the first row according to the following rule: label the columns $0, \ldots, n-1$, and then

$$Q_{1,j}=1 \qquad \text{if and only if} \qquad Q_{1,n-j}=0.$$

An example of this construction is the following adjacency matrix, A, of an (n, k, μ, λ, t)-graph with parameters (18, 8, 4, 3, 4):

$$A=\left(\begin{array}{ccccccccc|ccccccccc}
0&1&0&1&1&0&0&1&0&0&1&0&1&1&0&0&1&0\\
0&0&1&0&1&1&0&0&1&1&0&1&1&0&0&1&0&0\\
1&0&0&1&0&1&1&0&0&0&1&1&0&0&1&0&0&1\\
0&1&0&0&1&0&1&1&0&1&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&1&1&1&0&0&1&0&0&1&0&1\\
1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&1&1\\
1&1&0&0&1&0&0&1&0&0&1&0&0&1&0&1&1&0\\
0&1&1&0&0&1&0&0&1&1&0&0&1&0&1&1&0&0\\
1&0&1&1&0&0&1&0&0&0&0&1&0&1&1&0&0&1\\
\hline
0&1&0&1&1&0&0&1&0&0&1&0&1&1&0&0&1&0\\
1&0&1&1&0&0&1&0&0&0&0&1&0&1&1&0&0&1\\
0&1&1&0&0&1&0&0&1&1&0&0&1&0&1&1&0&0\\
1&1&0&0&1&0&0&1&0&0&1&0&0&1&0&1&1&0\\
1&0&0&1&0&0&1&0&1&0&0&1&0&0&1&0&1&1\\
0&0&1&0&0&1&0&1&1&1&0&0&1&0&0&1&0&1\\
0&1&0&0&1&0&1&1&0&1&1&0&0&1&0&0&1&0\\
1&0&0&1&0&1&1&0&0&0&1&1&0&0&1&0&0&1\\
0&0&1&0&1&1&0&0&1&1&0&1&1&0&0&1&0&0
\end{array}\right).$$

Here, $A^2+A=4J$; and Q is a 1-circulant with initial row (0, 1, 0, 1, 1, 0, 0, 1, 0), and P is as described above, in the remarks following Theorem 6.1.

7. Kronecker Product Constructions

We construct two families of solutions using the Kronecker matrix product (see [5] for an introduction and listing of useful identities). In this section, let J_k denote the $k \times k$ matrix of all 1's, and let I_k denote the $k \times k$ identity matrix, for any k.

Theorem 7.1. *Let A be the adjacency matrix of an (n, k, μ, λ, t)-graph $(A \neq J_n - I_n)$, and J_m the $m \times m$ matrix of all* 1's $(m > 1)$. *Then $A \times J_m$ is the adjacency matrix of an (n, k, μ, λ, t)-graph if and only if $t = \mu$. In this case the parameter set for $A \times J_m$ is $(nm, km, \mu m, \lambda m, tm)$. The result also holds for $J_m \times A$.*

Proof. Note that $A \times J_m$ is of order nm, and so we want necessary and sufficient conditions on A so that

$$(A \times J_m)^2 = -(\mu' - \lambda')(A \times J_m) + (t' - \mu')\, I_{nm} + \mu J_{nm}.$$

So we expand

$$\begin{aligned}(A \times J_m)^2 = A^2 \times J_m^2 &= (-(\mu - \lambda)A + tI_n + \mu(J_n - I_n)) \times mJ_m \\ &= -m(\mu - \lambda)(A \times J_m) + tm(I_n \times J_m) + \mu m((J_n - I_n) \times J_m).\end{aligned}$$

As $A \neq J_n - I_n$, some off-diagonal blocks of $(A \times J_m)^2 + (\mu' - \lambda')(A \times J_m)$ are $\mu m J_m$ (regardless of what $\mu' - \lambda'$ is), and the diagonal blocks are tmJ_m. As $m > 1$, some of the elements of the diagonal block are not on the diagonal. Hence, these blocks must be the same, so $t = \mu$. In this case,

$$(A \times J_m)^2 = -m(\mu - \lambda)(A \times J_m) + \mu m J_{nm}.$$

As $A \times J_m$ obviously has order mn and in- and out-degree kn, this proves the result for $A \times J_m$. The proof for $J_m \times A$ is similar. ∎

An example of this construction is the following adjacency matrix, $A' = A \times J_2$, where A is the adjacency matrix of the (6, 2, 1, 0, 1)-graph constructed in Section 4 and shown in Section 1.

$$A' = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}\right).$$

THEOREM 7.2. *Let A be the adjacency matrix of an (n, k, μ, λ, t)-graph $(A \neq J_n - I_n)$. Then*

$$A' = (A \times J_m) + (I_n \times (J_m - I_m))$$

is the adjacency matrix of an (n, k, μ, λ, t)-graph if and only if $\lambda = t - 1$. In this case, the parameter set for A' is $(nm, (k+1)m - 1, \mu m, (t+1)m - 2, (t+1)m - 1)$.

Proof. Once again, we start by expanding

$$\begin{aligned}(A')^2 = A \times &-((\mu - \lambda - 2)m + 2)J_m \\ &+ I_n \times ((t+1)m - 1)I_m + (t+1)m - 2)(J_m - I_m)) \\ &+ (J_n - I_n) \times \mu m J_m, \end{aligned} \tag{7.1}$$

and setting it equal to

$$\begin{aligned} -(\mu' - \lambda')A' &+ t'I_{nm} + \mu'(J_{nm} - I_{nm}) \\ &= -(\mu' - \lambda')(A \times J_m) + (I_n \times (J_m - I_m))) \\ &\quad + t'(I_n \times I_m) \\ &\quad + \mu'((I_n \times (J_m - I_m)) + ((J_n - I_n) \times J_m)) \\ &= A \times -(\mu' - \lambda')J_m \\ &\quad + I_n \times (t'I_m + \lambda'(J_m - I_m)) \\ &\quad + (J_n - I_n) \times \mu' J_m. \end{aligned} \tag{7.2}$$

As $A \neq J_n - I_n$, the two sums (7.1) and (7.2) must be equal componentwise (by A, I_n and $(J_n - I_n)$), and, as $m > 1$, none of the terms of the components are trivial (i.e., $J_n - I_n \neq 0$), so

$$\mu' - \lambda' = (\mu - \lambda - 2)m + 2,$$

$$\lambda' = (t+1)m - 2, \tag{7.3}$$

$$t' = (t+1)m - 1, \tag{7.4}$$

$$\mu' = \mu m. \tag{7.5}$$

So it is easy to see that this construction works if and only if

$$(\mu - \lambda - 2)m + 2 = \mu' - \lambda' = \mu m - (t+1)m + 2,$$

i.e., $\lambda = t - 1$. Clearly, $n' = nm$ and $k' = km + (m-1)$, and the other parameters of A' follow from Eqs. (7.3) through (7.5). ∎

An example of this construction is the following adjacency matrix, $A' = (A \times J_2) + (I_2 \times (J_2 - I_2))$, where A is the adjacency matrix of the $(6, 2, 1, 0, 1)$-graph constructed in Section 4 and shown in Section 1.

$$
A = \left(\begin{array}{cc|cc|cc|cc|cc|cc}
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}\right).
$$

Note that in each construction, the parameters of the new graph satisfy the conditions of the theorem, and thus may be expanded repeatedly by the same method.

8. The Family $t = \mu$, $\lambda = 0$

In this section, we find all possible parameter sets subject to the constraints $t = \mu$, $\lambda = 0$, and then construct examples for every possibility, starting with $t = \mu = 1$, then applying Theorem 7.1. Also, for the $t = \mu = 1$ case, a homomorphism, similar to the one in Section 4, from the constructed graph to a triangular association scheme strongly regular graph is demonstrated. (For the characterization of a triangular association scheme used here, see [16].)

Lemma 8.1. *If G is an (n, k, μ, λ, t)-graph with $t = \mu$, and $\lambda = 0$, then there is an integer b, such that $k = \mu b$ and $n = \mu b(b+1)$.*

Proof. The necessary conditions on the parameters found in Theorem 2.2, in this case, reduce to

$$k(k + \mu) = \mu n,$$

$$\mu = d$$

which leads to

$$2k/\mu \equiv 0 \pmod{2}.$$

Then, since $2k/\mu$ is an integer, $\mu \mid k$, and the lemma is proved. ∎

Lemma 8.2. *There is an (n, k, μ, λ, t)-graph with parameters $(k(k+1), k, 1, 0, 1)$.*

Proof. We construct the adjacency matrix A of such a graph. Let M_r be the following matrix ($r = 1, ..., k$):

$$(M_r)_{ij} = 1 \qquad \text{if and only if} \qquad i = r,$$

i.e., M_r has 1's in row r, and 0's elsewhere. Now let

$$A = \begin{bmatrix} 0 & M_1 & M_2 & \cdots & M_{k-1} & M_k \\ M_1 & 0 & M_2 & \cdots & M_{k-1} & M_k \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ M_1 & M_2 & M_3 & \cdots & M_k & 0 \end{bmatrix}.$$

It is not hard to see (especially with the aid of an example) that $A^2 + A = J$ and $AJ = JA = kJ$, and that A is thus the adjacency matrix of a $(k(k+1), k, 1, 0, 1)$-graph. ∎

An example is in order and will make the above construction much more understantable. Let $k = 3$, and then

$$A = \left(\begin{array}{ccc|ccc|ccc|ccc}
0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1\\
\hline
1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1\\
\hline
1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1\\
\hline
1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0
\end{array}\right).$$

Lemma 8.1 showed the necessity of the condition $\mu \mid k$. The following theorem shows sufficiency of this condition.

THEOREM 8.3. *There is an (n, k, μ, λ, t)-graph with $t = \mu$ and $\lambda = 0$ if and only if $\mu \mid k$ and $n = k(k+\mu)/\mu$.*

Proof. Lemma 8.1 states that if the graph exists, then $\mu \mid k$ and $n = k(k+\mu)/\mu$. Conversely, assume $k = b\mu$ and $n = b(b+1)\mu$. Use Lemma 8.2 to construct an adjacency matrix, A', of a graph with parameters $(b(b+1), b, 1, 0, 1)$. Then by Theorem 7.1, $A' \times J_\mu$ is the adjacency matrix of a graph with parameters $(b(b+1)\mu, b\mu, \mu, 0, \mu)$. ∎

Note that in the construction of Lemma 8.2, that any two rows are either identical or orthogonal. This will be our criterion for establishing a homomorphism from one of these graphs to an undirected graph. The undirected graph will turn out to be the triangular association scheme.

Let G be $(k(k+1), k, 1, 0, 1)$-graph such that

$a \to b$ or $a \leftrightarrow b$; $c \to d$ or $c \leftrightarrow d$; and $a \to d$ or $a \leftrightarrow d$ together imply that $c \to b$ or $c \leftrightarrow b$. (8.1)

for all a, b, c, d in G.

LEMMA 8.4. *Let G satisfy* (8.1), *and let $a \leftrightarrow b$ and $x \leftrightarrow y$ in G. Then the only possibilities for the set of other edges among these vertices are*:

(i) *there are no other edges*; *or*

(ii) *$a \to x$ and $y \to b$* (*or some combination isomorphic to this one*) (see Fig. 6).

Proof. If there are no other edges, then (i) is satisfied, and we are done. Also, as $t = 1$, each vertex has only one undirected edge, and in this case, these are $a \leftrightarrow b$ and $x \leftrightarrow y$, by assumption. So there are no additional undirected edges. Thus, we may assume, without loss of generality, that $a \to x$. We now claim that $y \to b$ and that there are no other additional edges among these vertices. From $a \to x$, $y \leftrightarrow x$, $a \leftrightarrow b$, and the hypothesis on G, we know that $y \to b$ or $y \leftrightarrow b$. But there are no additional undirected edges, so $y \to b$. The only other possible edges among these vertices are between a and y, and between b and x. Suppose $a \to y$. Then $a \to y$ and $a \to x \leftrightarrow y$, violating $\lambda = 0$. Similarly, all other possible edges violate $\lambda = 0$. ∎

Let G' be an undirected graph with vertex set $= \{\{a, b\}: a \leftrightarrow b \text{ in } G\}$. Lemma 8.4 allows us to define the following homomorphism:

$$\beta\colon G \to G'; \; a\beta = \{a, b\} \text{ such that } a \leftrightarrow b \text{ in } G$$

$a\beta \leftrightarrow x\beta$ in G' if and only if $a\beta = \{a, b\}$ and $x\beta = \{x, y\}$ are connected in G as described in Lemma 8.4.

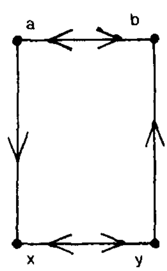

FIGURE 6

Fix a vertex $a\beta = \{a, b\}$ in G'. Partition the set of vertices of G' connected to $a\beta$, $\{x\beta : x\beta \leftrightarrow a\beta \text{ in } G'\}$, into two sets

$$R = \{x\beta = \{x, y\} : a \to x \text{ ot } a \to y\},$$

$$B = \{x\beta = \{x, y\} : b \to x \text{ or } b \to y\}.$$

This is a partition, by Lemma 8.4 and the definition of β. This partition will help prove that G' is isomorphic to a triangular association scheme.

THEOREM 8.5. *Let G be an (n, k, μ, λ, t)-graph with $t = \mu = 1$ and $\lambda = 0$, satisfying* (8.1). *Then there is an undirected graph G' isomorphic to a triangular association scheme $T(k+1)$, and a homomorphism β: $G \to G'$, as defined above.*

Proof. It will suffice to show that the G' defined above is isomorphic to $T(k+1)$. By a theorem of Shrikhande (found in [16]), it suffices to show that the vertices of R are connected in G', and that the vertices of B are connected in G'. We will only show this for B; the proof for R is identical.

So we need to show that if $x\beta$ and $z\beta$ are in B, then $x\beta \leftrightarrow z\beta$ in G'. So assume that $a \leftrightarrow b$, $x \leftrightarrow y$, and $z \leftrightarrow w$ in G; and that $b \to x$, and $b \to z$ in G. We show that $y \to z$ in G, and hence $x\beta = y\beta \leftrightarrow z\beta$ in G' (see Fig. 7). By Lemma 8.4, $y \to a$ and $w \to a$. By (8.1), $y \to a$, $w \to a$, and $w \leftrightarrow z$, we have $y \to z$. ∎

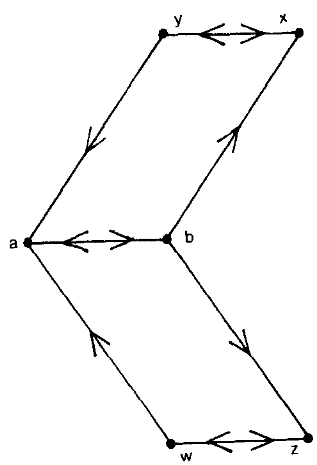

FIGURE 7

9. A Generalization of Addition Sets

In this section we construct another infinite family of (n, k, μ, λ, t)-graphs, using a generalization of addition sets. As used by Lam in [7], addition sets take place in the additive group Z_n. We modify that definition here so that the sets exist in certain **quasigroups**, sets with a binary operation that enjoys left and right cancellation, but is not necessarily associative. (For another generalization of addition sets applied to (0, 1)-matrices, see the last example in [2]).

Let H be a quasigroup. Since we have right cancellation in H, we may define

$$h_1 - h_2 = x \in H \qquad \text{if and only if} \qquad x + h_2 = h_1.$$

We define a **quasigroup-addition** set of a quasigroup H with binary operation $+$ and an element $h_0 \in H$ to be a subset $S \subseteq H$ such that the equation

$$s_1 + s_2 = h$$

has exactly μ solutions (s_1, s_2) with s_1, $s_2 \in S$ if $h \neq h_0$, and exactly t solutions (s_1, s_2) with $s_1, s_2 \in S$ if $h = h_0$. We will demonstrate an example below, after proving the general construction of this section.

Lemma 9.1. *Let A be a quasigroup circulant matrix, i.e.,*

$$A_{ij} = 1 \qquad \textit{if and only if} \qquad j - i \in S,$$

where S is a quasigroup-addition set of a quasigroup H that satisfies

$$(a - b) + (b - c) = a - c \tag{9.1}$$

for all a, b, $c \in H$. Then

$$A^2 = tI + \mu(J - I).$$

Proof. By the definition of matrix multiplication,

$$(A^2)_{ij} = \text{number of } m \text{ such that } m - i,\ j - m \in S.$$

Thus, it suffices to show that the number of m such that $m - i, j - m \in S$ is equal to the number of distinct ways to represent $j - i$ as the sum of two elements in S. Define a map γ from the set of m such that $m - i, j - m \in S$ into the set of representations of $j - i$ as the sum of two elements in S by the following rule: Suppose $m - i, j - m \in S$. Then by (9.1),

$$j - i = (j - m) + (m - i)$$

is a representation of $j-i$ as the sum of two elements in S. Let γ map m into this representation; it is clearly well defined. We show that γ is a bijection, which will complete the proof.

First we show γ is surjective. Assume

$$j-i=s+t, \qquad \text{where} \quad s,\ t\in S.$$

Let $m=t+i$, so $t=m-i$. Then

$$s+t=j-i=(j-m)+(m-i)=(j-m)+t.$$

By cancellation, $s=j-m$. As $s+t$ was an arbitrary representation, γ is surjective.

Now we show γ is injective. Assume $m_1-i=m_2-i=t$ in some representation. By cancellation, $m_1=m_2$, and γ is injective. ▮

We now give an example of such a quasigroup, and shortly, also a quasigroup-addition set of it. Let

$$H_k=Z_{k-1}\times Z_{k+1},$$

where $+$ in H_k is defined as

$$(a,b)+(c,d)=(a-c,\ b+d).$$

LEMMA 9.2. *H_k is a quasigroup satisfying* (9.1).

Proof. Simple verification. ▮

THEOREM 9.3. *If k is even, then*

$$S=\{(0,1),(1,2),...,(k-2,k-1),(0,k)\}$$

is a quasigroup-addition set of H_k with $\mu=1$, $t=2$, and $h_0=(0,0)$.

Proof. First note that $(0,k)=(0,-1)$.

We note that $|H_k|=k^2-1$, $|S|=k$, so $|S+S|=k^2$ (where $S+S$ is the quasigroup addition acting on the Cartesian product of S with itself), and it suffices to show that $(0,0)$ occurs at least twice in $S+S$, and that every other element of H_k occurs at least once in $S+S$. We construct these representations.

It is easy to see that $(0,0)=(0,1)+(0,-1)=(0,-1)+(0,1)$.

Next, we construct a representation for $(0,m)$, $m\neq 0$. As k is even, 2 has an inverse in Z_{k+1}, so $m/2$ exists, and is nonzero. Thus, there is an l such that $(l,m/2)\in S$, and $(0,m)=(l,m/2)+(l,m/2)\in S+S$.

Finally, we construct a representation for (l,m), $l\neq 0$. We construct it recursively. Our recursive steps go from $m-2$ to m, but this covers all of Z_{k+1}, as k is even. First,

$$(l, l) = (l, l+1) + (0, -1).$$

Now, suppose

$$(l, m-2) = h_1 + h_2,$$

where one of h_1, h_2 has first component different from 0 (if both have 0 in the first component, then $l = 0$, $\Rightarrow\Leftarrow$). Note that for any $h \in H_k$, $h = (a, a+1)$ for some a, unless $h = (0, -1)$. There are thus three cases to consider.

Case 1. $h_1 = (a, a+1)$, $h_2 = (b, b+1)$. Then $l = a - b$, and $m - 2 = a + b + 2$. So

$$(l, m) = (a+1, a+2) + (b+1, b+2).$$

Note that $(a+1, a+2)$, $(b+1, b+2)$ are in S as $(a, a+1)$, $(b, b+1)$ are in S, but not equal to $(0, -1)$.

Case 2. $h_1 = (a, a+1)$, $h_2 = (0, -1)$. Then $l = a$, and $m - 2 = a$. So

$$(l, m) = (a, a+1) + (0, 1).$$

Case 3. $h_1 = (0, -1)$, $h_2 = (a, a+1)$. Then $l = -a$, and $m - 2 = a$. So

$$(l, m) = (0, 1) + (a, a+1). \quad \blacksquare$$

Applying Lemma 9.1 and Theorem 9.3, we can construct an (n, k, μ, λ, t)-graph with parameters $(k^2 - 1, k, 1, 1, 2)$, for any even k. We give the following example with $k = 4$: $H_4 = Z_3 \times Z_5$,

$$S = \{(0, 1), (1, 2), (2, 3), (0, 4)\},$$

and

$$A = \left(\begin{array}{ccccc|ccccc|ccccc}
0&1&0&0&1&0&0&1&0&0&0&0&0&1&0\\
1&0&1&0&0&0&0&0&1&0&0&0&0&0&1\\
0&1&0&1&0&0&0&0&0&1&1&0&0&0&0\\
0&0&1&0&1&1&0&0&0&0&0&1&0&0&0\\
1&0&0&1&0&0&1&0&0&0&0&0&1&0&0\\
\hline
0&0&1&0&0&0&0&0&1&0&0&1&0&0&1\\
0&0&0&1&0&0&0&0&0&1&1&0&1&0&0\\
0&0&0&0&1&1&0&0&0&0&0&1&0&1&0\\
1&0&0&0&0&0&1&0&0&0&0&0&1&0&1\\
0&1&0&0&0&0&0&1&0&0&1&0&0&1&0\\
\hline
0&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\
0&0&0&0&1&1&0&1&0&0&0&0&0&1&0\\
1&0&0&0&0&0&1&0&1&0&0&0&0&0&1\\
0&1&0&0&0&0&0&1&0&1&1&0&0&0&0\\
0&0&1&0&0&1&0&0&1&0&0&1&0&0&0
\end{array}\right).$$

Note that the structure of H_k causes the adjacency matrix to be a back-circulant of circulant blocks.

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