ON SHARPLY $n$-TRANSITIVE SUPERSTABLE GROUPS

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We study sharply $n$-transitive superstable groups for $n=2,3$. For $n=3$ we get a complete classification, and for $n=2$ we prove that such groups split if we also assume the surjectivity principle. These results generalize the algebraic case.

1. Introduction

We will study infinite sharply $n$-transitive superstable groups for $n=2$ or 3. For $n>3$ the sharply $n$-transitive groups are completely classified, and are all finite [6,23], and for $n=2,3$ the finite ones have been classified [11,25,26]; cf. [9,19,24].

The ‘typical’ sharply 3-transitive group is $\text{PSL}_2(K)$ acting on the projective line over a field $K$, and the ‘typical’ sharply 2-transitive group is the group of affine transformations of the affine line, abstractly: $K_+ \rtimes K^\times$. For $K$ infinite, these groups are superstable in the model theoretic sense if and only if $K$ is algebraically closed. We will show that every sharply 3-transitive infinite superstable group is of the standard form $\text{PSL}_2(K)$ for some algebraically closed field $K$. We conjecture that the analogous result holds for $n=2$, and we get some partial results in this direction.

There are two rather different motivations for this research. Since algebraic groups over algebraically closed fields are superstable this work generalizes the algebraic case. At the same time our investigations may eventually lead to the discovery of nonalgebraic but superstable simple groups, which would be of great importance to model theory; this is our main reason for taking up the subject. It is well known that the study of multiply transitive groups was intimately connected with the classification of the finite simple groups, and that this interaction was very far from being one-sided. It is possible that similar developments may arise in the superstable context.

The organization of the paper is as follows. In Sections 2 and 3 we review background material from group theory and model theory, respectively. A general lemma
concerning the action of a definable involutory automorphism with finitely many fixed points on a superstable group is given in Section 4. Section 5 contains our analysis of infinite superstable sharply 2-transitive groups. We show that in the solvable case these groups have the anticipated form \( K \times K^\times \). We also give a partial result concerning the splitting of an arbitrary infinite superstable sharply 2-transitive group, and we explain the connection between this work and the main result of [4]. Section 6 contains the classification of arbitrary infinite superstable sharply 3-transitive groups.

We do not assume a priori that the permutation representations under consideration are interpretable in the given group, but we do show that this does occur in any case. In particular if we restrict our attention to algebraic groups (over algebraically closed fields), then the point stabilizers in sharply multiply transitive permutation representations must be algebraic.

2. Group theoretical preliminaries

Let the group \( G \) act transitively on the set \( X \). If \( H \) is the stabilizer of a point \( x \in X \), the action of \( G \) on \( X \) is equivalent to the action of \( G \) on the coset space \( G/H \). Such an action will be called definable iff the group \( H \) is definable in \( G \). If \( G \) is algebraic, an equivalent condition is that \( H \) is Zariski closed.

We say that the action of \( G \) is sharply \( n \)-transitive on \( X \) if:

For any \((x_1, ..., x_n), (y_1, ..., y_n)\) in \( X^n \) with \( x_i \neq x_j \) and \( y_i \neq y_j \) for all \( i \neq j \), there is a unique \( g \in G \) with \( g \cdot x_i = y_i \) for all \( i \).

Kerby’s monograph [13] contains a detailed treatment of sharply multiply transitive groups. We summarize here the main points needed for the present paper, making the group theoretical portion of the paper essentially self-contained modulo a few pages of [19]. In particular the following may be found in [19]:

Fact 2.1. If \( G \) is sharply \( n \)-transitive on \( X \) with \( n \geq 2 \) and \( H \) is the stabilizer of a point of \( X \), then \( H \) is sharply \( (n - 1) \)-transitive, and any point stabilizer is conjugate to \( H \). Furthermore \( G \) is primitive on \( X \), that is there is no nontrivial \( G \)-invariant equivalence relation on \( X \), or equivalently, \( H \) is a maximal subgroup of \( G \). If \( A \trianglelefteq G \) is abelian, then \( A \) acts regularly on \( X \), \( G = A \times H \), and the action of \( G \) on \( X \) may be identified with the action of \( G \) on \( A \) in which \( A \) acts by left multiplication and \( H \) acts by conjugation.

For the next statement, cf. [13].

Fact 2.2. Let \( G \) be sharply 2-transitive on \( X \), \( H \) the stabilizer of the point \( x \in X \) and \( y \in X - \{x\} \). Let \( w \in G \) switch \( x \) and \( y \). Then \( H \) has at most one involution, which will be central in \( H \) if it exists. The element \( w \) is an involution. For any \( g \in G \setminus H \),
there are unique elements $h_1, h_2 \in H$ with $g = h_1 w h_2$. If $A \triangleleft G$ is a proper abelian subgroup, then Fact 2.1 applies ($G = A \rtimes H$) and we say that $G$ is split; conversely if $G = A \rtimes H$ for some nontrivial $A$ then $A$ is abelian [18] and $A = a^H \cup \{1\}$ for any $a \in A^*$. If $H$ is abelian then $G$ necessarily splits [13,17].

**Notation.** For the rest of the present section we take $G$ sharply 2-transitive and fix the notation $H,w$ as in Fact 2.2. In addition we let $N^* = G \setminus \bigcup_{x \in G} H^x$, the set of elements of $G$ without fixed points, and $N = N^* \cup \{1\}$. We call $N$ the *Frobenius kernel* of $G$. We let $I$ be the set of involutions in $G$, and $I^2 = I \cdot I$.

**Fact 2.3.** If $G$ is sharply 2-transitive, then with the notation given above we have:

1. $h \in H^* \rightarrow C_G(h) \subseteq H$.
2. $g \in N^* \rightarrow C_G(g) \subseteq N$.
3. $H \cap H^x \neq \{1\} \Rightarrow g \in H$.
4. If $K \leq H$ with $K \neq \{1\}$, then $N_G(K) \subseteq H$.
5. $Z(G) = \{1\}$. □

This is all straightforward.

**Fact 2.4.** If $G$ is a finite sharply 2-transitive group then the Frobenius kernel $N$ is a subgroup. Whenever $G$ is sharply 2-transitive with the Frobenius kernel a subgroup, it is a normal abelian subgroup and Fact 2.2 applies. □

The first statement in Fact 2.4 is nontrivial, cf. [17; 19, Theorem 17.1].

**Fact 2.5** (Kerby [13, 2.1,3.2,4.1]). If $G$ is sharply 2-transitive, then with the notation given above we have:

1. $I = w^H \cup (I \cap H)$ (where $|I \cap H| \leq 1$).
2. $I^2 \subseteq N$.
3. If $H$ has an involution then $I^2 - \{1\}$ is a single conjugacy class. □

At times we will use the connection between split sharply 2-transitive groups and nearfields. If $G = A \rtimes H$ is a split sharply 2-transitive group, we will write $A$ additively, and we write the action of $H$ on $A$ multiplicatively, that is we write $h \cdot a$ for $a^h$ if $a \in A$, $h \in H$. Let $1 \in A^* = A \setminus \{0\}$ be some fixed element. Identify $h \in H$ with $h \cdot 1 \in A^*$. Then $G$ acts sharply 2 transitively on $A$ via $g \cdot x = h \cdot x + a$ when $g = (a,h) \in G$. Let $F(G)$ be the structure

$$(A;0,1,+,\cdot)$$

where $+$ is the group operation on $A$ and $\cdot$ is obtained by transferring the multiplication of $H$ to $A^*$, and defining

$$0 \cdot a = a \cdot 0 = 0 \quad \text{for } a \in A.$$
Then $F(G)$ is a nearfield \cite{13, 6.2, 7.1, 3.7}, that is, it satisfies the usual axioms for division rings with the exception of right distributivity. Conversely, the group of affine transformations over any nearfield acts sharply 2-transitively on the affine line.

A split sharply 2-transitive group $A \times H$ is called planar if $A$ equals the Frobenius kernel $N$. A number of equivalent conditions are known. In terms of nearfields, the planarity condition becomes \cite{13, 7.2}:

The equation $xa + b = x$ has a unique solution $x$ for all $a, b$ with $a \neq 1$.

For the geometrical background see \cite[V.3]{7}. Examples of non-planar split sharply 2-transitive groups are given in \cite{12, 27}.

Now we turn briefly to the 3-transitive case. Suppose $\mathcal{G}$ is a sharply 3-transitive group acting on the set $X$. Fix three distinct points $x, y, z$ of $X$. Let $G, H$ be the pointwise stabilizers of $\{z\}, \{z, x\}$ respectively, and let $w_1$ switching $x$ and $z$ and fixing $y$. Finally, set $B = H \cup w_1 H$.

Then $G$ acts sharply 2-transitively on $X \setminus \{x\}$, $H$ acts regularly on $X \setminus \{x, z\}$, $w_1$ is an involution normalizing $H$, and $B$ is a subgroup of $\mathcal{G}$. We will also use the notation $N, I$ for the Frobenius kernel and the set of involutions in $G$.

**Fact 2.6.** Suppose that $\mathcal{G}$ is a sharply 3-transitive group, and adopt the notation given above. Then:

1. $\mathcal{G} = G \cup G w_1 G$.
2. $N_{\mathcal{G}}(H) = B$.
3. $H = G \cap G^{w_1}$.
4. For $h \in H^*$, $g \in \mathcal{G}$, if $h^g \in H$, then $g \in B$.
5. For $a \in N^*$, $C_{\mathcal{G}}(a) \subseteq N$.
6. If $H$ has no involutions, then $C_B(w_1) = \langle w_1 \rangle$. If $H$ has an involution $i$, then $C_B(w_1) = \langle w_1, i \rangle$.

**Proof.** The proof of (1) is straightforward (and really only depends on 2-transitivity). For (2), the normalizer of $H$ is just the setwise stabilizer of $\{x, z\}$, which is clearly $B$. (3) and (4) are clear on similar grounds. For (5), note that $N^*$ is the set of elements of $\mathcal{G}$ whose fixed point set is exactly $\{x\}$; in particular if $a \in N^*$ and $g \in C_{\mathcal{G}}(a)$ then $g \in G$; hence $g \in N$ by Fact 2.3.

We prove (6). Let $g \in C_B(w_1)$. Replacing $g$ if necessary by $gw_1$, we may suppose that $g \in H$. As $w_1$ fixes $y$, $w_1$ also fixes $gy$ and $g^2 y$. Thus $y = g^2 y$, and $g^2 = 1$. The result follows, using Fact 2.2. \qed

### 3. Model theoretical preliminaries

Most of the results of the present paper on superstable groups have somewhat simpler proofs in the case of $\omega$-stable groups of finite Morley rank. On the other hand the natural level of generality for these results is usually the class of superstable...
groups, using the technology developed by Berline and Lascar in [2]; occasionally the additivity property of the U-rank even simplifies certain arguments. We review the Berline-Lascar technology here.

Let $G$ be a superstable group. It is customary in model theory to take this to mean that $G$ is a superstable structure carrying a distinguished operation $\cdot$ such that $(G; \cdot)$ is a group; it would actually be better to take $G$ to be just an $\omega$-definable group inside a superstable structure, where an $\omega$-definable subset is the intersection of an arbitrary family of definable subsets, taken inside a 'sufficiently saturated' model; if the intersection of $\kappa$ subsets is formed, the ambient model should be $\kappa^+$-saturated. By [20] any $\omega$-definable subgroup of a sufficiently saturated model will also be the intersection of definable subgroups. One simplification in the $\omega$-stable case is that $\omega$-definable subgroups are in fact definable in that case. In the algebraic case definable subgroups are just Zariski closed ones.

There is also a notion of dimension in the algebraic case, which generalizes to the superstable case as the notion of U-rank. In superstable structures every type $p$ has an ordinal U-rank $U(p)$, and a set $S$ is said to have U-rank $\alpha$ if $\alpha = \max_{p \in S} U(p)$. If this maximum does not exist then $S$ is usually not considered to have a U-rank. The same notions apply inside quotient structures, that is after factoring out a definable equivalence relation. U-rank is preserved by definable bijections, and the finite sets are exactly the sets of rank 0. In structures of finite U-rank, every definable subset has a U-rank.

Our notations for ordinal arithmetic are as follows. $\alpha \cdot \beta$ denotes the order type of $\alpha \times \beta$, lexicographically ordered. An ordinal $\alpha$ may be written in an essentially unique way as $\sum n_i \omega^{\gamma_i}$ with $n_i \in \mathbb{N}$ and $(\gamma_i)$ a decreasing sequence of ordinals; this is called the Cantor normal form. (The representation is unique if all $n_i$ are taken to be positive.) It is often convenient to write U-ranks in Cantor normal form. The natural sum $\alpha \oplus \beta$ of two ordinals is obtained by writing $\alpha, \beta$ in Cantor normal forms as $\sum a_i \omega^{\gamma_i}$, $\sum b_i \omega^{\gamma_i}$, and taking $\alpha \oplus \beta$ to be $\sum_i (a_i + b_i) \omega^{\gamma_i}$. The ordinal sum $\alpha + \beta$ is defined to be

$$a_1 \omega^{\gamma_1} + \cdots + a_r \omega^{\gamma_{r-1}} + (a_r + b_r) \omega^{\gamma_r} + b_{r+1} \omega^{\gamma_{r+1}} + \cdots + b_k \omega^{\gamma_k}$$

where $r$ is the least integer such that $b_r \neq 0$. An ordinal $\alpha$ is a monomial if it may be written in the form $a \omega^\gamma$ with $\gamma \in \mathbb{N}$; for monomials the ordinal sum and the natural sum coincide.

We will say that a superstable group has order $\alpha$ if $\omega^\alpha \leq U(G) < \omega^{\alpha+1}$. Equivalently, if U-rank$(G)$ is written in Cantor normal form as $\sum a_i \omega^{\gamma_i}$ with $a_i \neq 0$, then the order of $G$ is $\gamma_1$.

**Fact 3.1** (Berline and Lascar [2, III.8.2]). For all $\omega$-definable subgroups $H$ of a superstable group $G$, both $H$ and the coset structure $G/H$ have a U-rank [2], and

$$U(H) + U(G/H) \leq U(G) \leq U(H) \oplus U(G/H).$$
Fact 3.2 (Berline and Lascar [2, IV.2.7]). Let $G$ be a superstable group of order $\alpha$. Then $G$ has a definable normal subgroup of monomial rank, also of order $\alpha$. □

An $\omega$-definable subgroup $H$ of $G$ is called connected if it has no proper $\omega$-definable subgroup of finite index. The connected component $H^0$ of $H$ is the intersection of the $\omega$-definable subgroups of $H$ of finite index. $H^0$ is connected.

More generally $H$ is $\alpha$-connected if $H$ has no proper $\omega$-definable subgroup $K$ with $U(H/K) < \omega^{\alpha}$; for $\alpha = 0$ this is the usual notion of connectedness. One may define the $\alpha$-connected component $H^{(\alpha)}$ of $H$ similarly. If $H \triangleleft G$, then $H^{(\alpha)} \triangleleft G$ [2, IV.4.2]. For all of this, see [2, IV.4]. The critical value of $\alpha$ is the exponent occurring in the leading monomial when $U(H)$ is expressed in Cantor normal form; we will say that $H$ is of order $\alpha$ if $\omega^{\alpha} \leq U(H) < \omega^{\alpha+1}$. Then the rank of $H^{(\alpha)}$ is a monomial of the form $k \cdot \omega^r$ for some $k \in \mathbb{N}$ [2, IV.4.6].

A version of Zil'ber's indecomposability theorem [28, 29], which holds in this context was given in [2, V.3.1]. We use a special case of this: if $H$ is of order $\alpha$ and $(H_i : i \in i)$ is a family of $\alpha$-connected $\omega$-definable subgroups of $H$, then the $H_i$ generate an $\omega$-definable subgroup of $H$. Notice also that the normalizer of an $\omega$-definable subgroup is again $\omega$-definable [20]. As a consequence we have the following:

Fact 3.3 (Berline and Lascar [2, VI.2.4]). Let $G$ be superstable of order $\alpha$, $H \triangleleft G$ an $\alpha$-connected $\omega$-definable subgroup of $G$. Then $[G, H]$ is $\omega$-definable. □

If $H$ is an $\omega$-definable subgroup of a superstable group then a subset $S$ of $H$ is generic if $U(S) = U(H)$. If $H$ is connected then the intersection of two generic subsets of a connected group is again generic [21], and the product of two generic subsets is all of $H$.

We will say that $G$ satisfies the surjectivity principle iff every 1-1 function from a definable set $S \subseteq G$ to itself is surjective. This holds in particular if $G$ is algebraic [21].

We require one more fact which is not fully documented in the literature. The proof will be given in the Appendix.

Fact 3.4. Let $H$ be a superstable solvable $\alpha$-connected group of order $\alpha$. Then the derived subgroup $H'$ is nilpotent.

4. Definable involutory automorphisms with finitely many fixed points

Proposition 4.1. Let $\sigma$ be a definable involutory automorphism of a superstable group $G$ with $C_G(\sigma)$ finite. Then:

(1) There is a definable normal abelian subgroup $B$ of finite index in $G$ which is inverted by $\sigma$.

(2) Suppose $C_G(\sigma) \subseteq B \triangleleft G$, where $B$ is of finite index in $G$ and $B$ is abelian,
2-divisible, without 2-torsion, and is inverted by $\sigma$. Then $G$ is abelian and is inverted by $\sigma$.

(3) If $G$ is $\omega$-stable and contains no involutions, and $C_G(\sigma) = 1$, then $G$ is abelian and is inverted by $\sigma$.

Proof. (1) Define $f : G^0 / C_G^0(\sigma) \to G^0$ by
\[ f(g) = g^\sigma g^{-1}. \]
As $U(G) = U(G^0 / C_G^0(\sigma))$ and $f$ is 1-1, the image $A_0 = \{ g^\sigma g^{-1} : g \in G^0 \}$ is generic in $G^0$. Hence the set $A = \{ g \in G^0 : \sigma$ inverts $g \}$, which contains $A_0$, is generic.

If $g \in A$ and $h \in A \cap g^{-1}A$ then $\sigma$ inverts $g$, $h$, and $gh$, hence $g, h$ commute. Thus $A$ commutes with $A \cap g^{-1}A$. Since $A \cdot A - G^0$, $A \cap g^{-1}A$ is central in $G^0$, but $A \cap g^{-1}A$ is also generic, so $G^0$ is abelian and inverted by $\sigma$. It follows easily that there is an abelian definable normal subgroup of $G$ containing $G^0$ which is inverted by $\sigma$. Any definable subgroup of $G$ containing $G^0$ is of finite index.

(2) Let $B \leq G$ be abelian, 2-divisible, and 2-torsion free with $[G : B]$ finite, and with $\sigma$ inverting $B$. Let $\bar{G} = G / B$. Then $\bar{G}$ is a finite group on which $\sigma$ acts. We claim that $\sigma$ has no nontrivial fixed point on $\bar{G}$. Suppose that $g^\sigma = gb$ with $b \in B$. Let $a \in B$ with $a^2 = b$ and compute:
\[ (ga)^\sigma = ga, \quad so \quad ga \in B \quad and \quad g = 1. \]
As $\sigma$ is an involutory automorphism of $\bar{G}$ with no nontrivial fixed point, $\bar{G}$ is abelian of odd order, inverted by $\sigma$ [5, p. 2].

Now it suffices to show that $\sigma$ inverts $G$. Let $g \in G$. Then $g^\sigma = g^{-1}b$ for some $b \in B$. Applying $\sigma$, $g = (g^{-1}b)^\sigma = (g^{-1}b)^{-1}b^{-1} = b^{-1}gb^{-1}$, so $b^\sigma = b^{-1}$, and $g^2 \in C(b)$. But also some odd power of $g$ lies in $B \subseteq C(b)$, so $g \in C(b)$. Thus $b = b^{-1}$, $b = 1$, as required.

(3) It suffices to notice that an $\omega$-stable group $B$ without involutions is 2-divisble. This follows from [15] if $B$ is abelian, and follows in general by considering the group $ZC(g)$ for any $g \in B$. \( \square \)

We remark that the preceding lemma also holds if $G$ is an $\infty$-definable subgroup of a superstable group, but with $B \infty$-definable as well. Note that no stability hypothesis is used in part (2) of Proposition 4.1.

Example. We give an example of a non-abelian superstable group $G$ containing no involution, equipped with a definable involutory automorphism with no nontrivial fixed point. Let $G = \langle a \rangle \times \langle g \rangle \simeq \mathbb{Z} \times \mathbb{Z}$ with $a^8 = a^{-1}$. Define $\sigma$ by
\[ a^\sigma = a^{-1}, \quad g^\sigma = ag^{-1}. \]
Our group is $G = (G; \cdot, \sigma)$. This is a group with no involutions which is equipped with an involutory automorphism with no nontrivial fixed point. Finally, $G$ is superstable, as it is interpretable in the module $(\langle a, g^2 \rangle; g, \sigma)$ where $M = \langle a, g^2 \rangle = \mathbb{Z} \times \mathbb{Z}$ as an abelian group, and $g, \sigma$ induce commuting involutions in Aut $M$. 
5. Sharply 2-transitive superstable groups

Throughout this section $G$ is an infinite sharply 2-transitive superstable group acting on the set $X$. We retain the notation of Section 2, in particular the notation $H, w, N$ as defined there (with $w^2 = 1$). Notice that we may take $G$ to be arbitrary saturated, replacing the pair $(G, X)$ by an elementary extension $(G^*, X^*)$.

Lemma 5.1. The action of $G$ on $X$ is definable in the group structure on $G$.

Proof. Our claim is that $H$ is definable in this structure. For $h \in H$, as $C_G(h) = C_H(h)$, this group will be denoted simply $C(h)$. We assume that $H$ contains no involution, as otherwise $H = C(i)$ is definable.

Let $G$ be of order $a$, and suppose that for some $h \in H^*$, $C(h)$ is also of order $a$. Then $H_1 = \langle C(h)^a : h \in H^* \rangle$ is nontrivial and $\omega$-definable. As $1 < H_1 < H$, $H = N(H_1)$ is also $\omega$-definable. But $H$ is a maximal subgroup, therefore $H$ is definable [20].

Now suppose that for all $h \in H^*$ we have $U(C(h)) < \omega^a$. $G$ has a connected $\omega$-definable normal subgroup $A$ of monomial U-rank $k \omega^a$ for some $k$ (Fact 3.2). We make the following claims:

$$H \cap A = (1), \quad U(G) = 2 \cdot U(A).$$

It follows from (2) that $G$ has monomial U-rank. Then applying (2) with $A = G^0$ we get a contradiction.

For (1), suppose $h \in (H \cap A)^*$. Then:

$$U(C_A(h)) \leq U(C(h)) < \omega^a, \quad U(h^A) = U(A/C_A(h));$$

$$k \omega^a = U(A) \leq U(C_A(h)) \oplus U(h^A).$$

Thus $U(h^A) = k \omega^a$, $h^A$ is generic in $A$. As $A$ is connected it follows that $h_1^A = h_2^A$ for $h_1, h_2 \in (H \cap A)^*$, and hence $H \cap A$ has a unique nontrivial conjugacy class, since elements of $H$ which are $A$-conjugate are also $(H \cap A)$-conjugate. Therefore $H \cap A$ has a unique nontrivial conjugacy class. In particular $h^g = h^{-1}$ for some $h, g \in H \cap A \setminus \{1\}$. Then $g^2 \in C(h)$, but $g \notin C(h)$ (if not $h$ is an involution of $H$). Thus $h \in C_G(g^2) \setminus C_G(g)$ and $C_G(g) \subseteq C_G(g^2)$. On the other hand $g, g^2 \in H \cap A \setminus \{1\}$ are conjugated by some $x$. Now $C_G(g) \subseteq C_G(g^2)$. This contradicts the stability. (This last argument is a variant of Reineke's [22].)

As $A \ncong G$ we have $G = A \times H$ and $A$ is abelian. Fix $a \in A^*$. The map $g \mapsto a^g$ induces $G/A \leftrightarrow A^*$. We find

$$2U(A) = U(G/A) + U(A) \leq U(G) \leq U(G/A) \oplus U(A) = 2U(A),$$

proving (2). \qed
Problem. (1) Is $U(G)$ always a monomial?
(2) Does $N$ always have a $U$-rank?

Lemma 5.2. $G$ and $H$ are connected.

Proof. By the double coset decomposition $G = H \hat{\cup} HwH$ (and the uniqueness of the decomposition in $HwH$) there are elements $h_1, h_2 \in H$ such that $G^0 = h_1 H^0 \omega H^0 h_2$ modulo sets of lower rank. Conjugating by $h \in H$ and applying uniqueness, we find $H^0 h_2 h = H^0 h_2$, $h \in H^0 h_1 = H^0$. $H$ is connected, hence degree 1. By uniqueness, $H\omega H$ is also of degree 1, hence so is $G$: that is, $G$ is connected.◼

Lemma 5.3. If $G = A \times H$ is a split sharply 2-transitive superstable group of order $\alpha$, then $A, H, G$ are $\alpha$-connected, $U(G) = 2 \cdot U(H) = 2 \cdot U(A)$, and $G$ is planar (i.e., $A = N$).

Proof. As $A^* \leftrightarrow H$, $U(A) = U(H)$. If $B$ is the $\alpha$-connected component of $A$ then $B$ is a nontrivial normal abelian subgroup of $G$, so $G = B \times H$, forcing $A = B$, $A$ is $\alpha$-connected and hence of monomial rank. Then Fact 3.1 shows that $U(G) = 2 \cdot U(A)$ is also of monomial rank. As $G$ is also connected, it is $\alpha$-connected. Then $H = G/A$ is also $\alpha$-connected.

Now let $F$ be the associated nearfield. We will show directly that $F$ is planar. Fix $a \in F$ and define $\phi(x) = x - ax$; the claim is that $\phi$ is surjective. $\phi$ is a 1–1 homomorphism and $F$ is connected, so $\phi$ is indeed surjective.◼

We show in [4] the following:

Fact. Let $G = N \times H$ be a planar infinite sharply 2-transitive superstable group of finite $U$-rank. If the centralizer of $H$ in $\text{End } N$ is infinite, then $G$ is of the form $K_+ \times K^*$ with $K$ algebraically closed. In particular if $H$ has an infinite center, or if $N$ is of characteristic zero (i.e., torsion-free), then $G$ is of this form. (Recall that $N$ is the additive group of the nearfield $F(G)$ associated with $G$.)◼

As a variant, we can also prove the following:

Proposition 5.4. Let $G$ be a solvable sharply 2-transitive superstable group. Then $G = K \times K^*$ for some algebraically closed field $K$.

Proof. Let $A$ be a normal definable abelian subgroup of $G$. By Fact 3.4, $G'$ is nilpotent. By Fact 2.2, $G = A \times H$, so $(1) < A \cap G' < G'$. Therefore $A$ meets $ZG'$. If $a \in (A \cap ZG')^*$ then $a$ centralizes $H'$, so $H' = (1)$, $H$ is abelian. Therefore the nearfield $F(G)$ associated with $G$ is commutative, that is, $F(G)$ is a field. The result follows.◼
**Notation.** If $G$ has finite $U$-rank then we set

$$m = U(H), \quad n = U(N), \quad d = n - m.$$ 

**Lemma 5.5.** Assume that $G$ has finite $U$-rank, and $x \in N^*$. Then:

1. $U(G) = 2m$;
2. $m \leq U(x^G) \leq n$;
3. $m \geq U(C(x)) \geq m - d$;
4. $n < 2m$.

**Proof.** (1) follows from the double coset decomposition $G = H \cup HwH$ and uniqueness. For (2), notice that $H \ltimes x^H \subseteq x^G \subseteq N$. Then (3) follows using $U(G) = U(x^G) + U(C(x))$.

It remains to prove (4). Define functions $\alpha', \alpha : H^* \to H$ by

$$whw = \alpha'(h) wa(h)$$

for $h \in H^*$. Define $\xi : H^* \to H^*$ by $\xi(h) = h^{-1}\alpha(h)$. In [17] it is proved that $\xi$ is $1$–$1$ and:

$$(*) \quad \text{For } x, y \in H, \; xwy \in N \iff yx \in H \setminus \xi[H^*].$$

Let $S = H \setminus \xi[H^*]$. Then $U(S) < U(H)$, and $(*)$ implies that $N = (wS)^H \cup \{1\}$. Thus $U(N) \leq U(S) \oplus U(H) < 2m$. \(\square\)

**Lemma 5.6.** Let $G$ be a superstable sharply 2 transitive group of finite $U$-rank. Then the following are equivalent:

1. $G$ is split;
2. $G$ is planar (that is, $G = N \times H$);
3. $m = n$;
4. $U(C(x)) = n$ for some $x \in N^*$.

If the surjectivity principle holds in $H$ then these conditions hold.

**Proof.** By Lemma 5.3 (1) implies (2). Clearly (2) implies (3), and the equivalence of (3) and (4) is Lemma 5.5(3). We will now show that (4) implies (1).

Assume (4), and let $x_0 \in N^*$ be fixed with $U(C(x_0)) = n$, and let $X = x_0^G$. Since $U(C(x)) = n$ for $x \in X$ it is easy to see that:

For $x, y \in X, \; U(C(x, y)) = n \iff C(x)^0 = C(y)^0 \iff [C(x) : C(x, y)] < \infty$.

Let $C_x = \bigcap_{y \in X, U(C(x, y)) = n} C(y)$. Then $[C(x) : C_x] < \infty$ and $C_x$ is definable. The relation $E(x, y)$ given by $C_x = C_y$ is definable on $X$, and coincides with the relation: $U(C(x) \cap C(y)) = n$ on $X$. Thus $E$ has only finitely many classes on $X$, and $G$, being connected, must fix each one. We conclude that $C_x \leq G$, and as $C_x \leq N$, $G = C_x \times H$ is split.

Finally, if $H$ satisfies the surjectivity principle then it was shown in [17] that $G$ splits. \(\square\)
When $H$ contains an involution we will show that $G$ contains a definable subgroup which acts as a split sharply 2-transitive subgroup on one of its orbits (Proposition 5.8 below).

**Lemma 5.7.** Let $G$ be a superstable sharply 2-transitive group, and let $i, j$ be distinct involutions in $G$, with $i \in H$. Let $A = C(ij)$. Then $A$ is abelian, $A = iI \cap jI$, and $A$ is inverted by $i$ and by $j$.

**Proof.** Let $a = ij$. Then $a \in N^*$, so $A = C(a) \subseteq N$. As $i$ inverts $a$, $i$ acts on $A$ as an involutory automorphism, and as $i \in N$, $i$ has no nontrivial fixed points on $A$. By Proposition 4.1(1) there is $B$ definable abelian and of finite index in $A$ which is inverted by $i$ and $j$. For $b \in B$, $ib$ is an involution, and $b = i \cdot ib \in iI$. So $B \subseteq iI$, and similarly $B \subseteq jI$.

As $B \subseteq N$, $B$ has no 2-torsion. Our claim will follow from Proposition 4.1(2) once we show that $B$ is 2-divisible.

Since $iI \setminus \{1\} = (iw)^H$ (Fact 2.5.1) and $iI$ is closed under squaring, $iI$ is 2-divisible. We claim that $iI$ is uniquely 2-divisible. Indeed, if $k, k' \in I$ and $(ik)^2 = (ik')^2$, then $k'k \in C(i) \cap N = \{1\}$, so $k = k'$. Similarly $jI$ is uniquely 2-divisible, so $iI \cap jI$ is 2-divisible. Now $(iI \cap jI)/B$ is a 2-divisible subset of the finite group $A/B$, hence contains no 2-torsion. Thus for $b \in B$, we can solve $x^2 = b$ with $x \in iI \cap jI$, and then since $x^2 \in B$, we have $x \in B$, as required.

**Proposition 5.8.** Let $G$ be a superstable sharply 2-transitive group, and let $i, j$ be distinct involutions in $G$, with $i \in H$, $A = C(ij)$. Let $H_0$ be

$$
\{h \in H : (ij)^h \in A\}.
$$

Then $H_0 := N_H(A)$ and $G_0 := A \rtimes H_0$ is a sharply 2-transitive definable split subgroup of $G$, with $i \in H_0$. If $U(G)$ is finite then $A$ is infinite.

**Proof.** Let $a = ij$. We show first that $H_0 = N_H(A)$. Let $h \in H_0$, that is: $a^h \in C(ij)$. As $C(a)$ is abelian, $C(a) \subseteq C(a^h) = C(a)^h$, so by stability $C(a) = C(a)^h$, and $h \in N(A)$ as required. This argument also shows that $H_0 = \{h \in H : A \cap A^h \neq \{1\}\}$.

Evidently $A \rtimes H_0$ is a subgroup of $G$. It remains to prove that $H_0$ acts transitively on $A^*$. Let $b, b' \in A^*$. Then $ib, ib'$ are involutions distinct from $i$, hence conjugate under $H$. As $i \in ZH$, $b$ and $b'$ are also conjugate under $H$, as required.

If $U(G)$ is finite then $U(C(a)) \geq 2m - n > 0$ by Lemma 5.5. □

**Problem.** Is $C(ij)$ always infinite?

By the preceding result, when $G$ is a minimal nonstandard example in which $H$ contains an involution, then either $G$ is split, or $G$ contains a standard split subgroup $K_+ \times K^*$.
Proposition 5.9. Let $G$ be an infinite sharply 2-transitive $\omega$-stable group. Then $G$ is almost strongly minimal, hence $K_1$-categorical.

Proof. Let $X$ be a set on which $G$ has a sharply 2-transitive action. Let $K$ be a minimal normal definable subgroup of $G$. Then $K$ is almost strongly minimal [29] and transitive (since $G$ is primitive). Hence $X$ is almost strongly minimal, and since elements of $G$ are determined by quadruples from $X$, $G$ is almost strongly minimal. \hfill \Box

6. Sharply 3-transitive groups of finite Morley rank

Let $G$ be a sharply 3-transitive group of finite Morley rank acting on a set $X$. We use the notation established at the end of Section 2: $x, y, z, G, H, B, N, I, w_1$. The basic facts needed are found in Fact 2.6.


Proof. Let $G$ be of order $\alpha$. We show first that it suffices to show that at least one of these three groups is definable. Clearly if $G$ or $H$ is definable, then $G, H, B$ are all definable (using Lemma 5.1 and Fact 2.2). Conversely, suppose that $B$ is definable. By Fact 2.6(6) and Lemma 4.1 there is an abelian normal subgroup $A$ of $B$ of finite index, inverted by $w_1$. As the involutions of $A$ are fixed by $w_1$, there are finitely many involutions in $A$, and $U(2A) = U(A) - U(H)$, while $2A \leq H$ since $[H: B] = 2$. Thus $[H: 2A]$ is finite, and $H$ is definable.

A similar argument shows that if one of these groups is $\infty$-definable, then they all are; but as $G$ is a maximal subgroup it follows that $G$ is then definable, and hence $G, H, B$ are all definable.

We may suppose additionally:

1. $H$ contains no involutions;
2. for any $h \in H^*$, $U(C_g(h)) < \omega^\alpha$.

Indeed, if (3) fails then $B = C_g(i)$ by Facts 2.6(4) and 2.2. If (4) fails then consider the $\infty$-definable group

$$H_1 = \langle C_g(h)^{(\alpha)} : h \in H^* \rangle \triangleleft B.$$ 

By Fact 2.6(4) $N_g(B) = B$, and $B$ is $\infty$-definable.

Now let $K$ be any $\infty$-definable nontrivial normal subgroup of $G$. Then we claim that

$$1 < K \cap G \triangleleft H.$$ 

As $w_1$ normalizes $K \cap G$, we have $K \cap G \leq H$ by Fact 2.6(3). We must show also that $K \cap G$ is nontrivial. As $G$ is primitive, $K$ is transitive on $X$. Suppose $K \cap G = \{1\}$, so $G = K \times G$. Then $G$ acts sharply 2-transitively on $K^*$ (Fact 2.1), so $K$ is an
elementary abelian 2-group. Let $w$ be an involution of $G$. Then $w$ must centralize an infinite subgroup of $K$, contradicting the fact that $w$ has at most two fixed points on $X$.

Now let $K = \mathbb{G}^{(w)}$, $H_1 = H \cap K$. By (5) (1) $H_1$. By (4) $U(h^K) = U(K)$ for $h \in H_1^\ast$. So all elements of $H_1$ are conjugate under $K$, hence are conjugate under $B \cap K$. As $[B : H] = 2$, $H_1$ has at most two conjugacy classes.

Suppose $h \in H_1$ is conjugate to $h^{-1}$, say $h^g = h^{-1}$ with $g \in H_1$. Then $g^2 \in C(h)$, $g \in C(h)$, and $g, g^2$ are conjugate in $\mathbb{G}$, $C(g) < C(g^2)$, contradicting stability. So the elements $h, h^{-1}$ of $H_1$ belong to distinct conjugacy classes. So there are exactly two nontrivial conjugacy classes in $H_1$, and all elements of $H_1$ have the same order $p$, with $p$ either an odd prime, or $\infty$.

If $p = 3$ then $H_1$ is a group of exponent 3, hence nilpotent [10, III, § 6, Satz 6.6; 14]. Then $H_1$ has nontrivial center, and as $H$ has only two nontrivial conjugacy classes we find $H_1 = \mathbb{Z}/3\mathbb{Z}$; but $U(h^K) = U(K)$, a contradiction.

If $3 < p < \infty$ and $h \in H^\ast$, then we can solve $h^g = h^r = h$ for some $r \in \mathbb{N}$, $g \in H_1$. Then $h = h^g = h^r = h^t$, a contradiction.

Finally, suppose that $p = \infty$. For any $h \in H_1$, let $A_h$ be the minimal $\infty$-definable subgroup of $\mathbb{G}$ containing $h$. Then $A_h$ is an abelian subgroup of $H_1$, hence is torsion-free. Since $A_h - A_h^{-1}$, the groups $A_h$ are all conjugate for $h$ in $H_1$. Therefore they all have the same $U$-rank, and each is minimal $\infty$-definable, since if $A < A_h$, $a \in A$, then $A_a < A_h$ and $A_a, A_h$ are conjugate. Let $g, h \in H_1$ with $h^g = h^m$, $m > 1$. Then $g \in N(A_h)$, so $A_g \leq N(A_h)$. In this situation, with $U(A_g) = U(A_h)$, there is an algebraically closed field [1] $K$ with additive group $A_h$, and with $A_h \to K^\ast$ canonically; since $A_g$ is minimal, $A_g$ embeds in $K^\ast$, and since $A_g$ is torsion-free, $U(A_g) < U(K) = U(A_h)$, a contradiction. □

In particular the results of Section 4 apply to $G$.

**Theorem 6.3.** $\mathbb{G} = \text{PSL}_2(k)$ acting naturally on the projective line.

**Proof.** We know that $G$ and $H$ are definable, and $H$ is connected (Lemma 5.2). By Fact 2.6(6), $C_H(w_1)$ is finite. By Proposition 4.1(1), $H$ is abelian and $w_1$ acts on $H$ by inversion. Since $H$ is abelian, $G = N \times H$ is standard, that is, for some algebraically closed field $K$ we may identify $N$ with $K^\ast$, $H$ with $K^\ast$, and $X \setminus \{z\}$ with $K$, so that $x, y$ are identified with 0, 1 respectively, and $G$ is identified with $K^\ast \times K^\ast$ acting by affine transformations on $K$. We will also identify $z$ with the symbol $\infty$.

We know that $w_1$ inverts $H$ and fixes $y$, so $w_1$ acts on $K^\ast = X \setminus \{0, \infty\}$ by inversion, and switches 0 and $\infty$. Thus $\mathbb{G}$ acts on $X$ as a subgroup of $\text{PSL}_2(K)$, and as $G$ is 3-transitive, it must be the whole projective group. □

**Appendix. Superstable solvable groups**

If $G$ is a connected solvable group of finite Morley rank then the derived sub-
group $G'$ is nilpotent [16,28]. Here we will sketch the extension of this result to the superstable context. The ingredients of the proof are:

[16] The proof in the $\omega$-stable context;
[21] A streamlined version of [16];
[1,2] The technology reviewed in Section 3;
[8] The fact that a superstable field cannot have a nontrivial infinite definable group of automorphisms.

**Theorem (Fact 3.3).** Let $G$ be a solvable superstable $\alpha$-connected group of order $\alpha$. Then $G'$ is nilpotent.

**Proof.** By Fact 3.3, $G'$ and $G^{(2)}$ are definable subgroups. Taking $G$ to be a counterexample of minimal $U$-rank, as in [16] we may reduce to the case in which $G, G'$ are centerless. Let $A \subseteq Z(G^{(2)})$ be a minimal nontrivial $\omega$-definable normal subgroup of $G$, and let $B \leq A$ be a minimal $\omega$-definable $G'$-invariant subgroup of $A$. Then $A$ is abelian, and using the remarks following Fact 3.2 one sees that $A, B$ are definable and $\alpha$-connected.

View $A$ as a module over ring $R = \mathbb{Z}[G']/J$, where $J$ is the annihilator of $A$ in the group ring. This ring is not a priori interpretable in $G$, though each element of $R$ represents a definable endomorphism of $A$. By Schur's Lemma, $R$ induces a division ring $K$ on $B$. The division ring $K$ is interpretable in $G$ ([1], generalizing [29]), hence is an algebraically closed field.

For $g \in G$, the action of $R$ on $B^g$ induces another algebraically closed field which we call $K_g$. Also the action of $G$ on itself by inner automorphisms induces an action on the group ring and on $R$, and under this action $G$ permutes the kernels $J_g$ of the natural homomorphisms $R \to K_g$. One shows that

$$(*) \quad A = B^g_1 \oplus \cdots \oplus B^g_k$$

for some $g_1, \ldots, g_k \in G$,

and hence the set $\mathcal{J}$ of kernels $\{J_g : g \in G\}$ is finite. This crucial finiteness theorem allows the action of $G$ on the set of kernels to be interpreted in $G$ [21], and then the connectedness of $G$ implies that $G$ stabilizes each kernel. On the other hand the action of $G$ is transitive on $\mathcal{J}$ by construction, so all $K_g$ coincide and $A$ becomes a finite dimensional vector space over $R - K$. Now $G$ acts naturally as a group of automorphisms of $K$. By [8] no nontrivial element of $G$ fixes an infinite subfield of $K$. Arguing as in [21, Corollaire 3.6] we see that $G$ induces a finite group of automorphisms of $K$. As $G$ is connected, it acts trivially on $K$, that is $G$ acts $K$-linearly on $A$. Now we may conclude as in [16].

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