1. INTRODUCTION

The Brauer centralizer algebras, $B_n(Q)$, are finite-dimensional algebras indexed by a positive integer $n$ and a complex number $Q$. For integral values of $Q$, $B_n(Q)$ is the centralizer algebra for the orthogonal group or the symplectic group on the $n$th tensor powers of the natural representation. They were introduced by Richard Brauer in [Br], where many of their properties are given. Earlier, Schur had used the group algebra of the symmetric groups to study the corresponding centralizer algebras for the general linear groups. The algebras $B_n(Q)$ are defined for any value of the parameter $Q$. But unlike the symmetric group algebras, these algebras are not semisimple for certain values of $Q$. Brauer [Br], Brown [Brn], and Weyl [Wey] proved results about semisimplicity. In [HW1, 2, 3, 4] Hanlon and Wales studies the algebras in the cases in which the algebras are not semisimple. They found many surprising combinatorial conditions that helped to describe the radical for values of $Q$ when the radical was not zero. Together, these conditions led them to conjecture that the algebras are semisimple when $Q$ is not an integer. This was proved by Hans Wenzl [Wen]. In their work, Hanlon and Wales constructed certain matrices with polynomial entries that were the Gram matrices for certain representa-
tions. Semisimplicity meant that these matrices were nonsingular. The fact that $B_n(Q)$ is semisimple for $Q$ not an integer means the determinant of these matrices has only integral roots. Some of these roots have combinatorial descriptions, which led to open problems in finding descriptions for all of them. The main problem when the radical is nonzero is to find an effective description of the radical or to find an algorithm that finds the radical, or even its dimension. A related question is to find the dimensions of the irreducible representations.

Recent work of Paul Martin [M] on a related algebra, called the partition algebra, has suggested new techniques for studying the Brauer algebras. The partition algebras have a similar definition and are defined in terms of a parameter $Q$. The multiplication can be defined in a similar way, and in both cases the irreducible representations can be defined and parameterized by certain partitions. Indeed, the Brauer algebra is a subalgebra of the partition algebra with the same parameter $Q$. The purpose of this paper is to extend these techniques to the study of the Brauer algebras. In so doing, we are able to reprove Wenzl's result about the semisimplicity for noninteger values of $Q$. We are also able to give conditions that must be satisfied for the existence of certain embeddings of the generic modules into others. In particular, we show that a necessary condition for the embedding of one generic irreducible into another is that $Q$ be an integer, which implies Wenzl's result. These embeddings were enough in the partition algebra case to determine an algorithm to find the dimensions of all of the irreducible representations. This does not yet seem to be the case for the Brauer algebras.

The main new tools introduced by Martin are Frobenius reciprocity and the use of two functors, $F$ and $G$. In the Brauer algebra case, these functors connect the modules for $B_n(Q)$ to those of $B_{n+2}(Q)$. Martin used related functors extensively in his work.

1.1. Notation

An integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a weakly decreasing sequence of positive integers. If the sum of the $\lambda_i$'s is $n$, then this is denoted by $\lambda \vdash n$ and $|\lambda| = n$. The integers $\lambda_i$ are called the parts of $\lambda$. The Ferrers diagram associated with a partition $\lambda$ is the collection of boxes $[\lambda] = \{(i, j) : 1 \leq j \leq \lambda_i\}$ in $\mathbb{Z}^2$ using matrix-style coordinates. A partition is even if all of its parts are even. For a box $p = (i, j)$ in $[\lambda]$, the content of $p$, denoted $c(p)$, is the value $j - i$. If $\lambda$ and $\mu$ are two partitions such that $\lambda_i \geq \mu_i$ for all $i$, then we say $\lambda$ contains $\mu$, written $\mu \subseteq \lambda$. If $\mu \subseteq \lambda$ (so as sets $[\mu] \subseteq [\lambda]$), then the skew-partition $\lambda/\mu$ is the set $[\lambda] \setminus [\mu]$. A special case is when $\lambda/\mu$ contains one box. In this case, we say $\lambda$ covers $\mu$, denoted $\lambda \triangleright \mu$ (or $\mu$ is covered by $\lambda$, denoted $\mu \triangleleft \lambda$).
1.2. Brauer Centralizer Algebra

The Brauer centralizer algebra, $B_n(Q)$, is defined for every integer $n$ and any complex number $Q$. A basis for $B_n(Q)$ is the set of all 1-factors on $2n$ points. A 1-factor on $2n$ points is a graph with $2n$ vertices in which every vertex has degree 1. The set of 1-factors on $2n$ points is denoted by $\mathcal{F}_n$. We view elements of $\mathcal{F}_n$ by arranging the $2n$ points in two rows, each containing $n$ points, with the rows arranged one on top of the other. For example, a typical element of $\mathcal{F}_7$ is

![Graph](image)

When the 1-factor, $\delta$, is arranged this way, the top row of $\delta$ is denoted $\text{top}(\delta)$. Similarly, we denote the bottom row by $\text{bot}(\delta)$. For reference purposes, the points of the 1-factor are numbered 1 to $n$ from left to right in both the top and bottom. Lines joining two points, both of which are in the top or both of which are in the bottom, are called horizontal lines. Lines joining a point in the top to one in the bottom are called vertical lines.

The multiplication of two 1-factors, $\delta_1$ and $\delta_2$, is defined by placing $\delta_1$ above $\delta_2$. Now draw an edge from point $i$ in $\text{bot}(\delta_1)$ to point $i$ in $\text{top}(\delta_2)$ for all $i$. The resulting graph consists of paths that start and finish in $\text{top}(\delta_1)$ and $\text{bot}(\delta_2)$, as well as some cycles that use only points in the middle two rows. Let $\gamma(\delta_1, \delta_2)$ denote the number of these internal cycles. Form the 1-factor $\delta$ in $\mathcal{F}_n$ by using the paths in the resulting diagram that start and stop on the bottom or the top. The product $\delta_1 \cdot \delta_2$ in $B_n(Q)$ is defined to be $Q^{\gamma(\delta_1, \delta_2)}\delta$.

For example, if

![Graph](image)
then \( \gamma(\delta_1, \delta_2) = 1 \) and \( \delta_1 \cdot \delta_2 = Q^2 \delta \), where

The algebra \( B_n(Q) \) is associative with identity and has dimension \( |\mathcal{T}_n| = (2n - 1)!! = (2n - 1) \cdot (2n - 3) \cdots 3 \cdot 1 \). The identity is the 1-factor with each point \( i \) on the top joined to point \( i \) on the bottom.

As shown by Brauer [Br] and used extensively in [HW1,2,3,4], there is a series of ideals in \( B_n(Q) \). In particular, let \( B_n^m(Q) \) be the span of the elements in \( \mathcal{T}_n \) that have \( m \) or fewer vertical lines. As shown by Brauer, these elements form an ideal in \( B_n(Q) \). The notation here is different from that in earlier work, where the index was the number of horizontal lines on the top or the bottom. The difference in notation is made to be in conformity with the partition algebra, where the number of vertical lines is the relevant parameter. Notice these are an increasing series of ideals as \( B_n^{m-2}(Q) \subset B_n^m(Q) \).

These ideals give a filtration of \( B_n(Q) \). In particular, let \( I_n^m(Q) = B_n^m(Q) / B_n^{m-2}(Q) \) be the quotients of this filtration. A basis for \( I_n^m(Q) \) is the set of 1-factors that have exactly \( m \) vertical lines. The multiplication of \( \delta \) on this module is exactly as in \( B_n(Q) \), unless the resulting product has fewer than \( m \) lines, in which case the result is 0. So the multiplication by any element that is in \( B_n^{m-2}(Q) \) gives 0.

There are certain elements in \( B_n^{n-2}(Q) \), denoted \( X_{i,j} \), which are needed later. The element \( X_{i,j} \) has one horizontal line on the top and one on the bottom from point \( i \) to point \( j \). The remaining lines join point \( k \) on the top to point \( k \) on the bottom for \( k \neq i \) or \( j \).
1.3. Representation Theory of the Symmetric Group

The symmetric group on \( n \) objects is denoted by \( \text{Sym}(n) \). Notice that \( I^\mu_n(Q) \) is isomorphic to the group algebra \( C[\text{Sym}(n)] \) because it is spanned by all of the 1-factors that have \( n \) vertical lines, which are themselves permutations. In fact these elements span a subalgebra isomorphic to \( \text{Sym}(n) \), which we refer to as \( \text{Sym}(n) \).

Any \( B_\nu(Q) \)-module can be viewed as a \( \text{Sym}(n) \)-module by restricting to \( \text{Sym}(n) \). Notice that \( \text{Sym}(n) \), along with \( X_i \) for \( 1 \leq i < j \leq n \), generates \( B_\nu(Q) \). Thus, to show that two \( B_\nu(Q) \)-modules are isomorphic, it suffices to show that they are isomorphic as \( \text{Sym}(n) \)-modules and the action of each \( X_i \) is the same in both modules.

Several results on the representation theory of the symmetric group are needed. Here, we review some definitions and notation. For more details, see [J], [J K], and [S]. For \( \lambda \vdash n \), let \( S^\lambda \) denote the irreducible \( \text{Sym}(n) \)-module corresponding to \( \lambda \). The dimension of \( S^\lambda \) is denoted \( f^\lambda \) and its character by \( \chi^\lambda \).

In the group algebra \( C[\text{Sym}(n)] \), let \( \{e_\lambda : \lambda \vdash n\} \) be a set of irreducible orthogonal idempotents where the indexing is such that \( S^\lambda = C[\text{Sym}(n)] e_\lambda \). A combinatorial description of one choice for \( e_\lambda \) is as follows. Let \( R_\lambda \) be the row stabilizer of the Ferrers diagram \( [\lambda] \). So, \( R_\lambda \equiv \text{Sym}(\lambda_1) \times \cdots \times \text{Sym}(\lambda_l) \). Let \( C_\lambda \) be the column stabilizer of \( [\lambda] \); then

\[
e_\lambda = \frac{f^\lambda}{n!} \sum_{\sigma \in C_\lambda} \sum_{\tau \in R_\lambda} \text{sgn}(\sigma) \sigma \tau,
\]

where \( \text{sgn}(\sigma) \) is the sign of the permutation \( \sigma \). If \( V \) is a \( \text{Sym}(n) \)-module, then for any \( v \in V \), \( e_\lambda \cdot v \) is in the \( \lambda \)-isotopic component of \( V \).

Let \( \mu \vdash n \) and \( \eta \vdash m \). The Littlewood–Richardson coefficients \( c^\lambda_{\mu, \eta} \) are defined by

\[
S^\mu \otimes S^\eta \uparrow_{\text{Sym}(n) \times \text{Sym}(m)} S^{\mu + \eta} = \bigoplus_{\lambda \vdash n + m} c^\lambda_{\mu, \eta} S^\lambda,
\]

where the embedding of \( \text{Sym}(n) \times \text{Sym}(m) \) in \( \text{Sym}(n + m) \) is the obvious one. There is a combinatorial rule for computing these coefficients.
Expositions of this "Littlewood–Richardson Rule" are given in Section 2.8 of [JK] and Section 4.9 of [S]. A basic result of this rule is that $c_{\mu,\lambda}^{\nu} \neq 0$ implies $\mu \subseteq \lambda$. One special case that we need is $\eta = (2)$. In this case, $c_{\mu,\eta}^{\nu} = 1$ if $\mu \subset \lambda$, where $\lambda/\mu$ contains two boxes that are in different columns, and is zero otherwise.

2. GENERIC IRREDUCIBLES OF THE BRAUER ALGEBRA

This section gives a brief description of certain modules $S_\mu(n)$ of the Brauer centralizer algebras. The modules depend on the parameter $Q$. They are irreducible modules of $B_\mu(Q)$, except for finitely many values of $Q$, as seen, for example, in [HW1]. Moreover, they are all irreducible if and only if $B_\mu(Q)$ is semisimple. In the nonsemisimple case, the irreducibles are quotients of these generic irreducibles by maximal submodules.

Let $n = m + 2k$. Define $I^m_n(Q)$ to be the submodule of $I^m_n(Q)$ generated by elements with the fixed bottom

\[
\begin{array}{ccccccc}
\bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet \\
m & & & & & & k
\end{array}
\]

If $\delta_1$ is any 1-factor on $2n$ points, and $\delta_2$ is a 1-factor with bottom being (1), then either the product $\delta_1 \cdot \delta_2$ has bottom (1) or possibly two of the first $m$ points in the bottom are now joined. In the second case, since this multiplication is taking place in $I^m_n(Q)$, the result is 0. So, $I^m_n(Q)$ is indeed a submodule of $I^m_n(Q)$.

Let $\text{Sym}(m)$ be the subgroup of $\text{Sym}(n)$, fixing all but the first $m$ points $1, 2, \ldots, m$. Notice $\text{Sym}(m)$ acting on the right takes $I^m_n(Q)$ to itself. Now given $\mu \vdash m$, define $S_\mu(n)$ to be the $B_\mu(Q)$-module

\[
I^m_n(Q) \otimes_{\text{Sym}(m)} S^\mu,
\]

with the multiplication being $\delta_1 \cdot (\delta_2 \otimes v) = (\delta_1 \otimes (\delta_2 \cdot v)) \cdot v$. The term $\delta_1 \cdot (\delta_2 \otimes v)$ is 0 if $\delta_1 \cdot \delta_2$ is 0 in $I^m_n(Q)$. If it is not 0 in $I^m_n(Q)$, then $\delta_1 \cdot \delta_2$ has $m$ vertical lines. These must be the ones connecting the first $m$ points on the bottom to the top and in $B_\mu(Q)$, $\delta_1 \cdot \delta_2 = \delta'$, where $\delta' \in I^m_n(Q)$. The $B_\mu(Q)$-module $S_\mu(n)$ is generically irreducible. This means that for all but finitely many values of $Q$, $(S_\mu(n) : |\mu| \leq n, 2(n - |\mu|))$ is the set of distinct irreducible $B_\mu(Q)$-modules. See [HW1].
Next we give an explicit basis for $\mathcal{P}(n)$. A gain set $n = m + 2k$. An $(m, k)$ partial 1-factor on $n$ points is a labeled graph with $n$ vertices and $k$ edges such that each vertex has degree 0 or 1. The $m$ vertices of degree 0 are called free. Let $\mathcal{R}_{m, k}$ denote the set of $(m, k)$ partial 1-factors. A subset with each $x$ in $\mathcal{R}_{m, k}$ an element $f(x)$ in $\mathcal{I}^n_m(Q)$ by letting the top of $f(x)$ be $x$ itself and the bottom be (1), and connect the $m$ free vertices in the top and bottom from left to right in order. So, the vertical edges in $f(x)$ do not cross. The action of $\delta$ on $f(x) \otimes v_i$ is $\delta(f(x) \otimes v_i) = \delta \mathcal{I}^n_m(Q)f(x) \otimes v_i$, and is 0 if $\delta \mathcal{I}^n_m(Q)f(x) = 0$. Otherwise it is $\delta(f(x) \otimes v_i) = Q^{f(x)}(f(y) \otimes \sigma v_i)$, where $\sigma \in \operatorname{Sym}(m)$ is the unique permutation that straightens the vertical edges of $\delta \cdot f(x)$ and $y \in \mathcal{R}_{m, k}$ satisfies $f(y) = (\delta \cdot f(x))\sigma$.

**Proposition 2.1.** Take $\mu \vdash m \leq n$. Let $k = \frac{1}{2}(n - m)$ and $(v_1, \ldots, v_m)$ be a basis for the $\operatorname{Sym}(m)$-module $S^\mu$. Then $(f(x) \otimes v_i : x \in \mathcal{R}_{m, k}, 1 \leq i \leq f^m)$ is a basis for the $B_\lambda(Q)$-module $\mathcal{P}(n)$. In particular, $\dim(\mathcal{P}(n)) = (\frac{1}{2})^m(2k - 1)!!$.f^mu.

When $\lambda \vdash n$, the structure of $\mathcal{P}(n)$ is essentially that of the $\operatorname{Sym}(n)$-module $S^\lambda$. The space $\mathcal{I}^n_m(Q)$ is 1-dimensional and is generated by the identity 1. When $\sigma \in \operatorname{Sym}(n) \subseteq B_\lambda(Q)$, $\sigma(1 \otimes v) = 1 \otimes (\sigma v)$. Thus as a $\operatorname{Sym}(n)$-module it is isomorphic to $S^\lambda$. If $\delta \in B_\lambda(Q)$ has a horizontal edge, then $\delta \cdot 1 = 0$ in $\mathcal{I}^n_m(Q)$. Thus $\delta(1 \otimes v) = \delta \mathcal{I}^n_m(Q)1 \otimes v = 0$. Because the structure of $\mathcal{P}(n)$ does not depend on $Q$ in this case and is generically irreducible, it is always irreducible. Thus every nontrivial $B_\lambda(Q)$-homomorphism from $\mathcal{P}(n)$ into another $\mathcal{P}(n)$ is an embedding. Furthermore, given $\mathcal{P}(n)$, an embedding of $\mathcal{P}(n)$ in $\mathcal{P}(n)$ is a subspace $W \subseteq \mathcal{P}(n)$, such that $W \equiv S^\lambda$ as a $\operatorname{Sym}(n)$-module and $X_{i, j} = 0$ for all $w \in W$ and $1 \leq i < j \leq n$.

The description here gives a way to identify $B_\lambda(Q)$ as a cellular algebra in the same of [GL]. A basis for $\mathcal{I}^n_m(Q)$ can be given by all $(x_1, x_2, \sigma)$, where $\sigma$ is in $\operatorname{Sym}(m)$ and $x_i$ are $(m, k)$ partial 1-factors, where $m + 2k = n$. Here a diagram, $\delta_{x_1}$, with exactly $m$ vertical lines is represented by $\text{top}(\delta_{x_1}) = x_1$, $\text{bot}(\delta_{x_1}) = x_2$, and $\sigma$ is the permutation that maps the $i$th free point from the left to $\text{bot}(\delta_{x_1})$ to the $\sigma(i)$th free point in $\text{top}(\delta_{x_1})$. Let $\mathcal{I}^n_m$ be the complex vector space, with the basis being diagrams in $\mathcal{I}^n_m(Q)$ in which $\sigma$ is the identity. Now $\mathcal{I}^n_m(Q)$ can be identified with the vector space $\mathcal{J}^n_m \otimes \operatorname{Sym}(m)$, where $V$ is a space representing the regular representation of $\operatorname{Sym}(n)$. In particular, identify $(x_1, x_2, \sigma)$ with $\delta_{x_1, x_2} \otimes \sigma$, where $\delta_{x_1, x_2}$ is the diagram in $\mathcal{I}^n_m$ with top $x_1$, bottom $x_2$, and $\sigma$ is taken in $V$. Now pick a Kazhdan–Lusztig basis $w_{i_1, i_2}$ for $V$ as described in [GL], which is labeled by pairs of standard Young tableaux, $t_1, t_2$, of shape $\mu$ for all partitions $\mu$ of $m$. The cellular basis described in [GL] is $\{\delta_{x_1, x_2} \otimes w_{i_1, i_2}\}$. 

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**PROPOSITION 2.1.** Take $\mu \vdash m \leq n$. Let $k = \frac{1}{2}(n - m)$ and $(v_1, \ldots, v_m)$ be a basis for the $\operatorname{Sym}(m)$-module $S^\mu$. Then $(f(x) \otimes v_i : x \in \mathcal{R}_{m, k}, 1 \leq i \leq f^m)$ is a basis for the $B_\lambda(Q)$-module $\mathcal{P}(n)$. In particular, $\dim(\mathcal{P}(n)) = (\frac{1}{2})^m(2k - 1)!!f^\mu$.
for $m$ ranging over all $m$, with $0 \leq m \leq n$, for which $2(n-m)$. The important condition $C3$ in [GL, Definition 1.1] follows from the action of $\delta$. In particular, we have seen above that $\delta(f(x) \otimes w_i) = \delta \gamma^\mu_{\lambda}(Q) f(x) \otimes w_i$ is $0$ if $\delta \gamma^\mu_{\lambda}(Q) f(x) = 0$, and is $\delta(f(x) \otimes w_i) = Q^{\mu, \lambda}(f(x)) \otimes \sigma w_i$ where $\sigma \in \text{Sym}(m)$ is as above. The same result is true here if we replace $w_i$ with $w$. The crucial fact that makes this a cellular basis is that the action of $\delta$ does not depend on which bottom, $x$, has been chosen. This leads to the defining relations $C3$ in $\text{GL}$, as shown in Section 5. When the action gives $0$, it is modulo the span of diagrams with fewer vertical lines that are lower in the order $\Lambda$, as given in [GL, Section 5]. This set $\Lambda$ is the set of $\mu$ where $\mu \vdash m$ for which $2 \mid (n-m)$. When the action does not give $0$, the relation $C3$ in [GL] follows because the $w$ are a cellular basis for $\text{Sym}(m)$. Some of the results referred to in [HW1,2,3,4] are properties common to cellular algebras. Furthermore, the representations $S^\mu(n)$ given here are the same as the modules $W(\mu)$ for the elements of $\Lambda$.

3. NECESSARY CONDITIONS

In this section, we give necessary conditions for an embedding of $\mathcal{P}(n)$ into $\mathcal{P}(n)$ when $\lambda \vdash n$. In the case of the partition algebra, Martin studied these embeddings and was able to characterize in [M] exactly when they occur. This is the first step that allowed him to describe the irreducible modules in that case. Let $\langle \mathcal{P}(n), \mathcal{P}(n) \rangle$ denote the value of $\dim(\text{hom}_{B_n(Q)}(\mathcal{P}(n), \mathcal{P}(n)))$ for any $\lambda$ and $\mu$ with $2(n-|\lambda|)$ and $2(n-|\mu|)$. In Section 5, following Martin’s work on the partition algebra, we show that for $Q \neq 0$, $\langle \mathcal{P}(n + n_0), \mathcal{P}(n + n_0) \rangle \neq 0$ if and only if there is an embedding of $\mathcal{P}(n)$ into $\mathcal{P}(n)$ where $\lambda \vdash n$. Thus, it suffices to study only the case $\lambda \vdash n$.

Recall that the subalgebra of $B_n(Q)$ generated by 1-factors with $n$ vertical lines is isomorphic to $\mathbb{C} [\text{Sym}(n)]$. We refer to this subalgebra as $\text{Sym}(n)$. When $\lambda \vdash n$, $\mathcal{P}(n)$ is essentially the $\text{Sym}(n)$-module $S^\lambda$. Thus, if $\langle \mathcal{P}(n), \mathcal{P}(n) \rangle \neq 0$, then $\mathcal{P}(n)$, when viewed as a $\text{Sym}(n)$-module by restriction, must contain $S^\lambda$ as a constituent. A decomposition of $\mathcal{P}(n)$ as an $\text{Sym}(n)$-module is given in [HW3] (Theorem 4.1).

**Theorem 3.1.** Given $\mu \subseteq \lambda$. The multiplicity of $S^\lambda$ in the decomposition of $\mathcal{P}(n)$ as an $\text{Sym}(n)$-module is

$$\sum_{\eta \vdash 2k, \eta \text{ even}} c^\mu_{\lambda, \eta}.$$
where $2k = |\lambda/\mu|$ and $c_{\mu, \eta}^\lambda$ is a Littlewood–Richardson coefficient. If $\mu \nsubseteq \lambda$, the multiplicity is 0.

Thus a necessary condition for the embedding $\mathcal{R}_\mu(n) \hookrightarrow \mathcal{R}_\mu(n)$ is that $\mu \subseteq \lambda$. In the particular case $|\lambda/\mu| = 2$, this multiplicity is one if the two boxes in $\lambda/\mu$ are in different columns and zero otherwise.

We now give a stronger necessary condition for there to exist a $B_n(Q)$ embedding of $\mathcal{R}_\mu(n)$ into $\mathcal{R}_\mu(n)$ for $\lambda \vdash n$ due to the annihilation by all $X_i,j$'s. Let $\mu \vdash m$ where $n = m + 2k$. Suppose $\mathcal{R}_\mu(n) \hookrightarrow \mathcal{R}_\mu(n)$. Then there is a subspace $W$ of $\mathcal{R}_\mu(n)$, which as a $B_n(Q)$-module is $\mathcal{R}_\mu(n)$. We know that by just considering Sym$(n)$ $\subseteq B_n(Q)$, $S^\lambda$ must be the action, and so $\mu \subseteq \lambda$ by Theorem 3.1. A further condition is that $W$ is annihilated by $B_n^{-2}(Q)$.

To this end we form the element

$$T = \sum_{1 \leq i < j \leq n} X_{i,j}.$$ 

Recall $X_{i,j}$ has points $i$ and $j$ joined on the top and bottom, and the point $l$ in the top is joined to the point $l$ on the bottom for $l \neq i$ or $j$. Thus the element is in $B_n^{-2}(Q)$ and acts as the identity, except at $i$ and $j$. We first discuss how $T$ acts on $f(x)$ for $x$ an $(m, k)$ partial 1-factor in $R_{m,k}$.

**Lemma 3.2.** For a fixed $(m, k)$ partial 1-factor $x$, let $y = f(x)$ be the corresponding 1-factor in $I_{m}^{(n)}(Q)$. Then in $I_{m}^{(n)}(Q)$,

$$Ty = \left(k(Q - 1) + \sum_{1 \leq i < j \leq n} (i, j) - \sum_{1 \leq a < b \leq m} (a, b)\right)y,$$

where here the $(i, j)$ are transpositions in Sym$(n)$ acting on the left of $y$ and $(a, b)$ are transpositions in Sym$(m)$ acting on the right of $y$ and on the first $m$ positions only.

**Proof.** We consider how $X_{i,j}$ acts on $y$. There are four cases.

(a) Points $i$ and $j$ are joined in $x$. In this case, $X_{i,j}y = Qy$.

(b) Points $i$ and $j$ are both free points in $x$. Here $X_{i,j}y = 0$, as the resulting 1-factor has at least one more horizontal line and thus at most $m - 1$ vertical lines. Such terms are 0 in $I_{m}^{(n)}(Q)$.

(c) In $x$, one of the points $i$ and $j$ is free, say point $i$, and the other, point $j$, is joined to a third point $l$.
Then $X_{i,j}y$ is the same as $y$, except that point $i$ in the top is joined to point $j$ in the top and point $l$ in the top is joined to the point in the bottom that had been joined to point $i$. In other words, the line $(jl)$ becomes the line $(ji)$, and $l$ is joined to what $i$ had been joined to. That is, $X_{i,j}y = (i,l)y$.

(d) Neither point $i$ nor $j$ is free in $x$. Suppose points $i$ and $j$ are joined to points $l$ and $m$, respectively:

Here points $i$ and $j$ become joined, and points $l$, $m$ become joined. This could have been effected by interchanging $j$ and $l$ or $i$ and $m$, i.e., $X_{i,j}y = (j,l)y = (i,m)y$. Notice $X_{l,m}y$ gives the same term.

We now consider the action of $T$ on $y$. First partition $((i,j): 1 \leq i < j \leq n)$ into four sets $A$, $B$, $C$, and $D$, based on which of the four cases apply for this particular $x$. Now

$$Ty = \sum_{(i,j) \in A} X_{i,j}y + \sum_{(i,j) \in B} X_{i,j}y + \sum_{(i,j) \in C} X_{i,j}y + \sum_{(i,j) \in D} X_{i,j}y.$$  

Since there are $k$ edges in $x$, $\sum_{(i,j) \in A} X_{i,j}y = kQy$. For each $(i,j) \in B$, $X_{i,j}y = 0$, and thus the second term makes no contribution.

For any $(i,j) \in C$ or $D$, the effect of $X_{i,j}$ on $y$ is to multiply by a transposition $\tau_{i,j}$ on the left. For $(i,j)$ in $D$, recall that $X_{i,j}y = X_{l,m}y = (j,l)y = (i,m)y$, where points $i$ and $j$ are joined to points $l$ and $m$, respectively. In this case, we leave the exact association between the $(i,j)$ in $D$ and transpositions $\tau_{i,j}$ vague. All that matters is that setwise, both are accounted for. Thus as a first approximation to the sum over $C$ and $D$, we consider

$$\sum_{(i,j) \in C} X_{i,j}y + \sum_{(i,j) \in D} X_{i,j}y = \sum_{1 \leq i < j \leq n} (i,j)y.$$  

However, the right-hand side (RHS) contains terms that do not arise from an $(i,j) \in C \cup D$. There are two sources of overcounting. First, if points $i$
and \( j \) in the top of \( y \) are joined, then \((i, j)y = y\) is counted in the RHS but not in the LHS. To counteract this, we subtract \( ky \) from the RHS, since there are \( k \) edges in the top of \( y \). The second source of overcounting arises if both points \( i \) and \( j \) in the top of \( y \) are joined to points in the bottom. Say point \( i \) in the top is joined to point \( a \) and \( j \) to \( b \). Notice that \((i, j)y = y(a, b)\). To compensate for this overcounting, we subtract

\[
\sum_{1 \leq a < b \leq m} y(a, b)
\]

from the RHS. This gives

\[
\sum_{(i, j) \in C} X_{i,j}y + \sum_{(i, j) \in D} X_{i,j}y = \sum_{1 \leq i < j \leq n} (i, j)y - ky - \sum_{1 \leq a < b \leq m} y(a, b).
\]

Adding everything together, we get the desired result. \( \blacksquare \)

**Theorem 3.3.** Take \( \lambda \vdash n \) and \( \mu \vdash m < n \). If \( \langle \mathcal{R}(n), \mathcal{S}(n) \rangle \neq 0 \), then

\[
k(Q - 1) + \sum_{p \in [\lambda/\mu]} c(p) = 0,
\]

where \( n = 2k + m \).

**Proof.** Now suppose \( \langle \mathcal{R}(n), \mathcal{S}(n) \rangle \neq 0 \). Here, we view \( \mathcal{S}(n) \) as a \( \text{Sym}(n) \times \text{Sym}(m) \)-module by

\[
(\pi, \pi') : (\delta \otimes \upsilon) = \pi : (\delta \otimes (\pi' \upsilon)),
\]

where \( \pi \in \text{Sym}(n) \), \( \pi' \in \text{Sym}(m) \), \( \delta \in \mathcal{L}_n(Q) \), and \( \upsilon \in S^\mu \). The image of \( \mathcal{S}(n) \) is in the \( S^\lambda \otimes S^\mu \) component of \( \mathcal{S}_n(n) \). The sums \( \sum_{i,j}(i, j) \) and \( \sum_{(a,b)}(a, b) \) each act as a scalar, as each is in the center of the respective group algebras. The values are the corresponding \( \omega_\lambda(1,2) \) and \( \omega_\mu(1,2) \). As is well known (See Chapter 1 of [D], for example),

\[
\omega_\lambda(1,2) = \sum_{p \in [\lambda]} c(p)
\]

and

\[
\omega_\mu(1,2) = \sum_{p \in [\mu]} c(p),
\]

where here \( c(p) \) is the content of a box in \( \lambda \) or \( \mu \).
If \( \nu \neq 0 \) is in the \( S^\lambda \otimes S^\mu \) constituent of \( \mathcal{R}_\mu(n) \),
\[
T\nu = \left( k(Q - 1) + \sum_{p \in [\lambda]} c(p) - \sum_{p \in [\mu]} c(p) \right) \nu.
\]
Since \( \nu \neq 0 \), for this to be 0 we need
\[
k(Q - 1) + \sum_{p \in [\lambda/\mu]} c(p) = 0.
\]

If \( n = m + 2k \) with \( k \geq 0 \), \( \lambda \vdash n \), \( \mu \vdash m \), and \( \mu \subset \lambda \), we say the pair \((\lambda, \mu)\) satisfies \((\star)\) if \( k(Q - 1) + \sum_{p \in [\lambda/\mu]} c(p) = 0 \), where as usual here \( c(p) \) is the content of \( p \). In the case \( k = 1 \), for \((\lambda, \mu)\) to satisfy \((\star)\), we also require that \( \lambda/\mu \) has its two boxes in different columns. In general, satisfying \((\star)\) does not imply that there is an embedding, but in the case \( k = 1 \) it does.

Remark. Theorem 3.3 also follows from [N, p. 671, formula before (2.13)] when \( Q \) is a positive integer. They show that \( \mathfrak{T} - \sum_{1 \leq i \leq m} (i, j) \) is central and acts as the appropriate scalar on each generic irreducible, and the result follows similarly. Our proof is direct and does not use the isomorphism with \( B_n(Q) \) and the centralizer algebras, or any facts about their representations. It also works when \( Q \) is not a positive integer.

**Theorem 3.4.** Let \( \lambda \vdash n \). If \( |\lambda/\mu| = 2 \) and \((\lambda, \mu)\) satisfies \((\star)\), then there exists an embedding of \( \mathcal{R}(n) \) into \( \mathcal{R}_\mu(n) \).

**Proof.** This follows for the case in which \( \lambda/\mu \) is not in a row from results in [HW1] and [HW3] Theorem 4.8. However, an entirely new proof for \( |\lambda/\mu| = 2 \) comes by using Theorem 3.3. The main arguments in [HW3] applied to this case involve computing the terms of a degree 1 polynomial. The fact it is degree 1 means there is a root, and this means there is an embedding. Now Theorem 3.4 specifies the value of \( Q \) by \( Q - 1 + c(p) + c(q) = 0 \), which gives the result.

As described in [HW1], the module \( \mathcal{R}(n) \) has a unique maximal submodule that is annihilated by all elements in \( B_n(Q) \), called the radical of \( \mathcal{R}(n) \). When \( m = n - 2 \), the radical is annihilated by all \( X_{i,j} \), and so any \( \text{Sym}(n) \) irreducible subspace of the radical is one of the irreducible \( \mathcal{R}(n) \) for some \( \lambda \vdash n \). It is for this reason that when there is a radical for \( \mathcal{R}_\mu(n) \) with \( \mu \vdash n - 2 \) that nontrivial maps from \( \mathcal{R}(n) \) occur. This is not the case when \( |\mu| < n - 2 \), except in special circumstances.

The computations for the case of \( \mu \vdash n - 2 \) in [HW3] use the matrix \( Z(x) = Z_{n-2,1}(x) \), which is a square matrix with basis the labeled \((n-2,1)\)
partial 1-factors. Here if \( m + 2k = n \), a labeled \((m, k)\) partial 1-factor is an \((m, k)\) partial 1-factor for which the free points have been labeled with the integers \( 1, 2, \ldots, m \). Here the \( n - 1 \) free points are labeled with \( 1, 2, \ldots, n - 2 \). The matrix \( Z(x) \) has \( x \) on the diagonals and constants off. Let the vector space with these as basis be \( V = V_{n-2} \). As described in [HW1,3], there is a natural action of \( \text{Sym}(n) \times \text{Sym}(n - 2) \) on \( V \), where \( \text{Sym}(n) \) permutes the vertices and \( \text{Sym}(n - 2) \) permutes the labels. The matrix \( Z(x) \) commutes with this action.

To identify the action of \( X_{i,j} \), let \( \tilde{\delta} \) be an \((n - 2, 1)\) partial 1-factor and let \( \delta \) be a labeled \((n - 2, 1)\) partial 1-factor for which the free points of \( \delta \) have been labeled with the integers \( 1, 2, \ldots, n - 2 \). Suppose the order when read from left to right is \( \sigma(1), \sigma(2), \ldots, \sigma(n - 2) \). Here \( \sigma \) is a permutation in \( \text{Sym}(n - 2) \). Identify \( \tilde{\delta} \) with \( \tilde{f}(\delta) \sigma \), where here \( \sigma \) acts from the right and on the first \( n - 2 \) positions. Recall \( f(\delta) \) has a line from \( n - 1 \) to \( n \) on the bottom and the vertical lines from the top to bottom do not cross. Then consider \( X_{i,j} \cdot f(\tilde{\delta}) \sigma \). Recall this is 0 if \( i \) and \( j \) are both free; otherwise it is the usual \( B_n(Q) \) multiplication. In this case it is of the form \( f(\tilde{\delta}) \sigma \sigma_n \) or \( Q \) times it for an appropriate \( \delta_n \) and \( \sigma_n \). This, in turn, can be identified with \( \delta_n \) for a unique \((n - 2, 1)\) labeled partial 1-factor. This gives the action of \( X_{i,j} \) by identifying \( X_{i,j} \tilde{\delta} \) with either \( \delta_n \) or \( Q \delta_n \), being the second case. As explained in [HW3], if \( \tilde{\delta}_1 \) corresponding to \( f(\tilde{\delta}_1) \sigma_1 \) is another \((n - 2, 1)\) partial 1-factor, the \((\tilde{\delta}_1, \tilde{\delta}_1)\) entry of \( Z(x) \) is obtained by evaluating \( w = (f(\tilde{\delta}_1) \sigma_1 \sigma_2) f(\tilde{\delta}) \sigma \). Here \((\tilde{\delta}_1, \sigma_1)\) turned upside down. The map that turns the basis elements upside down generates an antismorphism of \( B_n(Q) \). The entry is 0 unless \( w = X_{n-1,n} \) or \( Q X_{n-1,n} \). It is 1 in the first case and \( x \) in the second. The second case occurs only on the diagonal.

With the definitions as above, \( Z(Q) \) for any value of \( Q \) has the important property that \( X_{i,j} \tilde{\delta} = Z(Q) \tilde{\delta} \). Now for a given value \( Q \) of \( x \), elements in the null space of \( Z(Q) \) give rise under this correspondence to elements in \( I^{n-2}_x(Q) \), which are annihilated by all \( X_{i,j} \). Let \( U \) be a vector space on which \( \text{Sym}(n - 2) \) acts as the regular representation. Now replace \( \sigma \) by a vector space transformation representing the action of \( \sigma \) on \( U \). The space \( V \) can be identified with the space spanned by all \( \delta \otimes_{\text{Sym}(n-2)} U \), where the \( \delta \) are \((n - 2, 1)\) partial 1-factors. The matrix \( Z(x) \) commutes with the \( \text{Sym}(n - 2) \) action for any value of \( x \). Breaking \( V \) into irreducible \( \text{Sym}(n - 2) \) invariant subspaces gives \( \mathcal{P}_x(n) \) with multiplicity \( f^\mu \). Each is isomorphic to the action on the span of all \( \delta \otimes S^\mu \), where \( \delta \) are the \((n - 2, 1)\) partial 1-factors and \( S^\mu \) is a representation space for \( S^\mu \). Within each \( \mathcal{P}_x(n) \) is a unique copy of \( S^\lambda \). As \( Z(x) \) commutes with this action, \( Z(x) \) must be a scalar matrix when acting on this subspace, with the scalar being a polynomial. As \( Z(x) \) has \( x \) on the diagonals, the determi-
nant of the action on this subspace is the characteristic polynomial, and so the polynomial is linear of degree 1. The intersection of the null space of \( Z \) with this submodule is in the radical of \( \mathcal{R}(n) \). But then when \( Q \) is the root of degree 1 polynomial, the subspace is in the radical. This means the \( S^1 \) subspace is in the radical for some value of \( Q \). But as above, this is an embedding, and the value of \( Q \) is given by Theorem 3.4.

4. RESTRICTION IN THE BRAUER ALGEBRA

In this section, we give a complete description of the restriction of the generic irreducibles of \( B_\lambda(Q) \). The standard embedding of \( B_\lambda(Q) \) in \( B_\nu \) is given as follows. For any 1-factor \( \delta \in B_\mu \), append one point in both the top and bottom rows of \( \delta \) and draw a vertical line between these points. Call this \( \delta \). Extend this map linearly to get an embedding of \( B_\lambda(Q) \) in \( B_\mu \). Given a \( B_\lambda(Q) \)-module \( M \), the restriction of \( M \) to \( B_\mu \) via this embedding is denoted \( M \downarrow \). In the semisimple case, it is well known that

\[
\mathcal{R}(n) \downarrow = \bigoplus_{\mu \in \lambda} \mathcal{R}(n-1) \oplus \bigoplus_{\nu \triangleright \lambda} \mathcal{R}(n-1).
\]

However, when \( B_\mu \) is not semisimple, \( \mathcal{R}(n) \downarrow \) need not decompose into a direct sum of irreducibles. However, we show there is a two-step filtration of \( \mathcal{R}(n) \downarrow \) in which the first terms in the sum (2) appear as constituents of a subspace, and the terms in the second sum appear in the quotient.

**Theorem 4.1.** Fix \( n = m + 2k \) and take \( \lambda \vdash m \) and assume \( m < n \). Let \( V = \mathcal{R}(n) \downarrow \). As a \( B_\lambda \)-module, \( V \) contains a submodule \( W \equiv \bigoplus_{\mu \in \lambda} \mathcal{R}(n-1) \). Furthermore, \( V/W \equiv \bigoplus_{\nu \triangleright \lambda} \mathcal{R}(n-1) \).

**Proof.** Let \( f = f^\lambda \) and \( \{ v_1, \ldots, v_j \} \) be a basis for \( S^\lambda \). Partition \( \mathcal{R}_{m,k} \) into two sets \( A \) and \( B \), where \( A \) is the set of \((m,k)\) partial 1-factors in which vertex \( n \) is free (i.e., has degree 0), and \( B \) is those in which vertex \( n \) is not free (i.e., has degree 1).

Define \( W \) to be the space generated by \( \{ f(a) \otimes v_i : a \in A, 1 \leq i \leq f \} \). We show first that \( W \) is invariant under any permutation in \( \text{Sym}(n-1) \) and all \( X_{i,j} \) for \( 1 \leq i < j \leq n-1 \). As these generate \( B_\mu \), this shows \( W \) is invariant. Suppose \( \sigma \) is a permutation in \( \text{Sym}(n-1) \). When embedded in \( B_\mu \), these permutations have a vertical line from point \( n \) in the top to point \( n \) in the bottom. Thus, for any \( a \in A \), \( \sigma(f(a)) = f(a')\sigma' \), where \( a' \) is another \((m,k)\) partial 1 factor and \( \sigma' \) acts on the right and rearranges the vertical lines so they do not cross. Notice that in \( f(a) \) the
line from point \( n \) on the top to point \( m \) in the bottom has not been affected by the action of \( \sigma \), so \( \sigma' \) fixed \( m \). Furthermore, \( \sigma' \) is in \( A \). This means that \( \sigma(f(a) \otimes v_i) = f(a') \sigma' \otimes v_i \), which is \( f(a') \otimes \sigma' v_i \). Therefore, \( W \) is preserved by \( \sigma \).

As in the proof of Lemma 3.2, there are four cases for the value of \( X_{i,j} \cdot f(a) \), depending on whether or not the vertices \( i \) and \( j \) are free or are on lines in \( a \). The result is either 0 if \( i \) and \( j \) are free, \( Qf(a) \) when \( i, j \) is a line in \( a \), or \( \tau f(a) \) for a transposition \( \tau \), which is \( (i,m) \) or \( (j,m) \). In each case, \( W \) is preserved. This shows that \( W \) is a \( B_{n-1}(Q) \)-module.

We now identify \( W \) as a \( B_{n-1}(Q) \)-module by finding an isomorphism between \( W \) and \( I_{n-1}^m(Q) \otimes_{\text{Sym}(n-1)} S^k \). In particular, for any \( a \in A \), let \( \varphi(a) \) be a point \( n \) removed. Notice this is a bijection from \( A \) to the set of \((m-1,k)\) partial 1-factors. Now let \( \varphi^m_1(a) = f(\varphi(a)) \). For example,

\[
\varphi^m_1(a) = \begin{pmatrix} 
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{pmatrix}
\]

Now let \( \varphi \) be the map from \( W \) to \( I_{n-1}^m(Q) \otimes_{\text{Sym}(n-1)} S^k \) given by \( \varphi(f(a) \otimes v_i) = \varphi^m_1(a) \otimes v_i \).

This map is clearly an isomorphism of vector spaces as \( \varphi^m_1 \) is a bijection. To show that it is a \( B_{n-1}(Q) \)-module map, we show that it commutes with permutations in \( \text{Sym}(n-1) \) and with all \( X_{i,j} \) for \( 1 \leq i < j \leq n-1 \). We saw above how \( X_{i,j} \) acts on \( f(a) \). The same description applies to the action of \( X_{i,j} \) on \( \varphi^m_1(a) \). The four cases are identical, and the actions are identical. In particular, if one is 0 or is multiplied by \( Q \), so is the other. In the other cases, they are multiplied by the same transposition. Now we must check that \( X_{i,j} (f(\varphi(a) \otimes v_i) = \varphi(X_{i,j},(f(a) \otimes v_i)) \). The actions of \( X_{i,j} \) are determined by the actions on \( f(a) \) and on \( f(\varphi(a)) \). This is immediate for the two cases being multiplied by 0 or \( Q \), and once we show permutations commute with \( \varphi \), we will have shown that \( X_{i,j} \) commutes.

Suppose then that \( \sigma \in \text{Sym}(n-1) \). We have seen that \( \sigma f(a) = f(a') \sigma' \). Now \( \varphi(\sigma f(a) \otimes v_i) = \varphi(f(a') \otimes \sigma' v_i) = \varphi^m_1(a') \otimes \sigma' v_i \). On the other hand, \( \sigma \varphi(a) = \varphi(a') \), as \( \sigma \) does not move \( n \). This means that \( \sigma f(\varphi(a)) = f(\varphi(a')) \sigma' \), which is just \( \sigma \varphi^m_1(a) = \varphi^m_1(a') \sigma' \). Now \( \sigma \varphi(f(a) \otimes v_i) = \sigma \varphi^m_1(a) \otimes v_i = \varphi^m_1(a') \sigma' \otimes v_i \). But this is \( \varphi^m_1(a') \otimes \sigma' v_i \). This shows that the actions commute.
To treat the action on $V/W$, we introduce a map similar to $\varphi_1$. Let $\varphi_2$ be the map from $B$ to $\mathcal{R}_{m+1,k-1}$, where $\varphi_2(b)$ is $b$ with vertex $n$ and the edge incident to it removed. Recall that $b$ is an $(m,k)$ partial 1 factor with vertex $n$ joined to another vertex. One of the vertices has been freed up, and a line taken away, so the resulting 1 factor is in $\mathcal{R}_{m+1,k-1}$. Given $b' \in \mathcal{R}_{m+1,k-1}$, the elements of $\{b \in B : \varphi_2(b) = b'\}$ are obtained by appending an additional vertex to $b'$, called vertex $n$, and adding an edge from vertex $n$ to any of the $m+1$ free vertices of $b'$. Hence, $\varphi_2$ is an $(m+1)$-to-1 map.

Using $\varphi_2$, we define a map $\varphi^\circ_2$ from $B$ to $I_{m-1}^{m+1}(Q)$ by letting the top of $\varphi^\circ_2(b)$ be $\varphi_2(b)$, draw an edge from the point in the top that was connected to vertex $n$ in $b$ to point $m+1$ in the bottom, and draw in the remaining vertical lines so that they do not cross. For example,

\[
\begin{array}{c}
\varphi^\circ_2 \\
\end{array}
\]

More formally, let $\sigma_j = (j,m+1,m,\ldots,j+1)$ be the permutation in $\text{Sym}(m+1)$ that maps $j$ to $m+1$ and then shifts the integers between $j+1$ and $m+1$ down by one. Suppose in $b$, vertex $n$ is joined to a vertex $l$, and suppose there are $j-1$ free points to the left of $l$. Then $\varphi^\circ_2(b) = f(\varphi_2(b))\sigma_j$. Also from the description above, for any $b' \in \mathcal{R}_{m+1,k-1}$ and $1 \leq j \leq m+1$, there is a $b \in B$ such that $\varphi^\circ_2(b) = f(b')\sigma_j$.

To prove $V/W$ has the desired structure, we need some facts about induced representations in the symmetric group. For $1 \leq i \leq m$, let $\tau_i = (i,m+1)$ and let $\tau_{m+1} = 1$. These form a set of coset representatives for $\text{Sym}(m+1)/\text{Sym}(m)$. A basis for $S^\Lambda \uparrow$ is indexed by $\{v_1,\ldots,v_f\} \times \{1,\ldots,m+1\}$. Take $\sigma \in \text{Sym}(m+1)$. The action of $\sigma$ on $S^\Lambda \uparrow$ is given by

\[
\sigma \cdot (v_i,j) = \left( (\tau_i \sigma \tau_j)^{-1} v_i, k \right),
\]

where $k$ is the unique value such that $\tau_i \sigma \tau_j \in \text{Sym}(m)$. We claim that

\[
\varphi : V/W \rightarrow I_{m-1}^{m+1}(Q) \otimes_{\text{Sym}(m+1)} S^\Lambda \uparrow
\]

\[
f(b) \otimes v \mapsto \varphi_2^\circ(b) \otimes (v, m+1)
\]

is a $B_{m-1}(Q)$-isomorphism.

We first show that it is a bijection. Since $\varphi_2$ is an $(m+1)$-to-1 map and $	ext{dim}(S^\Lambda \uparrow) = (m+1)\text{dim}(S^\Lambda)$, $V/W$ and $I_{m-1}^{m+1}(Q) \otimes S^\Lambda \uparrow$ have the same dimension. Hence, it suffices to show that $\varphi$ is onto.
A generating set for \( I_{n+1}^1(Q) \otimes S^4 \) is \((f(b') \otimes (v,j)): b' \in \mathcal{P}_{m+1,k-1}, v \in S^4, 1 \leq j \leq m + 1\) Choose \( b', v, \) and \( j \). Notice that \( \sigma_j^{-1} \tau_j^{-1} \tau_j = m + 1 \) or, equivalently, that \( \tau_j^{-1} \tau_j \in \text{Sym}(m) \). So, \( \sigma_j^{-1} \tau_j^{-1} (v,j) = (v', m + 1) \) for some \( v' \in S^4 \). Then \( f(b') \otimes (v,j) = f(b') \otimes s_j \cdot (v', m + 1) \) if \( b \in B \) be such that \( \varphi(b) = f(b')s_j \). Thus \( \varphi(b \otimes v') = f(b') \otimes (v,j) \), which shows that \( \varphi \) is onto.

Now we show that \( \varphi \) is a \( B_\mu(Q) \)-homomorphism. Pick \( b \in B \) and \( v \in S^4 \). Arguing as above for \( W \), notice that for all \( \sigma \in \text{Sym}(n - 1) \), \( \sigma \varphi(f(b) \otimes v) = \sigma \varphi(f(b) \otimes v) \). So, \( \varphi \) is a \( \text{Sym}(n - 1) \)-isomorphism. Thus, we only need to show that the action of \( X_{i,j} \) is the same on both sides for all \( 1 \leq i < j \leq n - 1 \), as these generate \( B_\mu(Q) \). Let \( r \) be the vertex that is connected to vertex \( n \) in \( b \). In \( \varphi(b) \), point \( r \) in the top is connected to point \( m + 1 \) in the bottom. If neither \( i \) nor \( j \) is equal to \( r \), then it is clear that \( X_{i,j} \varphi(f(b) \otimes v) = \varphi X_{i,j}(f(b) \otimes v) \). Suppose \( i = r \). Then there are two cases, based on whether vertex \( j \) in \( b \) is free or connected to some other vertex.

Suppose first that vertex \( j \) is free in \( b \). Then in \( X_{i,j}f(b) \) point \( n \) in the top is connected to a point in the bottom. Thus, \( X_{i,j}(f(b) \otimes v) \in W \) and is equal to zero in \( V/W \). On the other side, both points \( i \) and \( j \) in the top of \( \varphi(b) \) are connected to points in the bottom. So, \( X_{i,j}\varphi(b) = 0 \), because the resulting multiplication in \( I_{n+1}^1(Q) \) has fewer vertical lines. Thus, \( X_{i,j}\varphi(f(b) \otimes v) = 0 \), as desired. It may help to consult the picture above, form \( f(b) \), and multiply by \( X_{1,3} \). This shows also how the quotient plays a role.

In the case in which vertex \( j \) is connected to, say, vertex \( k \) in \( b \), \( X_{i,j}(f(b) \otimes (r,k)f(b)) \). So, \( \varphi X_{i,j}(f(b) \otimes (r,k)) = \varphi(f(b) \otimes v) = X_{i,j}\varphi(f(b) \otimes v) \).}

**Example.** We work an example that illustrates Theorem 4.1 and the situation for the next theorem. Let \( n = 4 \) and \( \mu = 2 \). Let \( V \) be \( \mathcal{S}_\mu(4) = \{t_{1,1}, t_{1,2}, t_{1,3}, t_{1,4} \} \). The (2,1)-partial 1 factors are the six elements in \( \{t_{1,1}, \ldots, t_{1,4} \} \), where \( t_{1,j} \) has a line from vertex \( 1 \) to \( 2 \) and the remaining two vertices are free. A basis for \( \mathcal{S}_\mu(4) \) is the set \( f(t_{1,j}) \otimes v \), where \( v \) spans the \( \text{Sym}(2) \) module \( S^2 \), which is the trivial module. Let \( w_1 = f(t_{1,2}) \otimes v \), \( w_2 = f(t_{1,3}) \otimes v \), and \( w_3 = f(t_{1,4}) \otimes v \). Also let \( v_1 = f(t_{1,2}) \otimes v \), \( v_2 = f(t_{1,3}) \otimes v \), and \( v_3 = f(t_{1,4}) \otimes v \).

Now \( W \) is the span of \( \{w_1, w_2, w_3\} \). Notice that \( \text{Sym}(3) \) acts naturally on \( w_1, w_2, \) and \( w_3 \) by permuting subscripts. It does the same for \( v_1, v_2, \) and \( v_3 \). Also note \( X_{1,2}w_3 = Qw_3 \) and \( X_{1,2}w_2 = X_{1,2}w_1 = w_3 \). Similar relations hold for \( X_{1,3} \) and \( X_{2,3} \). The identification with the action of \( B_\mu(Q) \) on \( \mathcal{S}_\mu(3) \) is now clear, as Theorem 4.1 predicts. The quotient by \( W \) is the direct sum of \( S^3 \) and \( S^2 \). The vector \( y = v_1 + v_2 + v_3 \) is fixed by \( \text{Sym}(3) \). Notice \( X_{1,2}v_1 = Qv_3, X_{1,2}v_2 = w_3, \) and \( X_{1,2}v_3 = 0 \). Similar relations hold
for $X_{1,3}$ and $X_{2,3}$. These show that mod $W$, the span of $(y)$, is $\mathcal{S}_3(3)$. The span of $(v_1 - v_3, v_2 - v_3)$ mod $W$ gives $\mathcal{S}_{2,3}(3)$. Notice that both partitions that cover $\mu$ occur, as Theorem 4.1 says they should.

This also shows that $y + W$ mod $W$ does not split directly. However, it is possible that for some different choice of coset, representative $y + w$ for some $w \in W$ splits. As $\text{Sym}(3)$ would fix $y + w$, $\text{Sym}(3)$ would have to fix $w$, and so $w = \gamma(w_1 + w_2 + w_3)$. Now notice $X_{1,2}(y + \gamma(w_1 + w_2 + w_3)) = (2 + \gamma(Q + 2))w_3$. For this to be 0, we need $2 + \gamma(Q + 2) = 0$. Exactly the same equations must be satisfied if $X_{1,3}$ or $X_{2,3}$ is used where $w_1$ or $w_2$ occurs instead of $w_3$. This can be solved if and only if $Q \neq -2$. But if $Q - 2 = 0$, the subspace spanned by $\{w_1 + w_2 + w_3\}$ is an invariant module isomorphic to $\mathcal{S}_3(3)$, which is an embedding of $\mathcal{S}_3(3)$ into $\mathcal{S}_2(3)$.

Notice the pair $(3, 1)$ satisfies $(\star)$ exactly when $Q = -2$. This shows that $\mathcal{S}_2(3)$ is a submodule of $\mathcal{S}_2(4)$ when $Q = -2$ or not. It occurs in different ways in the sense that if $Q \neq -2$, it splits from $W$. If $Q = -2$, it does not split from $W$, but occurs as a submodule of $\mathcal{S}_3(3)$, which is itself a submodule. This phenomenon is quite general, which we demonstrate with the following theorem.

We now consider the case in which $\lambda \vdash n - 2$ and examine $\mathcal{S}(n)$ more carefully. Using the notation of Theorem 4.1, we know $V/W \cong \bigoplus_{\nu \triangleright \lambda} \mathcal{S}_\nu(n - 1)$. We may pick coset representatives for the cosets of $V/W$ in the span of $f(a) \otimes v_i$, where here $a \in A$ and $1 \leq i \leq f^\lambda$ by Theorem 4.1. Choose a specific $\nu \triangleright \lambda$ and find the subspace of $V/W$ isomorphic to $\mathcal{S}_\nu(n - 1)$. Set $f = f^\lambda$. Now pick coset representatives $v_1, v_2, \ldots, v_f$ in the span of $f(a) \otimes v_i$ for which $v_1 + W, v_2 + W, \ldots, v_f + W$ is a basis for this subspace. This is how we chose them in the example above. There are now two possibilities:

1. For the given $\{v_1, \ldots, v_f\}$ there exist element $\{w_1, \ldots, w_f\}$ in $W$ such that $\{v_1 + w_1, \ldots, v_f + w_f\}$ is a basis for $\mathcal{S}(n - 1)$ in $V$. This is a splitting of $\mathcal{S}_\nu(n - 1)$.

2. The module $\mathcal{S}_\nu(n - 1)$ appears as a submodule for some $\mathcal{S}_\mu(n - 1)$ with $\mu \triangleleft \lambda$. Since $|\nu/\mu| = 2$, this occurs if and only if $(\nu, \mu)$ satisfy condition $(\star)$. In particular, there is at most one value of $Q$ for which this happens.

We show that exactly one of these conditions holds for any given $\nu \triangleright \lambda$, $Q$, and choice of $v_i$ as indicated. Notice, incidentally, that for $\lambda \vdash n$, $\mathcal{S}(n)$ is just the usual restriction for the symmetric group, and in the Brauer algebra, $n - |A|$ must be divisible by 2, so this is the highest nontrivial case.

**Theorem 4.2.** Given $\lambda \vdash n - 2$ and $\nu \triangleright \lambda$, then exactly one of the conditions (1) or (2) above holds for $v_i$ as described above.
Proof. Let $t_{i,j}$ be the $(n-2,1)$ partial 1-factor with an edge from vertex $i$ to vertex $j$ and $T_{i,j} = f(t_{i,j}) \in I_n^2 (Q)$. A gain set $f = f^\lambda$. A basis for $H(n)$ is $\{T_{i,j} \otimes s_k\}$, where $\{s_1, \ldots, s_k\}$ is a basis for $S^n$. Denote $H(n) \downarrow V$ by $V$. From Theorem 4.1 the $B_{n-1}(Q)$-module $W$ is generated by $\{T_{i,j} \otimes s_k : 1 \leq i, j \leq n-1\}$ and is isomorphic to $\bigoplus_{\mu \in \lambda} H_\mu(n-1)$. The quotient space $V/W$ has coset representatives $\{T_{i,n} \otimes s_k\}$ and is isomorphic to $\bigoplus_{\mu \in \lambda} H_\mu(n-1)$ as a $B_{n-1}(Q)$-module.

Choose coset representative $(v_1, \ldots, v_f)$ in the span of $\{T_{i,n} \otimes s_k\}$ such that $(v_1 + W, v_2 + W, \ldots, v_f + W)/W \equiv H_\mu(n-1)$. To satisfy (1), we want to find $(w_1, \ldots, w_f)$ in $W$ such that $\{v_1 + w_1, \ldots, v_f + w_f\}$ is a basis for $H_\mu(n-1)$ in $V$. First, for $\sigma \in \text{Sym}(n-1)$, $\sigma \cdot v_i = \Sigma a\sigma_{ij} v_j$, where $[a\sigma_{ij}]$ is the matrix representation for $\sigma$ in $S^n$. Thus, we want $\sigma \cdot (v_i + w_i) = \Sigma a\sigma_{ij} (v_j + w_j)$, which implies $\sigma \cdot w_i = \Sigma a\sigma_{ij} w_j$. This means that as a $\text{Sym}(n-1)$-module, $\langle w_1, \ldots, w_f \rangle \equiv S^n$. From comments after Theorem 3.1, there is such a unique subspace in $H_\mu(n-1)$ for each $\mu \in \lambda$ such that the two boxes in $\mu/\mu$ are in different columns. When the two boxes of $\nu/\mu$ are in the same column, Theorem 3.1 says that even as $\text{Sym}(n-1)$-modules, $H_\mu(n-1)$ cannot be embedded in $H_\nu(n-1)$. We discuss the consequences of this later. Write $w_i = \Sigma_{\mu \in \lambda} (w_i)_\mu$, where $(w_i)_\mu$ is the $\mu$-isotropic component of $w_i$ in $W$. In general, for any $w \in W$, $(w)_\mu$ stands for its $\mu$-isotropic component. By Schur’s Lemma, the $(w_i)_\mu$ are unique up to a scalar multiple. Thus for any $\mu$ with $\nu/\mu$ in different columns, there is a unique choice of scalar $\alpha_\mu$ so that $\alpha_\mu w_1, \alpha_\mu w_2, \ldots, \alpha_\mu w_f$ gives the splitting. If $\nu/\mu$ are in the same column, there is no choice. Up to this point, nothing in the proof has depended on $Q$.

We have shown that $H_\mu(n-1)$ splits as a $\text{Sym}(n-1)$-module for any choice of $\alpha_\mu$. To show that it splits as a $B_{n-1}(Q)$-module, we need to show that there is some choice of scalars $\alpha_\mu$ such that

$$X_{r,s}(v_r + \Sigma_{\mu \in \lambda} \alpha_\mu (w_i)_\mu) = 0$$

for $1 \leq r, s \leq n-1$, and $1 \leq i \leq f$. We examine this component by component. That is, we show that

$$(X_{r,s}(v_r))_\mu + \alpha_\mu X_{r,s}(w_i)_\mu = 0$$

for all $\mu$. The choice of $\alpha_\mu$‘s depends on $Q$. In certain cases there are no solutions for the $\alpha_\mu$‘s (i.e., condition (1) does not hold). However, this happens exactly when $H_\mu(n-1)$ is a submodule of one of the $H_\mu(n-1)$ (i.e., condition (2) holds).

Since $(v_1, \ldots, v_f)$ is a basis for $H_\mu(n-1)$ in $V/W$ and $\nu \vdash n-1$, we have $X_{r,s} v_r \in W$. Write this as $X_{r,s} v_r = \Sigma_{\mu \in \lambda} (X_{r,s} v_r)_\mu$. Because of
our choice of \( v_i \) spanned by linear combinations of \( T_{r,n} \otimes s_k \) and the fact
that \( r, s < n \), notice \( X_{r,s}(T_{s,n} \otimes s_k) = T_{r,s} \otimes s* \) and \( X_{r,s}(T_{i,n} \otimes s_k) = 0 \)
if \( i \) is not \( r \) or \( s \). This means \((X_{r,s}, v_i)_\mu = T_{r,s} \otimes s_* \) for some \( s_* \) in the \( S^\mu \)
component of \( S^\lambda / \). 

Write \((w_i)_\mu = \sum \beta_{k,l,m}(T_{k,l} \otimes s_m)\). Since \( W \) as a \( B_{n-1}(Q) \)-module breaks
up as a direct sum of \( \mathcal{S}_\mu(n) \), \((X_{r,s}, w_i)_\mu = X_{r,s},(w_i)_\mu \). Break the sum for
\((w_i)_\mu \) into two parts: \((k, l) = (r, s) \) and \((k, l) \neq (r, s) \). Let

\[
(w_i)_\mu = \sum_m \beta_{r,s,m}(T_{r,s} \otimes s_m) + \sum_m \sum_{(k,l) \neq (r,s)} \beta_{k,l,m}(T_{k,l} \otimes s_m) = T_{r,s} \otimes s_* + \sum_m \sum_{(k,l) \neq (r,s)} \beta_{k,l,m}(T_{k,l} \otimes s_m),
\]

where \( s_* = \sum_m \beta_{r,s,m} s_m \).

The action of \( X_{r,s} \) is different on these two parts. In particular, the
terms with \( T_{r,s} \) are multiplied by \( Q \). The action of multiplication on the
left by \( X_{r,s} \) is

\[
X_{r,s}(T_{k,l} \otimes s) = \begin{cases} 
Q(T_{r,s} \otimes s), & \text{if } \{k,l\} = \{r,s\}, \\
T_{r,s} \otimes s, & \text{if } |\{k,l\} \cap \{r,s\}| = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

So, \( X_{r,s}(w_i)_\mu = Q(T_{r,s} \otimes s_*) + (T_{r,s} \otimes s_{**}) \). It is important to notice
that each of these terms is independent of \( Q \).

To find a solution to \((X_{r,s}, v_i)_\mu + \alpha_\mu X_{r,s},(w_i)_\mu = 0\), we need

\[
s_* + Q\alpha_\mu s_* + \alpha_\mu s_{**} = 0. \tag{3}
\]

Remember, \( s_* \), \( s_{**} \), and \( s_{***} \) are just vectors in the \( S^\mu \) component of
\( S^\lambda / \). Because \( B_{n-1}(Q) \) is semisimple for almost all \( Q \), a solution must
exist for almost all \( Q \). Suppose \( s_* \) and \( s_{**} \) were linearly independent.
Then, by (3), \( s_* \) is in their linear span and \( s_* = cs_{**} + ds_{***} \) for unique \( c \) and \( d \). But then the only time (3) has a solution is for \( \alpha_\mu = -d \) and
\( Q = c/d \). This contradicts the fact that there are an infinite number of solutions. Thus \( s_{**} \) and \( s_{***} \) are linearly dependent. By (3), it follows
that \( s_* \), \( s_{**} \), and \( s_{***} \) are linear multiples of one another.

Suppose now the boxes of \( \nu/\mu \) are in the same column. There are no choices for \( (w_i)_\mu \) except 0. This means \( s_{**} = s_{***} = 0 \), as the \( \mathcal{S}_\mu(n) \)
splits for infinitely many \( Q \). It follows that \((X_{r,s}, v_i)_\mu = 0 \) and the \( v_i \) are
already split from this component.

We now assume the boxes of \( \nu/\mu \) are in different columns. Recall that
the value \( \alpha_\mu \) does not depend on \( i \). We need a technical fact about
\((X_{r,s}, v_i)_\mu \), which we prove below.
Claim. For each $\mu \subset \lambda$, with $\nu/\mu$ not in a column, there exists an $i$ such that $(X_{r,i},v)_\mu \neq 0$.

We assume that this holds and continue with the proof. From the claim, we pick an $i$ for which $(X_{r,i},(v))_\mu \neq 0$. This implies $s_* \neq 0$. So (3) becomes

$$s_* (1 + Q \alpha_\mu \gamma_1 + \alpha_\mu \gamma_2) = 0,$$

where $s_{**} = \gamma_1 s_*$ and $s_{***} = \gamma_2 s_*$. This applies for all $i$ for which $(X_{r,i},(v))_\mu \neq 0$. Because it can be solved for almost all $Q$, and because $\alpha_\mu$ is determined by it, $\gamma_1$ and $\gamma_2$ are the same for these $i$. This is a remarkable fact. In working examples, there seems to be no reason that these values should be the same. It is true, as we indicate, because of the solutions for infinitely many $Q$.

Given an $i$ for which $(X_{r,i},(v))_\mu = 0$, $s_{**}$ and $s_{***}$ must also be 0, or Eq. (3) could not be solved for more than one $Q$. So

$$\alpha_\mu = \frac{-1}{Q \gamma_1 + \gamma_2}$$

gives the desired solution to (3) unless $Q \gamma_1 + \gamma_2$ happens to be 0. If this were the case, then $X_{r,i}(w_i)_\mu = 0$ for all $i$. In other words, $(w_1)_\mu, \ldots, (w_j)_\mu$ is an embedding of $\mathcal{S}(n - 1)$ in $\mathcal{S}(n - 1)$ (Condition (2) holds). Furthermore, if there is such an embedding, then $Q \gamma_1 + \gamma_2$ must be 0. So, Condition (1) holds if and only if $Q \gamma_1 + \gamma_2 \neq 0$, and Condition (2) holds if and only if $Q \gamma_1 + \gamma_2 = 0$. Therefore, exactly one of these conditions holds.

Proof of Claim. We now prove the technical claim. For each $\mu \subset \lambda$, with $\nu/\mu$ not in a column, there exists an $i$ such that $(X_{r,i},v)_\mu \neq 0$. Since $\{v_1, \ldots, v_r\}$ spans the $S^\nu$ component of $V/W$, all that is needed is to exhibit an element $v$ in the $S^\nu$ component such that $(X_{r,i},v)_\mu \neq 0$. Since $X_{1,2} = \sigma X_{r,\sigma^{-1}}$ for some $\sigma \in \text{Sym}(n - 1)$, $(X_{r,v})_\mu \neq 0 \Leftrightarrow (\sigma(X_{r,v}))_\mu \neq 0 \Leftrightarrow (\sigma X_{r,\sigma^{-1}}(\sigma v))_\mu \neq 0 \Leftrightarrow X_{1,2}(\sigma v)_\mu \neq 0$. Thus, it suffices to show $(X_{r,v})_\mu \neq 0$ for just one pair $(r,s)$.

Since $\nu = n - 1$, $\mathcal{S}(n - 1)$ is essentially the $\text{Sym}(n - 1)$-module $S^\nu$. Thus given an element $T_{i,n} \otimes v$ for $V/W$ where $v \in S^\nu$,

$$e_v \cdot (T_{i,n} \otimes v)$$

is in the $\mathcal{S}(n - 1)$ component of $V/W$. There is a vector $v$ in the $S^\nu$-component, for which $v = e_v \cdot v$. By moving $e_v$ through the tensor product, we get $e_v \cdot (T_{i,n} \otimes v) = (e_v \cdot T_{i,n} \cdot e_v) \otimes v$. Thus, it suffices to
show that

$$X_{r,s}(e_v \cdot T_{i,n} \cdot e_\mu) \neq 0$$

for some choice of $r$, $s$, and $i$.

We are assuming that the boxes of $\nu/\mu$ are in different columns. We visualize the terms in $e_v \cdot T_{i,n} \cdot e_\mu$ as fillings of the Ferrers diagram of $\nu$. Start by associating the boxes in $[\nu]$ with the first $n - 1$ points in any fashion, subject to the constraint that the boxes in the skew diagram $\nu/\mu$ are associated with point $i$ (the point connected to point $n$) and the point in the top that is connected to point $n - 2$ in the bottom. Now given a term in $e_v \cdot T_{i,n} \cdot e_\mu$, its associated diagram is obtained by looking at points 1 to $n - 1$ in the top and in its associated box, writing the number $i$ if it is connected to point $i$ in the bottom for $1 \leq i \leq n - 3$, writing $a$ if it is connected to point $n$ in the top, and writing $b$ if it is connected to point $n - 2$ in the bottom.

For example, with $n = 10$, $\nu = (4, 3, 1, 1)$, and $\mu = (4, 2, 1)$, the diagram might be the following:

```
  4  7  6  2
  5  1   a
  3   b
```

The idempotent $e_v \in C[Sym(n - 1)]$ acts on these diagrams by permuting positions, and $e_\mu \in C[Sym(n - 3)]$ acts by permuting values. In this framework, the row and column stabilizers used to define $e_v$ and $e_\mu$ are clear. They are just the row and column stabilizers of the diagram.

Now we take a particular filling of the boxes of $\mu$ with the top of $T_{i,n}$ so that the diagram has 1 to $n - 3$ entered in order from left to right, then top to bottom. Then place $a$ and $b$ in $\lambda/\mu$ so that $a$ is in a column to the right of $b$. The diagram looks like the following:

```
  1  2   \cdots   \mu_1
\mu_1 + 1   \cdots   \mu_2
  \vdots   \vdots   a
  \vdots   \vdots   b
  m - 3
```
Choose \( r, s \) to be the values of points associated with the two boxes \( v/\mu \).
The first thing to notice is that the only terms in \( e_r \cdot T_{i,n} \cdot e_s \) that are not
zeroed by \( X_{r,s} \) have \( a \) in \( v/\mu \), since otherwise the resulting multiplication
would have two horizontal lines and thus would be zero. When a term has
\( a \) in one of the boxes of \( v/\mu \), the effect of \( X_{r,s} \) is to replace the edge in
the top from point \( n \) to one of the points corresponding to \( v/\mu \) with an
edge between the points \( r \) and \( s \), and replace the vertical edge that uses
the other point in \( v/\mu \) with one between point \( n \) in the top and the same
point in the bottom:

To show that \( X_{r,s}(e_r \cdot T_{i,n} \cdot e_s) \) is not zero, we examine just the coeffi-
cient on \( X_{r,s}T_{i,n} \). In the picture above, \( X_{r,s}T_{i,n} \) has a line from \( a \) to \( b \), and
all other entries are as displayed. Suppose

\[
X_{r,s}((\text{sgn}(\sigma)\sigma\tau)T_{i,n}(\text{sgn}(\sigma^*)\sigma^*\tau^*)) = X_{r,s}T_{i,n},
\]

where \( \sigma \in C_v, \tau \in R_v, \sigma^* \in C_\mu, \) and \( \tau^* \in R_\mu \). There does exist such a
choice of \( (\sigma, \tau, \sigma^*, \tau^*) \); all four being the identity. For (4) to hold, the
effect of \( (\sigma, \tau, \sigma^*, \tau^*) \) on \( T_{i,n} \) must be to either flip the positions of \( a \) and
\( b \) or leave them fixed, and leave all of the other boxes fixed. Only \( \sigma \) and \( \tau \)
affect the boxes that initially contain \( a \) and \( b \). Recall that \( a \) and \( b \) are
in different columns. If they are also in different rows, then there is no
choice of \( \sigma \) and \( \tau \) that flips \( a \) and \( b \). If \( a \) and \( b \) are in the same row, then
it is possible for \( \tau \) to flip the two. However, \( \sigma \) cannot move either of these
positions. There are many ways for \( \sigma |_\mu \tau |_\mu \sigma^* \tau^* = \text{id} \). In each case, \( \text{sgn}(\sigma |_\mu) = \text{sgn}(\sigma^*) \); in fact, they must have the same cycle type. Since \( \sigma \) cannot
move either of the boxes in \( v/\mu \), \( \text{sgn}(\sigma) = \text{sgn}(\sigma |_\mu) = \text{sgn}(\sigma^*) \). Thus, all
of the terms that contribute to the coefficient on \( X_{r,s}T_{i,n} \) have positive
sign, and there is at least one. So, \( X_{r,s}(e_r \cdot T_{i,n} \cdot \mu) \neq 0 \), which shows the
claim.

We can now use Theorem 4.2 to determine the top and bottom con-
stituents of \( \mathcal{S}(n) \downarrow \) for \( \lambda \vdash n - 2 \). A bottom constituent \( \Gamma \) of a module \( U \)
is a submodule \( U_\Gamma \) of \( U \) for which restriction to \( U_\Gamma \) is \( \Gamma \). Theorem 4.1
proves that \( \mathcal{S}(n - 1) \) is a bottom constituent of \( \mathcal{S}(n) \downarrow \) if \( \mu < \lambda \). If \( \nu \triangleright \lambda \),
we have shown that \( \mathcal{S}(n - 1) \) is also a bottom constituent if there are no
\( \mu < \lambda \) for which \( \nu / \mu \) satisfies (\( \star \)), as it splits off as a direct summand of \( \mathcal{S}(n) \). However, if there is such a \( \mu \), then \( \mathcal{S}(n-1) \) is also a bottom constituent, as it is embedded in \( \mathcal{S}_\mu(n-1) \). In particular, this proves that \( \mu < \lambda \) or \( \mu > \lambda \), then \( \langle \mathcal{S}(n-1), \mathcal{S}(n) \rangle = 1 \).

This analysis also enables us to determine the top constituents of \( \mathcal{S}(n) \). A top constituent \( \Gamma \) of a module \( U \) is a quotient of \( U \) that is isomorphic to \( \Gamma \). That is, there exists a submodule \( U' \) of \( U \) for which \( U/U' \) affords \( \Gamma \). The top constituents of \( \mathcal{S}(n) \) are, of course, all \( \mathcal{S}_n(n-1) \) where \( \nu > \lambda \). Furthermore, if there are no \( \nu > \lambda \) for which \( \nu / \mu \) satisfies (\( \star \)), then \( \mathcal{S}_\mu(n-1) \) is a top constituent, as all \( \mathcal{S}_s \) split from it. On the other hand, if there is some \( \nu > \lambda \) for which \( \nu / \mu \) satisfies (\( \star \)), then \( \mathcal{S}_\mu(n) \) is not a top constituent of any indecomposable constituent by Theorem 4.2 and so cannot be a top constituent of \( V \). This means all \( \mathcal{S}_\mu(n-1) \) for \( \mu < \lambda \) or \( \mu > \lambda \) are bottom constituents. However, not all \( \mathcal{S}_\mu(n-1) \) are top constituents. We summarize:

**Corollary 4.3.** Suppose \( \lambda \vdash n \).

1. For all \( \mu > \lambda \) or \( \mu < \lambda \), \( \langle \mathcal{S}_\mu(n-1), \mathcal{S}_\mu(n) \rangle = 1 \).
2. For all \( \nu > \lambda \), \( \langle \mathcal{S}_\nu(n), \mathcal{S}_\nu(n-1) \rangle = 1 \).
3. If \( \mu < \lambda \) and for all \( \nu > \lambda \), \( (\nu, \mu) \) does not satisfy (\( \star \)); then \( \mathcal{S}_\mu(n) \) has \( \mathcal{S}_\mu(n-1) \) as a top constituent. If there is such a \( \nu \), then it does not.

Analyzing the above proves the following corollary.

**Corollary 4.4.** For \( \lambda \vdash n \) and \( \nu > \lambda \), \( \mathcal{S}_\nu(n-1) \) is a direct summand of \( V = \mathcal{S}_\nu(n) \) if and only if there is no \( \mu < \lambda \) for which \( \nu / \mu \) satisfies (\( \star \)).

**Remark.** If \( \lambda \vdash n \), then \( \mathcal{S}_\nu(n) \) is the restriction for \( \mathcal{S}_\nu \) as a \( \text{Sym}(n) \)-module and thus is isomorphic to \( \bigoplus_{\mu < \lambda} \mathcal{S}_\mu(n-1) \).

5. The Functors \( F \) and \( G \)

In this section we introduce the functors \( F \) and \( G \), which relate \( B_{n-1}(Q) \) modules to those of \( B_n(Q) \). The ideas make use of the work of Green [G, Section 6.2] and were fully utilized by Martin [M]. These are general arguments that apply to algebras with an associated idempotent. Here we use the idempotent \( e = (1/Q)X_{n-1,n} \). When \( Q \neq 0 \), we cannot divide by 0, and so we define the functors differently and provide different arguments when necessary.
Let $B_n(Q)$-mod be the category of $B_n(Q)$-modules and assume $n > 2$. If $Q \neq 0$ we define $F$ and $G$ by

$$F: B_n(Q) \text{-mod} \to B_{n-2}(Q) \text{-mod}$$

$$M \mapsto eM$$

and

$$G: B_{n-2}(Q) \text{-mod} \to B_n(Q) \text{-mod}$$

$$M \mapsto B_n(Q) e \otimes_{B_{n-2}(Q)} M.$$ 

Properties of these functors can be found in Green [G]. The composition $FG$ is the identity. The functor $F$ also has the property of being exact. That is, if $0 \to V_1 \to V_2 \to V_3 \to 0$ is an exact sequence of $B_n(Q)$-modules, then $0 \to F(V_1) \to F(V_2) \to F(V_3) \to 0$ is an exact sequence of $B_{n-2}(Q)$-modules.

When $Q = 0$, the functor $F$ is defined as above, except that we replace $e$ by $X$. As $B_n(Q)$ commutes with $X$, $X$ is a submodule of $M$. To define $G$, let $H_n(Q)$ be the left ideal of $B_n(Q)$ spanned by diagrams for which $n$ in the bottom is joined to $n-1$ in the bottom. Notice that if $Q \neq 0$, this is just $B_n(Q)e$, as above. In any case, it is $B_n(Q)X$ as $X$ commutes with $B_n(Q)$, there is a natural right action of $B_{n-2}(Q)$ on $H_n(Q)$. For $Q = 0$ and $M$ a left $B_{n-2}(Q)$ module defines $G(M) = H_n \otimes_{B_{n-2}(Q)} M$.

It is shown in [G, 6.2(d)] that when $Q \neq 0$ and $m \neq 0$ in $M$, then $e \otimes m \neq 0$. In any case, notice there is a map from $H_n(Q) \otimes M$ to $M$ by taking $\delta \otimes m$ to $(X_{n-1,n}^\delta)_m$, for any diagram $\delta$ in $H_n(Q)$. Here $(X_{n-1,n}^\delta)_m$ is $(X_{n-1,n}^\delta)_{-m}$, with the $(n-1)$st and $n$th vertices removed in both the top and the bottom. Now take $\delta^* = (n-2,n-1)X_{n-1,n}$. The image of $\delta^* \otimes m$ is $m$, and so $\delta^* \otimes m$ is not 0 in $G(M)$. We now investigate some further properties of $G$.

**Lemma 5.1.** Let $\varphi: M_1 \to M_2$ be a $B_{n-2}(Q)$-homomorphism. Then

$$\varphi^*: G(M_1) \to G(M_2)$$

$$\delta \otimes m \mapsto \delta \otimes \varphi(m)$$

is a $B_n(Q)$-homomorphism for elements $\delta$ in $H_n(Q)$. Moreover, if $\varphi$ is nonzero, then so is $\varphi^*$. Furthermore, there are elements $\delta^*$ in $H_n(Q)$ for which $\delta^* \otimes m \neq 0$ for any $m \neq 0$ in $M_1$ or in $M_2$.

**Proof.** Take $\delta^*$ as defined immediately above the statement of the theorem. If $Q = 0$ we can also take $\delta^* = e$. Let $\delta_1$ be 1-factors in $B_n(Q)$,
let $\delta$ be a diagram in $H_n(Q)$, and let $m \in M_1$. Then the computation
$$
\varphi_*(\delta_1(\delta \otimes m)) = \varphi_*((\delta_1 \delta) \otimes m)
= (\delta_1 \delta) \otimes \varphi(m)
= \delta_1(\delta \otimes \varphi(m))
= \delta_1 \varphi_*(\delta \otimes m)
$$
shows that $\varphi_*$ is a $B_1(Q)$-homomorphism. If $\varphi$ is not zero, then $\varphi(m_1) = m_2 \neq 0$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Now $\varphi_*(\delta^* \otimes m_1) = \delta^* \otimes \varphi(m_1) = \delta^* \otimes m_2 \neq 0$, which shows that $\varphi_*$ is also not zero.

We show that $F$ and $G$ connect $\mathcal{S}(n - 2)$ and $\mathcal{S}(n)$ when $|\lambda| \leq n - 2$.

**Proposition 5.2.** $G(\mathcal{S}(n - 2)) = \mathcal{S}(n)$ and $G(\mathcal{S}(n - 2) \downarrow) = \mathcal{S}(n) \downarrow$.

**Proof.** We do the case $Q \neq 0$ first.

Claim. $I^m_n(Q) \equiv B_n(Q)e \otimes \delta e \otimes I^m_{n-2}(Q)$. First, we examine the elements in $U \equiv B_n(Q)e \otimes \delta e \otimes I^m_{n-2}(Q)$. Suppose $\delta_1 e \otimes \delta_2 \in U$. Let $e$ be the element in $I^m_{n-2}(Q)$ for $1 \leq i \leq m$, where point $i$ on the top is joined to point $i$ on the bottom. The remaining horizontal lines are also directly above one another in positions $(m + 1, m + 2), (m + 3, m + 4), \ldots, (n - 3, n - 2)$. In particular, the top and the bottom are (1) with one line removed. For $n = 9$ and $m = 3$,

Choose a $\sigma \in S_{n-2}$, for which $\sigma \delta_2 = e$. This gives $\delta_1 e \otimes \delta_2 = \delta_2 \sigma^{-1} e \otimes e$. Since $X_{i,j+1} e = Q e$ for $i = m + 1, m + 3, \ldots,$
$$
\delta_1 e \otimes \delta_2 = (1/Q^\delta) \delta_1 \sigma^{-1} e \otimes (X_{m+1,m+2} X_{m+3,m+4} \cdots X_{n-3,n-2}) e
= \delta_1 \sigma^{-1} (X_{m+1,m+2} X_{m+3,m+4} \cdots X_{n-3,n-2}) e \otimes e.
$$
Thus, any element of $U$ can be written as a linear combination of terms of the form $\delta_3 e \otimes e$, where $\delta_3$ has horizontal lines $(m + 1, m + 2), \ldots, (n - 3, n - 2), (n - 1, n)$ in the bottom. Furthermore, if there other horizontal lines, say $(i, j) 1 \leq i, j \leq m$, in the bottom, then $\delta_3 e \otimes e = (1/Q) \delta_3 X_{i,j} e \otimes e = (1/Q) \delta_3 e \otimes \delta_2 e \otimes e = 0$. Thus, we need only consider $\delta_3$'s with the bottom being (1). It is routine to show that the map $\delta_3 e \otimes e \leftrightarrow \delta_3$ gives the desired isomorphism.
Now a computation gives the result:

\[
G(\mathcal{A}(n - 2)) = B_n(Q) \otimes_{B_{n-2}(Q)} \left( T_{n-2}^{m^*}(Q) \otimes_{\text{Sym}(m)} S^A \right)
= (B_n(Q) \otimes_{B_{n-2}(Q)} T_{n-2}^{m^*}(Q)) \otimes_{\text{Sym}(m)} S^A
= T_{n}^{m^*}(Q) \otimes_{\text{Sym}(m)} S^A
= \mathcal{A}(n).
\]

A similar computation gives \(G(\mathcal{A}(n - 2) \downarrow) = \mathcal{A}(n) \downarrow\).

We do the case \(Q = 0\) in a similar way. Just as above, the essence is to investigate \(H_n(Q) \otimes_{B_{n-2}(Q)} (Q) I_n^{m^*} \). We want to show that this has the same dimension as \(I_n^{m^*}(Q)\). In particular, suppose \(\delta_1\) is a diagram in \(H_n(Q)\). This means it has a line joining \(n - 1\) and \(n\) in the bottom. Consider \(\delta_1 \otimes_{B_{n-2}(Q)} \delta_2\). Here \(\delta_2\) is in \(I_n^{m^*}\). We will show many of these terms are zero because of the tensor product action.

We will start by showing we need only take \(\delta_1\) as a diagram in \(I_n^{n-2}\). To be in \(H_n(Q)\) means it has a line from \(n - 1\) in the bottom to \(n\) in the bottom, so the bottom and top each have at least one horizontal line. Suppose there are more with \(i, j\) as a line in the bottom other than the one from \(n - 1\) to \(n\). Let \(i', j'\) be a line on the top. Let \(\delta_1'\) be a diagram that is the same as \(\delta_1\), except that the lines from \(i'\) to \(j'\) and from \(i\) to \(j\) are replaced by vertical lines joining \(i\) to \(i'\) and \(j\) to \(j'\). Notice \(\delta_1 = \delta_1'X_{i,j}\).

Now \(\delta_1 \otimes \delta_2 = \delta_1'X_{i,j} \otimes \delta_2 = \delta_1 \otimes X_{i,j} \delta_2\). Continue until the resulting \(\delta_2^*\) has \(n - 2\) vertical lines. We can now act by an element of \(\text{Sym}(n - 2)\) acting on \(\text{top}(\delta_2)\), so the lines are all lined up directly above the lines in \((1)\), which are in positions \((m + 1, m + 2)\), \((m + 3, m + 4)\), \ldots, \((n - 3, n - 2)\) and act on the other side of the tensor with the inverse. Act further so that all of the vertical lines join \(j\) on the bottom to \(j\) on the top for \(1 \leq j \leq m\). We want to identify this \(\delta_1 \otimes \delta_2\) with an element of \(f(d)\sigma\), where \(d\) is an \((m, k)\) partial 1-factor where \(m + 2k = n\). Notice \(\delta_2^*\) already has one horizontal line in its top, which we take to be one of the lines in \(d\). For the others identify \(\text{top}(\delta_2)\) with \(\text{bot}(\delta_2)\) in positions \(m + 1, m + 2, \ldots, n - 2\). Let the \(k - 1\) further lines for \(d\) be the images under \(\delta_1\) of the lines in \(\text{top}(\delta_2)\). This gives \(k - 1\) further lines and so a \(d\) with the right parameters. Now identify the elements \(i\) in \(\text{top}(\delta_2)\) with the elements \(i\) in \(\text{bot}(\delta_1)\) for \(1 \leq i \leq m\). This gives an element in \(I_n^{m^*}\). The dimension is correct.

It is only left to show that the left actions are the same. This is certainly true for elements of \(\text{Sym}(n)\), as the points and lines are permuted in the same way. Now examine the action of \(X_{i,j}\). Once these agree the action is the same, as these generate \(B_n(Q)\). We first show that when \(i, j\) is a line in \(d\), the action gives 0. This is certainly true for \(X_{i,j}f(d)\), as it is multiplied
by \( Q \), which is 0. We must consider \( X_{i,j}(\delta_1 \otimes \delta_2) = (X_{i,j} \cdot \delta_1) \otimes \delta_2 \). If \( i, j \) is the horizontal line in \( \text{top}(\delta_1) \), the product \( X_{i,j} \cdot \delta_1 = Q \delta_1 = 0 \). If \( i, j \) is one of the other horizontal lines in \( d \), it means that \( i, j \) are joined by vertical lines in \( \delta_1 \), to say, \( l, l + 1 \), which is one of the lines in \( \text{top}(\delta_1) \). Now \( X_{i,j} \cdot \delta_1 = \delta_X \). Now passing this through the tensor gives \( \delta_1 \otimes X_{i,j+1}(\delta_2) \). As \( Q = 0 \), this last term is 0. The other situation giving 0 in \( X_{i,j}(d) \) is when \( i, j \) are both isolated in \( d \). In this case \( X_{i,j} \cdot \delta_2 = 0 \). In this case neither \( i \) nor \( j \) is part of a line in \( \delta_1 \), and so both are joined to, say, \( i', j' \) in \( \text{bot}(\delta_1) \), where \( i' \) and \( j' \) are at most \( m \). Again \( X_{i,j}(\delta_1) = \delta_X \), and passing through the tensor gives 0 as \( X_{i',j'} \cdot \delta_2 = 0 \). The remaining cases involve \( i, j \), for which at least one of \( i, j \) (say \( i \)) is part of a line in \( d \). In this case \( X_{i,j}(d) = (i, j)f(d) \). It is only necessary to show that the same is true for \( X_{i,j}(\delta_1 \otimes \delta_2) \). Again one must distinguish between the different cases for \( i \). Either \( i \) is part of the one horizontal line in \( \text{top}(\delta_1) \) or it is an \( i \) in \( \text{top}(\delta_1) \) that is joined to a point \( i' \) that is part of a line in \( \text{top}(\delta_2) \). A short computation relating the various transpositions involved shows that this action is the same and we are done.

**Proposition 5.3.** Suppose \(|\lambda| \leq n - 2\). Then \( F(\mathcal{P}(n)) = \mathcal{P}(n - 2) \) and \( F(\mathcal{P}(n)_{\downarrow}) = \mathcal{P}(n - 2)_{\downarrow} \).

**Proof.** In \( Q \neq 0 \) we know from Proposition 5.2 that \( G(\mathcal{P}(n - 2)) = \mathcal{P}(n) \). Applying \( F \) to both sides and using the fact that \( FG \) is the identity gives \( \mathcal{P}(n) = F(\mathcal{P}(n)) \). The second statement is proved similarly. We give a different argument for \( Q = 0 \), which applies equally well for \( Q \neq 0 \) and is more direct than the above.

Recall that \( \mathcal{P}(n) \) is spanned by \( I_{n_1}^{m_{i_1}} \otimes_{\text{Sym}(m)} v_i \), where \( v_i \) is a basis for \( S^m \). If \( \delta \) is a diagram in \( I_{n_1}^{m_{i_1}} \), consider \( X_{i-1,n} \cdot \delta \). This is zero if \( \delta \) has a line between \( n - 1 \) and \( n \) or if \( n - 1 \) and \( n \) are both free. If not it is \((i, j)\delta \), where \( j = n - 1 \) or \( n \) and \( i < n - 1 \). This element can be considered in \( I_{n-2}^{m_{i_1}} \) by just taking away the four right-hand nodes that join \( n - 1 \) to \( n \) in top and bottom. In particular, \( X_{i-1,n}(\delta \times v_i) \) is naturally an element of \( \mathcal{P}(n - 2) \). It is onto, as any element \( \delta_1 \) in \( I_{n-2}^{m_{i_1}} \) can be obtained by extending it to \( I_{n_1}^{m_{i_1}} \) by adjoining four nodes with \( n - 1 \) and \( n \) joined in both the top and bottom, and then acting by a transposition that moves \( n - 1 \) to \( j \) in the top for any \( j < n - 2 \).

**Theorem 5.4.** Let \( \mu \vdash n \) and \( n_0 \) be a nonnegative even integer, and if \( Q = 0 \), suppose \( \mu \) is not the empty partition (i.e., assume \( m \neq 0 \)). Then \( \langle \mathcal{P}(n + n_0), \mathcal{P}(n + n_0) \rangle \neq 0 \) if and only if \( \mathcal{P}(n) \) can be embedded in \( \mathcal{P}(n) \).

**Proof.** Suppose first \( Q \neq 0 \) and suppose \( \langle \mathcal{P}(n + n_0), \mathcal{P}(n + n_0) \rangle \neq 0 \). Let \( \varphi \) map \( \mathcal{P}(n + n_0) \) to \( \mathcal{P}(n + n_0) \) with \( W \) the image of \( \varphi \). Apply the functor \( F \) with \( e = (1/Q)X_{n+n_0-1,n+n_0} \). If \( eW \neq 0 \), this gives a non-
trivial map from $\mathcal{S}_\mu(n + n_0 - 2)$ to $\mathcal{S}_\mu(n + n_0 - 2)$, and we can apply induction to get a nontrivial map from $\mathcal{S}_\mu(n)$ into $\mathcal{S}_\mu(n)$. It is an embedding as $\mathcal{S}_\mu(n)$ is irreducible. We suppose then that $eW = 0$.

Notice $W$ is a $B_{n + n_0}(Q)$ submodule of $\mathcal{S}_\mu(n + n_0)$, which is annihilated by $X_{n + n_0 - 1, n + n_0}$. By applying elements of $\text{Sym}(n + n_0)$ we see $X_{i,j}W = 0$ also for any $i, j$. In particular, $X = X_{1,2}X_{3,4} \cdots X_{n_0-1,n_0}W = 0$.

Because $W$ is the image of $\varphi, W = \mathcal{S}(n + n_0)/U$ for some submodule $U$ of $\mathcal{S}(n + n_0)$. By the results of [HW1], $\mathcal{S}(n + n_0)$ contains a unique maximal submodule annihilated by all elements in $B_{n + n_0}(Q)$, particularly by $X$. Suppose first $Q \neq 0$. Let $y$ be the $(n, n_0/2)$ partial 1-factor with lines $(1, 2), (3, 4), \ldots, (n_0 - 1, n_0)$ and with free points $n_0 + 1, n_0 + 2, \ldots, n_0 + n$. Notice that $v = f(y) \otimes s_k$ is a nonzero element of $\mathcal{S}(n + n_0)$. Notice also that $Xv = Q^{n_0/2}v \neq 0$. This means $v$ is not in $U$. Also $X(v + U) = Q^{n_0/2}(v + U)$, which is not 0 modulo $U$. This is a contradiction and shows $eW \neq 0$, which does this part.

In the case $Q = 0$, recall that $\mu$ is not empty. Now let $y$ have free point 1, lines between $(2, 3), (4, 5), \ldots, (n_0, n_0 + 1)$, and let the remaining points be free. As above, let $v = f(y) \otimes s_k$. Now $Xv \neq 0$ and argue as above.

For the other direction, given an embedding of $\mathcal{S}_{\mu}(n)$ in $\mathcal{S}_{\mu}(n)$, apply the functor $G_{n_0/2}$ times. By Proposition 5.2, this gives a map from $\mathcal{S}(n + n_0)$ to $\mathcal{S}_{\mu}(n + n_0)$. By Lemma 5.1, this map is not zero, which shows the other direction.

It should be noted that for $n_0 > 0$, these maps obtained by applying $G$ are not necessarily embeddings.

**Theorem 5.5.** Let $\lambda \vdash m \leq n$. If $\mu \lessdot \lambda$ or $\mu \triangleright \lambda$, then $\langle \mathcal{S}(n - 1), \mathcal{S}(n) \downarrow \rangle = 0$.

**Proof.** Proof by induction on $n - m$. We have already considered the cases $n - m = 0$ or 2 in Corollary 4.3 and the remark following it, so suppose $n - m \geq 4$. Recall $n - m$ is even. Suppose $\mu \lessdot \lambda$ or $\mu \triangleright \lambda$. By the induction hypothesis, $\langle \mathcal{S}(n - 3), \mathcal{S}(n - 2) \downarrow \rangle \neq 0$. Thus there is a nonzero homomorphism $\phi: \mathcal{S}(n - 3) \to \mathcal{S}(n - 2) \downarrow$. By Lemma 5.1, the homomorphism $\phi_\mu: G(S_{\mu}(n - 3)) \to G(\mathcal{S}(n - 2) \downarrow)$ is nonzero and, by Proposition 5.2, $\phi_\mu: \mathcal{S}(n - 1) \to \mathcal{S}(n) \downarrow$. So, $\langle \mathcal{S}(n - 1), \mathcal{S}(n) \downarrow \rangle \neq 0$.

Notice that for $|\lambda| = n$ or $n - 2$, this occurs because $\mathcal{S}(n - 1)$ is embedded in $\mathcal{S}(n) \downarrow$. For smaller $|\lambda|$ this need not be true. However, a homomorphic image of $\mathcal{S}(n - 1)$ is embedded in $\mathcal{S}(n) \downarrow$.

There is a stronger statement here that holds for certain values of $\mu$. It makes use of Corollary 4.4 and some properties of the tensor product. In general, if $W_1$ is a $B_{n-\lambda}(Q)$ submodule of $W$, the universality of the tensor product gives a map from $B_n(Q) \otimes_{B_{n-\lambda}(Q)} (W_1)$ into $B_n(Q) \otimes_{B_{n-\lambda}(Q)} W$. 
However, it is not, in general, an embedding. However, if \( W_1 \) is a direct summand, this is indeed the case. In particular, if \( W \equiv W_1 \oplus W_2 \), then

\[
B_n(Q) \otimes_{B_{n-2}(Q)} (W) \equiv B_n(Q) \otimes_{B_{n-2}(Q)} (W_1) \oplus B_n(Q) \otimes_{B_{n-2}(Q)} (W_2).
\]

This gives rise to a strengthening of the previous theorem for certain \( \mu \).

**Theorem 5.6.** Suppose \( \lambda \vdash \mu \) and \( \mu \vdash \lambda \). Suppose further that there are no \( \mu' \prec \lambda \) for which \( (\mu, \mu') \) satisfies \( \boxdot \). Then \( \mathcal{S}_\mu(n-1) \) is a direct summand of \( \mathcal{S}_\lambda(n) \). If, on the other hand, \( \mu' \prec \lambda \) and there are no \( \mu \vdash \lambda \) for which \( (\mu, \mu') \) satisfies \( \boxdot \), then \( \mathcal{S}_\mu(n) \) is a direct summand of \( \mathcal{S}_\lambda(n) \).

**Proof.** This follows from Corollary 4.4 and the property stated above.

### 6. Induction in the Brauer Algebra

Given a \( B_n(Q) \)-module \( M \), define the induced \( B_{n+1}(Q) \)-module \( M \uparrow \) to be \( B_{n+1}(Q) \otimes_{B_n(Q)} M \). In the semisimple case,

\[
\mathcal{S}_\lambda(n) \uparrow \equiv \bigoplus_{\mu \vdash \lambda} \mathcal{S}_\mu(n+1) \oplus \bigoplus_{\nu \vdash \lambda} \mathcal{S}_\nu(n+1).
\]

As with restriction, when \( B_{n+1}(Q) \) is not semisimple, such a direct sum decomposition need not exist.

Here is a very nice method for connecting induction and restriction using the functor \( G \).

**Theorem 6.1.** Let \( M \) be any \( B_n(Q) \)-module. Then \( M \uparrow \equiv G(M) \downarrow \) as \( B_{n+1}(Q) \)-modules.

**Proof.** First, we show that \( H_{n+2}(Q) \equiv B_{n+1}(Q) \times B_n(Q) \)-modules, where \( B_{n+1}(Q) \) acts on the left and \( B_n(Q) \) acts on the right. The isomorphism is given taking \( \delta \in B_{n+1}(Q) \), moving point \( n+1 \) in the bottom of \( \delta \) along with its adjacent edge to the point \( n+2 \) in the top row and adding an edge in the bottom between points \( n+1 \) and \( n+2 \). The rest of the diagram remains the same. For example, \( \delta \) gets mapped to \( \delta' \). It is straightforward to check that this is a \( B_{n+1}(Q) \times B_n(Q) \)-isomorphism. The key is to notice that the "moved" point is effected by neither the right
$B_n(Q)$ action when in the bottom of $\delta$ nor by the left $B_{n+2}(Q)$ action when in the top when mapped into $H_{n+2}(Q)$. Thus its physical location does not matter.

Now recall that $M \uparrow \equiv B_{n+1} \otimes_{B_n(Q)} M$. Furthermore, $G(M) \equiv H_{n+2} \otimes_{B_n(Q)} M$ when $Q = 0$ and $B_{n+2}(Q)e \otimes_{B_n(Q)} M$ when $Q \neq 0$. In the second case, $B_{n+2}(Q)e$ is $H_{n+2}(Q)$. Since both of the tensor products are right $B_n(Q)$-actions, under this $B_{n+1}(Q) \times B_n(Q)$-isomorphism, we get

$$B_{n+1} \otimes_{B_n(Q)} M \equiv H_{n+2}(Q) \otimes_{B_n(Q)} M$$

or

$$B_{n+1} \otimes_{B_n(Q)} M \equiv B_{n+2}(Q)e \otimes_{B_n(Q)} M$$

as left $B_{n+1}(Q)$-modules. This means $M \uparrow \equiv G(M) \downarrow$. This argument works both when $Q = 0$ and when it is not.

**Corollary 6.2.** Let $|\lambda| \leq n$. Then

$$\mathcal{S}_\lambda(n) \uparrow \equiv \mathcal{S}_\lambda(n+2) \downarrow$$

as $B_{n+1}(Q)$-modules.

**Proof.** Apply the previous theorem in the case $M = \mathcal{S}_\lambda(n)$ along with Proposition 5.2.

Applying Theorem 5.5 to this gives the following result.

**Theorem 6.3.** Let $\mu \vdash m \leq n$. The value of $\langle \mathcal{S}_\lambda(n+1), \mathcal{S}_\mu(n) \uparrow \rangle$ is not zero if $\mu \lessdot \lambda$ or $\mu \succ \lambda$.

To investigate induction in the next section, we examine the case $\lambda \vdash n$ with a little more care. We find results analogous to Theorems 4.1 and 4.2. The analogue of Theorem 4.1 is quite easy to prove.

**Corollary 6.4.** Let $\lambda \vdash n$ and $V = \mathcal{S}_\lambda(n) \uparrow$. As a $B_{n+1}(Q)$-module, $V$ contains a submodule $W = \bigoplus_{\mu \vdash \lambda} \mathcal{S}_\mu(n+1)$. Furthermore, $V/W \cong \bigoplus_{\mu \vdash \lambda} \mathcal{S}_\mu(n+1)$. 
Proof. Combining Corollary 6.2 and Theorem 4.1 gives a proof. 

As in Theorem 4.2 there are two cases. For a given \( \nu \triangleright \lambda \), either \( \mathcal{S}(n + 1) \) splits off from \( \mathcal{S}(n) \) or the module \( \mathcal{S}(n + 1) \) appears as a submodule of \( \mathcal{S}(n + 1) \) for some \( \mu < \lambda \). We know by Theorem 4.2 that exactly one of these occurs. Furthermore, from the previous section, the second occurs if and only if \( (\mu, \nu) \) satisfy condition (\( \star \)) and the two boxes in \( \nu/\mu \) are in different columns.

Theorem 6.1 provides a method for connecting induction and restriction. Information about restrictions from \( B_{n+2}(Q) \)-modules is related to induction from \( B_n(Q) \)-modules. In particular, when \( \lambda \vdash n \) and \( \mu < \lambda \),

\[
\langle \mathcal{S}_\lambda(n) \uparrow, \mathcal{S}_\mu(n + 1) \rangle = \langle \mathcal{S}_\lambda(n), \mathcal{S}_\mu(n + 1) \downarrow \rangle = 1.
\]

While this follows from Frobenius reciprocity, the knowledge about the structure of \( \mathcal{S}(n) \) uses Theorem 6.1. If one of the \( \nu \triangleright \lambda \) has \( \nu/\mu \) satisfying (\( \star \)), this comes because \( \mathcal{S}_\lambda(n) \) is a top constituent of \( \mathcal{S}(n) \) and a bottom constituent of \( \mathcal{S}(n + 1) \). However, if none of the \( \nu \triangleright \lambda \) has \( \nu/\mu \) satisfying (\( \star \)), \( \mathcal{S}_\mu(n + 1) \) is itself a top constituent of \( \mathcal{S}(n) \). We have proved the following.

**Theorem 6.5.** Let \( \lambda \vdash n \). Then

1. For all \( \nu \triangleright \lambda \), \( \langle \mathcal{S}_\lambda(n) \uparrow, \mathcal{S}_\mu(n + 1) \rangle \neq 0 \).
2. If \( \mu < \lambda \) and there does not exist a \( \nu \triangleright \lambda \) such that \( (\nu, \mu) \) satisfy (\( \star \)), then \( \langle \mathcal{S}_\lambda(n) \uparrow, \mathcal{S}_\mu(n + 1) \rangle \neq 0 \).

7. FROBENIUS RECIPROCITY AND \( \langle \mathcal{S}_\lambda(n), \mathcal{S}_\mu(n) \rangle \neq 0 \)

WHERE \( \lambda \vdash n \)

Besides (\( \star \)), there is another condition derived from Frobenius reciprocity that must be satisfied for \( \langle \mathcal{S}_\lambda(n), \mathcal{S}_\mu(n) \rangle \neq 0 \) for \( \lambda \vdash n \).

**Theorem 7.1.** Suppose \( \lambda \vdash n \) and \( \langle \mathcal{S}_\lambda(n), \mathcal{S}_\mu(n) \rangle \neq 0 \). Then for every \( \lambda' < \lambda \), \( \langle \mathcal{S}_\lambda(n - 1), \mathcal{S}_\mu(n - 1) \downarrow \rangle \neq 0 \). Furthermore, there is a \( \mu' < \mu \) or \( \mu' \triangleright \mu \) such that \( \langle \mathcal{S}_\lambda(n - 1), \mathcal{S}_\mu(n - 1) \rangle \neq 0 \).

**Proof.** Suppose \( \langle \mathcal{S}_\lambda(n), \mathcal{S}_\mu(n) \rangle \neq 0 \). Take \( \lambda' < \lambda \). From Theorem 6.5, \( \langle \mathcal{S}_\lambda(n - 1) \uparrow, \mathcal{S}_\mu(n) \downarrow \rangle \neq 0 \). By Frobenius reciprocity, this implies \( \langle \mathcal{S}_\lambda(n - 1), \mathcal{S}_\mu(n - 1) \downarrow \rangle \neq 0 \), giving the first statement.

As in Theorem 4.1, let \( V \) be \( \mathcal{S}_\mu(n) \downarrow \) and let \( W \) be the submodule of \( V \) isomorphic to \( \bigoplus \mathcal{S}_\mu(n - 1) \). Let \( S \) be the image of \( \mathcal{S}_\mu(n - 1) \) in \( V \). Suppose first \( S \subseteq W \). Since \( \lambda' \vdash n - 1 \), \( S \) is irreducible. Thus it must be a submodule of some \( \mathcal{S}_\mu(n - 1) \). This implies \( \langle \mathcal{S}_\lambda(n - 1), \mathcal{S}_\mu(n - 1) \rangle \neq 0 \).
Suppose otherwise that $S \not\in W$. Then $(S \otimes W)/W \cong S$ because $S$ is irreducible and is not in $W$. This gives a map in $\text{Hom}(\mathcal{P}_n(n-1), V/W)$, which is nontrivial, as $S \not\in W$. Indeed, $(S \otimes W)/W$ is isomorphic to $\mathcal{P}_n(n-1)$. Now by Theorem 4.1,

$$V/W \cong \bigoplus_{\mu \triangleright \lambda} \mathcal{P}_\mu(n-1).$$

Now just as before, $S \otimes W/W \subseteq \mathcal{P}_\mu(n-1)$ for some $\mu \triangleright \lambda$. Now we have $\langle \mathcal{P}_\lambda(n-1), \mathcal{P}_\mu(n-1) \rangle \neq 0$.

Remark. It is important to notice that the value of $Q$ does not change.

If $\lambda \vdash n$ and $\langle \mathcal{P}_\lambda(n), \mathcal{P}_\mu(n) \rangle \neq 0$ for a given value of $Q$, then $\langle \mathcal{P}_\lambda(n-1), \mathcal{P}_\mu(n-1) \rangle \neq 0$ for the same value of $Q$.

8. SEMISIMPLICITY OF $B_n(Q)$ FOR NONINTEGER $Q$

We now have enough information to show that if $Q$ is not an integer, $B_n(Q)$ is semisimple. This was first conjectured by Hanlon and Wales in [HW1] and proved by Wenzl in [Wen]. This has consequences for the integrality of roots of certain determinants, as described in [HW3]. This proof is quite different from Wenzl’s proof in a number of ways. In particular, it does not depend on the integrality of roots of certain polynomials associated with the orthogonal groups.

Theorem 8.1. If $Q$ is not an integer, $B_n(Q)$ is semisimple

Proof. Suppose $B_n(Q)$ is not semisimple. We show that $Q$ must be an integer. In [HW1, Sect. 4B] it is shown that $B_n(Q)$ not being semisimple implies $\langle \mathcal{P}_\lambda(n), \mathcal{P}_\mu(n) \rangle \neq 0$ for some $\lambda$ and $\mu$ with $|\mu| < |\lambda| \leq n$. This is also inherent in [GL, 3.8ii]. By Theorem 5.4, we can assume $\lambda \vdash n$ when $Q \neq 0$. If $Q = 0$, we are done.

Set $k = |\lambda/\mu|/2$. If $k = 1$, we are done, as condition $(\star)$ implies that

$$Q = 1 - \frac{1}{k} \sum_{p \in \lambda/\mu} c(p),$$

which is an integer. If $k > 1$, repeatedly apply Theorem 7.1 to obtain a pair $\lambda/\mu$ with $|\lambda/\mu| = 2$. Since each application of Theorem 7.1 reduces the size of the larger partition by one, this process will terminate in such a pair. This reduces the problem to the case $k = 1$, which we have just handled. \[\blacksquare\]
9. THE CASE $\mu = \emptyset$

The necessary conditions given in previous sections are not sufficient for showing the existence of an embedding. In general, we do not have a method for determining whether $\mathcal{R}(n)$ embeds in $\mathcal{S}(n)$. However, we can treat the special case when $\mu = \emptyset$. Let $a^b$ denote the partition with $b$ parts, all of which are equal to $a$.

**Corollary 9.1.** If $\lambda \vdash n$, $\langle \mathcal{S}(n), \mathcal{R}(n) \rangle \neq 0$, then $\lambda = a^b$, where $a$ is even and $Q = b - a + 1$. This means $\lambda$ has only one corner.

**Proof.** Suppose $\lambda$ has two or more outer corners. Let $p_1$ and $p_2$ be two of the outer corners of $\lambda$. Applying Theorem 7.1 to $\lambda \setminus p_i = \lambda^a \circ \lambda$, the only choice for $\mu^o$ is the unique partition of 1, (1). Thus, both $(\lambda \setminus p_1, (1))$ and $(\lambda \setminus p_2, (1))$ satisfy condition $(\star)$. Hence, by subtraction, $c(p_1) = c(p_2)$, which is a contradiction. Therefore, $\lambda$ must be a rectangle. We know from Theorem 3.1 that $\lambda$ must be an even partition on $n$ if $S^\lambda$ occurs as a constituent of $\mathcal{S}(n)$ as a $\text{Sym}(n)$-module. This means $a$ is even.

Now suppose $n = ab$ and $\lambda = a^b$ with $a$ even. A basis for $\mathcal{R}(n)$ is indexed by 1-factors on $n$ vertices. The 1-factor corresponds to the top, the bottom is given by (1), and there are no vertical lines. When decomposed as a $\text{Sym}(n)$-module, $\mathcal{R}(n)$ contains $S^\lambda$ with multiplicity 1. Let $W$ be the unique copy of $S^\lambda$ contained in $\mathcal{R}(n)$.

Following the proof of the technical claim in Theorem 4.2, we visualize the basis of $\mathcal{R}(n)$ as 1-factors of the boxes of the Ferrers diagram of $a^b$. The points of the top row are associated with the boxes of the Ferrers diagram so that points 1 to $a$ correspond to the boxes in the first row from left to right, points $a + 1$ to $2a$ correspond to the boxes in the second row also from left to right, and so on. Note that the points $n - 1$ and $n$ correspond to the two rightmost boxes in the bottom row. For example, with $a = 4$ and $b = 3$, the element

![Diagram of 1-factors](image)
is visualized as

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
\end{array} \]

In the following proof, the idempotent \( e_{\lambda} \) is used. Following these diagrams, the obvious choices are made for the row and column stabilizers.

**THEOREM 9.2.** Suppose \( Q = b - a + 1 \). Then \( X_{i,j}w = 0 \) for all \( w \in W \) and \( 1 \leq i < j \leq n \). This shows that \( \mathcal{S}(n) \) embeds in \( \mathcal{S}(n) \).

**Proof.** We begin with a reduction that is for notational convenience. By definition of \( X_{i,j} \), it is easy to see that

\[ \sigma X_{i,j} \sigma^{-1} = X_{\sigma(i), \sigma(j)} \]

for all \( \sigma \in \text{Sym}(n) \). Choose \( \sigma \) so that \( \sigma(i, j) = (n - 1, n) \). Then \( X_{i,j}W = 0 \iff \sigma X_{i,j} \sigma^{-1}W = 0 \iff X_{n-1,n}W = 0 \). So, it suffices to show that \( X_{n-1,n}W = 0 \).

Now consider \( W \) as a \( \text{Sym}(n - 2) \times \text{Sym}(2) \)-module by restriction. We view \( \text{Sym}(n - 2) \) as acting on the first \( n - 2 \) points and \( \text{Sym}(2) \) on the last 2. By the Littlewood–Richardson rule and Frobenius reciprocity, we have

\[ W \cong Y \oplus Z, \]

where \( Y \cong S^\mu \otimes S^{(1,1)} \) with \( \mu = a^{b-2}, (a - 1)^2 \) and \( Z \cong S^\nu \otimes S^{(2)} \) with \( \nu = a^{b-1}, a - 2 \). Here we are using the notation \( a^{b-2}, (a - 1)^2 \) to denote the partition with \( b - 2 \) parts \( a \), and \( a - 1 \) twice. Similarly \( a^{b-1}, a - 2 \) is the partition with \( b - 1 \) parts \( a \) and one part \( a - 2 \). Since \( S^{(1,1)} \) is the sign representation

\[ Y = \frac{1}{2}(id - (n - 1, n))Y. \]

Note that \( X_{n-1,n} = X_{n-1,n}(n - 1, n) \). So,

\[ X_{n-1,n}Y = \{X_{n-1,n}(id - (n - 1, n))\}Y = 0. \]

So, it suffices to show that \( X_{n-1,n}Z = 0 \).
Let $U$ be the subspace of $\mathcal{H}(n)$ spanned by all 1-factors in which point $n - 1$ and $n$ are adjacent. Notice that $U$ is a $\text{Sym}(n - 2) \times \text{Sym}(2)$-module, where again $\text{Sym}(n - 2)$ acts on the first $n - 2$ points and $\text{Sym}(2)$ on the last 2. The decomposition of $U$ as a $\text{Sym}(n - 2) \times \text{Sym}(2)$-module is

$$\bigoplus_{\eta \leq n - 2} S^\eta \otimes S^{(2)}.$$  

In particular, $U$ contains exactly one copy of the irreducible $S^r \otimes S^{(2)}$, which is denoted $Z$.

The map $X_{n-1,n}$ takes $W$ to $U$. Moreover, it is a $(\text{Sym}(n - 2) \times \text{Sym}(2))$-equivariant map. Thus it must map $Z$ into $\hat{Z}$. Notice the importance of the uniqueness of $Z$ in $W$ and $\hat{Z}$ in $U$. If this map is zero on $Z$, we are done. Otherwise, by Schur's lemma, it is a scalar. Up to this point, nothing in this proof has depended on the value of $Q$. However, the value of this scalar is a function of $Q$ and is denoted $a(Q)$. Pictorially, this is

$$W \xrightarrow{X_{n-1,n}} U \xrightarrow{a(Q)} \hat{Z}.$$  

Let $\delta$ be the special element of $\mathcal{H}(n)$ whose top is (1). For $a = 4$ and $b = 3$, the diagram of $\delta$ is

We show below that the coefficient on $\delta$ in $X_{n-1,n}e_A\delta$ is zero when $Q = b - a + 1$. 
We use this computation to show that \( \alpha(Q) = 0 \) for \( Q = b - a - 1 \). We know that \( e, d \in W \) because \( W \) is isomorphic to \( S^\lambda \) as a \( \text{Sym}(n) \)-module. Let \( \hat{z}_1, \ldots, \hat{z}_d \) be a basis for \( Z \) and \( \hat{z}_1, \ldots, \hat{z}_d \) be the corresponding basis for \( Z \) such that \( X_{n-1,n} e_d = \alpha(Q) \hat{z}_d \). Let \( e_d = x + \sum c_i \hat{z}_i \), where \( y \in Y \) and \( c_i \in \mathbb{C} \). Then

\[
X_{n-1,n} e_d = 0 + \alpha(Q) \sum c_i \hat{z}_i.
\]

Let \( m_i \) denote the coefficient of \( \hat{z}_i \). The coefficient of \( \hat{z}_i \) in \( X_{n-1,n} e_d \) is

\[
\alpha(Q) \sum c_i m_i.
\]

We show below that this is a nonzero multiple of \( Q - (b - a + 1) \). Hence so is \( \alpha(Q) \).

We now examine the coefficient of \( \hat{z}_i \) in \( X_{n-1,n} e_d \). Recall

\[
e_d = \frac{f^\lambda}{n!} \sum_{\sigma \in C_x} \sum_{\tau \in R_x} \text{sgn}(\sigma) \sigma \tau d
\]

\[
X_{n-1,n} e_d = \frac{f^\lambda}{n!} \sum_{\sigma \in C_x} \sum_{\tau \in R_x} \text{sgn}(\sigma) X_{n-1,n} \sigma \tau d.
\]

We need only consider \( \sigma, \tau \) for which \( X_{n-1,n} \sigma \tau d = Q' d \). The power \( j \) is at most 1, as \( X_{n-1,n} \) has only one horizontal line. We ignore the coefficient \( f^\lambda / n! \).

If \( \sigma \tau d \) has a line joining \( n - 1 \) and \( n \), \( X_{n-1,n} \sigma \tau d = Q \sigma \tau d \). This is \( Q d \) if and only if \( \sigma \tau d = d \). We count the number of pairs \( (\sigma, \tau) \) for which \( \sigma \tau d = d \). Let \( R \) be the subgroup of \( R_x \) that fixes \( d \). Suppose its size is \( r \). If \( \sigma \tau d = d \), \( \tau \) must be in \( R \). For if not, some line of \( \tau \) would not be in positions \( 2i + 1, 2i + 2 \) and no column permutations could put them in such positions. But all lines in \( d \) are in such positions. So for \( \sigma \tau d = d \), we must have \( \tau \in R \). This means \( \tau d = d \). Now we must count the number of \( \sigma \) for which \( \sigma d = d \). For this to be so, the column permutations acting on \( d \) must map horizontal lines to horizontal lines. In particular, if \( \sigma \) is the permutation in column \( 2i + 1 \), the permutation in column \( 2i + 2 \) must be the same. Suppose \( a = 2a' \). Then we may have any permutation in each odd column, as long as the same permutation occurs for the column to the right. There are \( b! \) such permutations for each odd column, and so \( (b!)^{a'} \) such terms. The total number is \( r(b!)^{a'} \). The sign is always \( + \), as the permutations come in pairs. The total contribution is \( Q' \cdot r \cdot (b!)^{a'} \).

Suppose now that \( \sigma \tau d \) does not have a line joining \( n - 1 \) and \( n \). Then \( \sigma \tau d \) has lines from \( n \) to \( i \) and from \( n - 1 \) to \( j \), say. Here \( j \) and \( i \) are
smaller than \( n - 1 \). Now \( X_{n-1,n} \sigma \tau \delta \) has lines from \( i \) to \( j \) and from \( n \) to \( n - 1 \). A part from these nodes, \( X_{n-1,n} \sigma \tau \delta \) is the same as \( \sigma \tau \delta \). If this is to be \( \delta \), \( i \) and \( j \) must be in the same row in positions \( 2t+1 \) and \( 2t+2 \). All of the remaining terms of \( X_{n-1,n} \sigma \tau \delta \) are the same as in \( \delta \). We must count the number of pairs \( \sigma, \tau \) that give this. The situation is different if the images of \( n \) and \( n - 1 \) are not in the two rightmost columns. Suppose this for now. In particular, suppose \( 2i+1 \) and \( 2i+2 \) are adjacent in the columns \( s \) and \( s+1 \), where \( s \) is odd and \( s \leq a-3 \). Suppose \( \sigma \) and \( \tau \) are such that \( \sigma \tau \delta \) has lines from the pair \( n-1 \), \( n \) to the pair \( 2i+1 \), \( 2i+2 \), and all other lines are as in \( \delta \). Looking at \( \tau \delta \) we see \( \tau \) that has interchanged the pair in the last two columns of some row, say row \( k \), with a pair \( 2t+1 \), \( 2t+2 \) in the same columns as \( 2i+1 \), \( 2i+2 \). Applying \( \sigma \) to \( \tau \delta \) puts these lines into the proper rows. In particular, \( 2i+1 \) is mapped to \( 2i+2 \) and \( 2i+2 \) is mapped to \( 2i+1 \). Moreover, the points above \( n-1 \), \( n \) in the \( k \)th row are mapped to \( n-1 \), \( n \). There are \( a'-1 \) such pairs of columns. There are two different ways to do each one: interchanging the term in the \( n-1 \)st column with the term in the column \( s \) or with the term in the column \( s+1 \). This could have been from any of the \( b \) columns. All other column permutations must act on the lines together. The number of these not in the column containing \( 2t+1 \), \( 2t+2 \) and the last two is \((b!)a'^{-2}\). The total number of possibilities is \( r(b!)a'^{-2} \).

We now must account for the lines that go from \( n-1 \), \( n \) to a pair \( i \), \( i+1 \), where \( i \) and \( i+1 \) are in the last two columns. We are assuming \( X_{n-1,n} \sigma \tau \delta = \delta \), and \( \sigma \tau \delta \) is the same as \( \delta \), except that there is a line from \( n \), \( n-1 \) to a pair of points in the last two columns. As \( \sigma \tau \delta \) has no vertical lines, \( n \) must be joined to a point in row \( s \), but columns \( a-1 \) and \( n-1 \) must be joined to a point in row \( s \) and column \( a \). As above, \( \tau \) must be in \( R \). Notice that if \( \gamma \) is the transposition interchanging \( n \) with the point in column \( s \) above it, \( \gamma \sigma \tau = \delta \). Let \( S \) be the subgroup of the \( C_\lambda \) that fixes \( \delta \). It follows that \( \sigma \) must be of the form \( \gamma \sigma' \), where \( \sigma' \) is in \( S \). This is in the same coset of \( S \) as the transposition interchanging \( n-1 \) with the point above it in the \( s \)th row. There are \( b-1 \) choices for the row. The size of \( S \) is \((b!)a'\). The contribution here is \( r \cdot (b!)a' \cdot (b-1) \). Here \( \text{sgn}(\sigma) = -1 \), because the sign of elements in \( S \) is even and \( \text{sgn}(\gamma) = -1 \).

Adding all contributions gives

\[
r((b!)a'(Q + 2(a' - 1) - (b - 1))) = r((b!)a'(Q + a - b - 1)).
\]

If \( Q + a - b - 1 = 0 \) we must have \( X_{n-1,n}W = 0 \). If \( Q + a - b - 1 \neq 0 \), \( X_{n-1,n}W \neq 0 \).
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