1. Introduction

In [1], Zadeh introduced the fundamental concept of fuzzy sets, which formed the backbone of fuzzy mathematics. In [2], he introduced yet another very useful concept, namely, fuzzy singletons.

In some previous studies [3–5], the basic concepts and general properties of fuzzy topologies were formulated and investigated. However, due to the lack of a proper fuzzification of point, one was unable to study convergence or local properties in fuzzy topology. Fortunately, Zadeh’s introduction of fuzzy singleton provided the key to this problem. Based on this notion, the concept of fuzzy points is introduced in the present paper and results concerning local countability, separability, and local compactness are obtained. Many subtle, sometimes surprising, departures from the theory of general topology are observed. Consequently, many seemingly simple results need more elaborate proofs.

The introduction of fuzzy points also makes a study of convergence meaningful. However, in this paper we shall limit ourselves to results on convergence in connection with $C_1$ and separable spaces only. No further studies on Moore–Smith convergence will be included. For completeness, a preliminary section is also included.

2. Preliminaries

Let $X = \{x\}$ be a space of points. A fuzzy set $A$ in $X$ is characterized by a membership function $\mu_A(x)$ from $X$ to the unit interval $[0, 1]$. 

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DEFINITION 2.1. Let $A$ and $B$ be fuzzy sets in $X$. Then:

1. $A = B$ if $\mu_A(x) = \mu_B(x)$ for all $x \in X$;
2. $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$;
3. $C = A \cup B$ if $\mu_C(x) = \max[\mu_A(x), \mu_B(x)]$ for all $x \in X$;
4. $D = A \cap B$ if $\mu_D(x) = \min[\mu_A(x), \mu_B(x)]$ for all $x \in X$;
5. $E = A'$ if $\mu_E(x) = 1 - \mu_A(x)$ for all $x \in X$.

More generally, for a family of fuzzy sets, $\mathcal{A} = \{A_i \mid i \in I\}$, the union $C = \bigcup_i A_i$, and the intersection, $D = \bigcap_i A_i$, are defined by

$$\mu_C(x) = \sup_i \{\mu_{A_i}(x)\}, \quad x \in X,$$

$$\mu_D(x) = \inf_i (\mu_{A_i}(x)), \quad x \in X.$$

The symbol $\emptyset$ will be used to denote the empty fuzzy set ($\mu_{\emptyset}(x) = 0$ for all $x$ in $X$). For $X$, we have by definition $\mu_X(x) = 1$ for all $x$ in $X$.

DEFINITION 2.2. A fuzzy topology is a family $\mathcal{T}$ of fuzzy sets in $X$ which satisfies the following conditions:

1. $\emptyset, X \in \mathcal{T}$,
2. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,
3. If $A_i \in \mathcal{T}$ for each $i \in I$, then $\bigcup_i A_i \in \mathcal{T}$.

$\mathcal{T}$ is called a fuzzy topology for $X$, and the pair $(X, \mathcal{T})$ is a fuzzy topological space (fts). Every member of $\mathcal{T}$ is called a $\mathcal{T}$-open fuzzy set (or simply open fuzzy set).

As in general topology, the indiscrete fuzzy topology contains only $\emptyset$ and $X$, while the discrete fuzzy topology contains all fuzzy sets.

DEFINITION 2.3. Let $f$ be a function from $X$ to $Y$. Let $B$ be a fuzzy set in $Y$ with membership function $\mu_B(y)$. Then the inverse of $B$, written as $f^{-1}[B]$, is a fuzzy set in $X$ whose membership function is defined by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \quad \text{for all } x \in X.$$ 

Conversely, let $A$ be a fuzzy set in $X$ with membership function $\mu_A(x)$. The image of $A$, written as $f[A]$, is a fuzzy set in $Y$ whose membership function is given by

$$\mu_{f[A]}(y) = \sup_{x \in f^{-1}[y]} \{\mu_A(x)\} \quad \text{if } f^{-1}[y] \text{ is not empty},$$

$$= 0, \quad \text{otherwise},$$

for all $y$ in $Y$, where $f^{-1}[y] = \{x \mid f(x) = y\}$. 

DEFINITION 2.4. A function from a fts \((X, \mathcal{T})\) to a fts \((Y, \mathcal{U})\) is \(F\)-continuous iff the inverse of each \(\mathcal{U}\)-open fuzzy set is \(\mathcal{T}\)-open.

DEFINITION 2.5. A function from a fts \((X, \mathcal{T})\) to a fts \((Y, \mathcal{U})\) is \(F\)-open iff it maps an open fuzzy set in \((X, \mathcal{T})\) onto an open fuzzy set in \((Y, \mathcal{U})\).

DEFINITION 2.6. Let \(\mathcal{T}\) be a fuzzy topology. A subfamily \(\mathcal{B}\) of \(\mathcal{T}\) is a base for \(\mathcal{T}\) iff each member of \(\mathcal{T}\) can be expressed as the union of some members of \(\mathcal{B}\).

DEFINITION 2.7. Let \(\mathcal{T}\) be a fuzzy topology. A subfamily \(\mathcal{J}\) of \(\mathcal{T}\) is a subbase for \(\mathcal{T}\) iff the family of finite intersections of members of \(\mathcal{J}\) forms a base for \(\mathcal{T}\).

DEFINITION 2.8. Let \((X, \mathcal{T})\) be a fts. A family \(\mathcal{A}\) of fuzzy sets is a cover of a fuzzy set \(B\) iff \(B \subseteq \bigcup \{A \mid A \in \mathcal{A}\}\). It is an open cover iff each member of \(\mathcal{A}\) is an open fuzzy set. A subcover of \(\mathcal{A}\) is a subfamily which is also a cover.

3. LOCAL COUNTABILITY AND SEPARABILITY

We need the concept of fuzzy points.

DEFINITION 3.1. A fuzzy point \(p\) in \(X\) is a fuzzy set with membership function
\[
\mu_p(x) = \begin{cases} 
  y, & \text{for } x = x_0, \\
  0, & \text{otherwise},
\end{cases}
\]
where \(0 < y < 1\). \(p\) is said to have support \(x_0\) and value \(y\).

DEFINITION 3.2. Let \(p\) be a fuzzy point and \(A\) a fuzzy set in \(X\). Then \(p\) is said to be in \(A\) or \(A\) contains \(p\), denoted \(p \in A\), iff \(\mu_p(x) < \mu_A(x)\) for all \(x \in X\).

THEOREM 3.1. If \(A = \bigcup_{i \in I} A_i\), where \(I\) is any index set, then \(p \in A\) iff \(p \in A_i\) for some \(i \in I\).

Proof. (\(\Rightarrow\)) Clear.

(\(\Leftarrow\)) Let the support of \(p\) be \(x_0\). Then
\[
\mu_A(x_0) = \sup_{i \in I} \mu_{A_i}(x_0).
\]
There are two cases: (i) there exists some \( i_0 \in I \), such that \( \mu_{A_{i_0}}(x_0) = \mu_A(x_0) \), and (ii) \( \mu_{A_i}(x_0) < \mu_A(x_0) \) for all \( i \in I \). In case (i), \( p \in A_{i_0} \). In case (ii), \( p \in A \) implies that \( \mu_p(x_0) < \mu_A(x_0) \) and since \( \mu_A(x_0) = \sup_{i \in I} \mu_A(x_0) \) it follows that \( \mu_p(x_0) < \mu_{A_{i_0}}(x_0) \) for some \( i_0 \). Thus \( p \in A_{i_0} \).

In ordinary set theory, Theorem 3.1 is trivial. But in the case of fuzzy set theory, this is not as trivial as one may imagine. In fact, if one replaces the inequality in Definition 3.2 by \( \mu_p(x) \leq \mu_A(x) \), then Theorem 3.1 is no longer true. On the other hand, should we restrict all fuzzy sets to take values 0, 1 and hope that Definitions 3.1 and 3.2 would reduce to the corresponding definitions in ordinary set theory, we should have used \( 0 < y \leq 1 \) and \( \mu_p(x) \leq \mu_A(x) \) instead of \( 0 < y < 1 \) and \( \mu_p(x) < \mu_A(x) \). In other words, our current definitions will not reduce to the ordinary case even if we impose the restriction that all fuzzy sets will take values 0, 1 only. Since Theorem 3.1 is necessary for some of our subsequent developments, we shall adhere to the present definitions.

**Theorem 3.2.** Let \((X, \mathcal{T})\) be a fts. Then a subfamily \( \mathcal{B} \) of \( \mathcal{T} \) forms a base of \( \mathcal{T} \) iff for every member \( A \) of \( \mathcal{T} \) and for every fuzzy point \( p \in A \), there exists a member \( B \) of \( \mathcal{B} \) such that \( p \in B \subseteq A \).

**Proof.** (\( \Rightarrow \)) From the fact that \( A \) is the union of members of \( \mathcal{B} \) and by Theorem 3.1, the result follows.

(\( \Leftarrow \)) Let \( X_1 \) be the subset of \( X \) such that \( \mu_A(x) > 0 \) for \( x \in X_1 \) and \( \mu_A(x) = 0 \) for \( x \in X - X_1 \). Let \( x_0 \in X_1 \), and \( y_0 = \mu_A(x_0) > 0 \). Since \( y_0 \leq 1 \), there exists a sequence of real numbers \( \{y_i\}, i = 1, 2, 3, ... \), such that \( 0 < y_i < y_0 \) for all \( i \) and \( \lim_{i \to \infty} y_i = y_0 \). Define a sequence of fuzzy points \( p_i \) by

\[
\mu_{p_i}(x) = y_i, \quad \text{for} \quad x = x_0, \\
= 0, \quad \text{otherwise}.
\]

Then \( p_i \in A \). By assumption, there exists a member \( B \) of \( \mathcal{B} \) such that \( p_i \in B \subseteq A \). Clearly, the union of all \( B \)'s over all the indices \( i \) and all points \( x_0 \in X_1 \) is exactly \( A \).

**Definition 3.3.** \( A \) fts \((X, \mathcal{T})\) is said to be \( C_\infty \) iff there exists a countable base \( \mathcal{B} \) for \( \mathcal{T} \).

**Definition 3.4.** Let \((X, \mathcal{T})\) be a fts and \( p \) a fuzzy point. A subfamily \( \mathcal{B}_p \) of \( \mathcal{T} \) is called a local base of \( p \) iff \( p \in B \) for every member \( B \) of \( \mathcal{B}_p \), and for every member \( A \) of \( \mathcal{T} \) such that \( p \in A \) there exists a member \( B \) of \( \mathcal{B}_p \) such that \( p \in B \subseteq A \).
Definition 3.5. A fts \((X, \mathcal{F})\) is said to be \(C_1\) iff every fuzzy point in \(X\) has a countable local base.

Theorem 3.3. If \((X, \mathcal{F})\) is \(C_{II}\), then it is \(C_1\).

Proof. Let \(p\) be a fuzzy point in \(X\). By assumption, \(\mathcal{F}\) has a countable base \(\mathcal{B}\). Let \(\mathcal{B}_p\) be the subfamily of \(\mathcal{B}\) defined by \(\mathcal{B}_p = \{B \mid B \in \mathcal{B}, p \in B\}\). Then \(\mathcal{B}_p\) is countable. Let \(A \in \mathcal{F}\) be such that \(p \in A\). By Theorem 3.2, there exists a member \(B\) of \(\mathcal{B}\) such that \(p \in B \subseteq A\). By definition, \(B\) is a member of \(\mathcal{B}_p\). Thus, \((X, \mathcal{F})\) is \(C_1\).

Definition 3.6. A fts \((X, \mathcal{F})\) is said to be separable iff there exists a countable sequence of fuzzy points \(\{p_i\}, i = 1, 2, \ldots\), such that for every member \(A\) of \(\mathcal{F}\) and \(A \neq \emptyset\), there exists a \(p_i\) such that \(p_i \in A\).

Theorem 3.4. If a fts \((X, \mathcal{F})\) is \(C_{II}\), then it is separable.

Proof. By assumption, \(\mathcal{F}\) has a countable base \(\mathcal{B} = \{B_i\}, i = 1, 2, \ldots\). For \(B_i \neq \emptyset\), there exists a point \(x_i \in X\) such that \(\mu_{B_i}(x_i) > 0\). Define a fuzzy point \(p_i\) as follows:

\[
\mu_{p_i}(x) = \begin{cases} \frac{1}{2} \mu_{B_i}(x_i), & \text{for } x = x_i, \\ 0, & \text{otherwise}. \end{cases}
\]

Clearly, \(p_i \in B_i\). The countable sequence \(\{p_i\}, i = 1, 2, \ldots\), is the required sequence for \((X, \mathcal{F})\) to be separable because every member \(A\) of \(\mathcal{F}\) will contain a member of \(\mathcal{B}\), say, \(B \subseteq A\). Consequently, \(p_i \in A\).

In [4], it is proved that a \(C_{II}\) fts is also Lindelöf. Together with the results just obtained, one sees that among the four types of countability properties, namely, \(C_{II}, C_1\), Lindelöf and separable, \(C_{II}\) is the strongest. In [4], it is also shown that the \(F\)-continuous image of a Lindelöf fts is also Lindelöf. Here, we can derive a similar result.

Theorem 3.5. Let \(f\) be an \(F\)-continuous function from a separable fts \((X, \mathcal{F})\) onto a fts \((Y, \mathcal{U})\). Then \((Y, \mathcal{U})\) is also separable.

Proof. Let \(\{p_i\}, i = 1, 2, \ldots\), be a countable sequence of fuzzy points so that for each member \(A\) of \(\mathcal{F}\) there exists some \(p_i\) such that \(p_i \in A\). Note that the family \(\{f[p_i]\}, i = 1, 2, \ldots\), forms a countable sequence of fuzzy points in \(Y\). Let \(B\) be a member of \(\mathcal{U}\). Then \(f^{-1}[B]\) is a member of \(\mathcal{F}\), and hence there exists a fuzzy point \(p_i\) such that \(p_i \in f^{-1}[B]\). Consequently, \(f[p_i] \in B\) since \(f\) is onto. Therefore \((Y, \mathcal{U})\) is separable.

In [5], it is demonstrated that the image of a \(C_{II}\) fts under an \(F\)-continuous and \(F\)-open function is \(C_{II}\). This holds for \(C_1\) space as well.
THEOREM 3.6. Let $f$ be an $F$-continuous function from a $C_1$ fts $(X, \mathcal{T})$ onto a fts $(Y, \mathcal{U})$. If $f$ is also $F$-open, then $(Y, \mathcal{U})$ is $C_1$.

Proof. Let $q$ be a fuzzy point in $Y$. Then $f^{-1}[q]$ is in general a fuzzy set in $X$, not necessarily a fuzzy point. More specifically, if $q$ has support $y_0$ and value $y$, then $f^{-1}[q]$ is the fuzzy set with membership function:

$$
\mu_{f^{-1}(q)}(x) = \begin{cases} 
1 & \text{for all } x \in f^{-1}[y_0], \\
0 & \text{otherwise}.
\end{cases}
$$

Define a fuzzy point $p$ in $X$ by the following membership function:

$$
\mu_p(x) = \begin{cases} 
1 & \text{for } x = x_0, \\
0 & \text{otherwise},
\end{cases}
$$

where $x_0 \in f^{-1}[y_0]$.

Clearly, $p \in f^{-1}[q]$ and $f[p] = q$. By assumption, $p$ has a countable local base, say, $\mathcal{B}_p$. We shall show that the countable family of fuzzy sets $\mathcal{V}_q = \{f[A] \mid A \in \mathcal{B}_p\}$ forms a local base of $q$. First, $f[A]$ is a member of $\mathcal{U}$ for all $A \in \mathcal{B}_p$ since $f$ is $F$-open. Second, let $B$ be a member of $\mathcal{U}$ such that $q \in B$. Then $f^{-1}[B]$ is a member of $\mathcal{T}$ and $p \in f^{-1}[B]$. Thus, there exists a member $A \in \mathcal{B}_p$ such that $p \in A \subset f^{-1}[B]$. Consequently, $q \in f[A] \subset B$, and the result follows.

4. CONVERGENCE, COUNTABILITY, AND SEPARABILITY

The introduction of fuzzy points enables us to discuss convergence in a meaningful way.

DEFINITION 4.1. Let $p_n$, $n = 1, 2, \ldots$, be a sequence of fuzzy points in a fts $(X, \mathcal{T})$ with supports $x_n$, $n = 1, 2, \ldots$. Let $p$ be a fuzzy point with support $x \neq x_n$, for all $n \geq n_0$, where $n_0$ is some number. Then $p_n$ is said to converge to $p$, written $p_n \rightarrow p$, iff for every member $A$ of $\mathcal{T}$ such that $p \in A$, there exists a number $m$, such that $p_n \in A$ for all $n \geq m$.

Note that the restriction on the supports is necessary to make the definition meaningful. Note also that if $p_n$ has value $y_n$ and $p$ has value $y$, in general $p_n \rightarrow p$ does not imply $y_n \rightarrow y$. In fact, we have the following observation. If $p_n \rightarrow p$ and $p$ has support $x_0$ and value $y$, then $p_n \rightarrow q$ for all fuzzy points $q$ with support $x_0$ and value $z \geq y$. In the theory of general topology, we have a similar situation. As a matter of fact, the uniqueness of limits of convergent nets is a characterization of a special type of topological space, namely, Hausdorff space.
DEFINITION 4.2. Let $p$ be a fuzzy point in $(X, \mathcal{F})$ with support $x_0$. Let $A$ be a fuzzy set in $X$. Then $p$ is an accumulation point of $A$ iff for every member $B$ of $\mathcal{F}$ such that $p \in B$, $B \cap A_p \neq \emptyset$, where $A_p$ is the fuzzy set with membership function

$$\mu_{A_p}(x) = 0, \quad \text{for } x = x_0,$$

$$\mu_{A_p}(x), \quad \text{otherwise.}$$

Similar to our previous remarks on convergence, we note that if $p$ is an accumulation point of $A$ and $p$ has support $x_0$ and value $y$, then all fuzzy points $q$ with the same support $x_0$ and value $z \geq y$ are accumulation points of $A$.

LEMMA 4.1. Let $(X, \mathcal{F})$ be a ft. Let $A$ be a fuzzy set and $p$ a fuzzy point in $X$. If there exists a sequence of fuzzy points $p_n$, $n = 1, 2, \ldots$, such that $p_n \in A$ and $p_n \to p$, then $p$ is an accumulation point of $A$.

Proof. Let $B$ be a member of $\mathcal{F}$ such that $p \in B$. By convergence, there exists $m$ such that for all $n \geq m$, $p_n \in B$. On the other hand, $p_n \in A$ and $p_n$ has support different from that of $p$ for all $n \geq n_0$, therefore, $p_n \in A_p$ for all $n \geq n_0$. It follows that $p_n \in B \cap A_p$ for $n \geq \max(n_0, m)$, implying that $B \cap A_p \neq \emptyset$.

LEMMA 4.2. If $(X, \mathcal{F})$ is $C_1$, then for every fuzzy point $p$ in $X$, there exists a countable local base $\mathcal{V} = \{V_i\}$, $i = 1, 2, \ldots$, of $p$ such that $V_1 \supset V_2 \supset V_3 \supset \cdots$.

Proof. By assumption, there exists a countable local base $\mathcal{B} = \{B_i\}$, $i = 1, 2, \ldots$, of $p$. Define

$$V_1 = B_1, \quad V_2 = B_1 \cap B_2, \ldots, \quad V_n = \bigcap_{i=1}^{n} B_i, \ldots$$

Clearly $V_1 \supset V_2 \supset V_3 \supset \cdots$. To show that they form a local base, let $A$ be a member of $\mathcal{F}$ such that $p \in A$. There exists a member of $\mathcal{B}$, say $B_{i_0}$, such that $p \in B_{i_0} \subset A$. By definition of local base, $p \in B_i$, $i = 1, 2, \ldots, i_0$. Thus

$$p \in \bigcap_{i=1}^{i_0} B_i = V_{i_0} \subset B_{i_0} \subset A.$$

Consequently, $\{V_i\}$, $i = 1, 2, \ldots$, forms a local base of $p$.

We can now prove the following theorem.
**Theorem 4.1.** Let $(X, \mathcal{T})$ be a $C_1$ fts. Let $A$ be a fuzzy set and $p$ a fuzzy point in $X$. Then $p$ is an accumulation point of $A$ iff there exists a sequence of fuzzy points $p_n$, $n = 1, 2, \ldots$, such that $p_n \in A$ and $p_n \to p$.

**Proof.** ($\Leftarrow$) It follows from Lemma 4.1.

($\Rightarrow$) By Lemma 4.2, there exists a local base $\{V_i\}$, $i = 1, 2, \ldots$, of $p$ such that $V_1 \supset V_2 \supset V_3 \supset \cdots$. That $p$ is an accumulation point of $A$ implies that $V_i \cap A_p \neq \emptyset$ for all $i$. Let $x_i \in X$ be such that $\mu_{V_i \cap A_p}(x_i) > 0$. Define a fuzzy point $p_i$ by the following membership function:

$$\mu_{p_i}(x) = \frac{1}{3} \mu_{V_i \cap A_p}(x_i), \quad \text{for } x = x_i,$$

$$= 0, \quad \text{otherwise.}$$

Then $p_i \in V_i \cap A_p$ and $p_i$ has different support from that of $p$ for all $i$. Furthermore, $p_i \to p$. To see this, let $B \in \mathcal{T}$ be such that $p \in B$, then there exists $V_m$ such that $p \in V_m \subseteq B$. But $V_m \supset V_i$ for all $i \geq m$, consequently, $p_i \in V_m \subseteq B$ for all $i \geq m$. The result therefore follows.

**Theorem 4.2.** Let $(X, \mathcal{T})$ be a fts. If there exists a countable sequence of fuzzy points $\{p_i\}$, $i = 1, 2, \ldots$, in $X$ such that every fuzzy point $p$ in $X$ is an accumulation point of the fuzzy set $A = \bigcup_i p_i$. Then $(X, \mathcal{T})$ is separable.

**Proof.** Clear.

However, the converse of the above theorem is in general not true, demonstrating yet another departure from general topology. We have the following counter example.

**Theorem 4.3.** There exists a separable fts $(X, \mathcal{T})$ and a fuzzy point $p$ in $X$ such that for any countable sequence $\{p_i\}$, $i = 1, 2, \ldots$, of fuzzy points in $X$, $p$ is not an accumulation point of the union $A = \bigcup_i p_i$.

**Proof.** Let $X$ be a space of points. Let $x_0 \in X$. Let $A_\alpha$, $0 \leq \alpha \leq 1$, be fuzzy sets in $X$ defined by

$$\mu_{A_\alpha}(x) = \alpha, \quad \text{for } x = x_0,$$

$$= 0, \quad \text{otherwise.}$$

Let $\mathcal{T} = \{\emptyset, X, A_\alpha, 0 \leq \alpha \leq 1\}$. Then $(X, \mathcal{T})$ is a fts. Consider the countable sequence of fuzzy points $\{p_\beta\}$ such that the support of each $p_\beta$ is $x_0$ and $\beta$ ranges over the set of rational numbers between 0 and 1. Any member $B$ of $\mathcal{T}$ such that $B \neq \emptyset$ will contain a member of $\{p_\beta\}$. Thus $(X, \mathcal{T})$ is separable. Let $p$ be a fuzzy point with support $x_0$ and value $a_0$, $0 < a_0 < 1$. Then $p$ is not an accumulation point of the union $A$ of any countable fuzzy points since $B \cap A_p = \emptyset$ for all $B \in \mathcal{T}$ containing $p$ and $B \neq X$. 

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5. Local Compactness

The localization of compactness is naturally local compactness, which forms the subject matter of discussion in this section.

**Definition 5.1.** A fts is compact iff each open cover of the space has a finite subcover.

**Definition 5.2.** A fts \((X, \mathcal{F})\) is said to be locally compact iff for every fuzzy point \(p\) in \(X\) there exists a member \(A \in \mathcal{F}\) such that (i) \(p \in A\) and (ii) \(A\) is compact, i.e. each open cover of \(A\) has a finite subcover.

Clearly, each compact fts is locally compact.

The next result demonstrates once more the ramification of fuzzy topology from general topology.

**Theorem 5.1.** A discrete fts \((X, \mathcal{F})\) is not locally compact.

**Proof.** Let \(p\) be a fuzzy point in \(X\) with support \(x_0\) and value \(y, 0 < y < 1\). Let \(A\) be any open fuzzy set such that \(p \in A\). Let \(\mu_A(x)\) be its membership function. Consider the family \(\mathcal{A}\) of all fuzzy sets \(\{A_i\}, i \in I\), such that \(\mu_{A_i}(x) < \mu_A(x)\) for all \(x \in X\). Clearly, \(\mathcal{A}\) forms a cover of \(A\). Since \((X, \mathcal{F})\) is discrete, any fuzzy set is open. \(\mathcal{A}\) is therefore an open cover of \(A\). However, \(\mathcal{A}\) does not have a finite subcover. Thus \((X, \mathcal{F})\) is not locally compact.

Like \(C_1\) fts, we have the following theorem.

**Theorem 5.2.** Let \(f\) be an \(F\)-continuous function from a locally compact fts \((X, \mathcal{F})\) onto a fts \((Y, \mathcal{U})\). If \(f\) is also \(F\)-open, then \((Y, \mathcal{U})\) is locally compact.

**Proof.** Let \(q\) be a fuzzy point in \(Y\) with support \(y_0\) and value \(y\). Define a fuzzy point \(p\) in \(X\) by

\[\mu_p(x) = \begin{cases} y, & \text{for } x = x_0, \\ 0, & \text{otherwise}, \end{cases}\]

where \(x_0 \in f^{-1}[y_0]\). Then \(f[p] = q\). By assumption, there exists a member \(A \in \mathcal{F}\) such that \(p \in A\) and \(A\) is compact. Now \(q \in f[A]\), and \(f[A]\) is a member of \(\mathcal{U}\) since \(f\) is \(F\)-open. Furthermore, \(f[A]\) is compact since \(f\) is \(F\)-continuous. Thus \((Y, \mathcal{U})\) is locally compact.

6. Product and Quotient Spaces

In this section we shall discuss product and quotient spaces generated by the spaces defined so far. We are particularly interested in the conditions for
which the properties of local countability, separability and local compactness are productive and divisible.

Let \( \{X_\alpha\}, \alpha \in I \), be a family of spaces. Let \( X = \prod_{\alpha \in I} X_\alpha \) be the usual product space and let \( P_\alpha \) be the projection from \( X \) onto \( X_\alpha \).

Further assume that each \( X_\alpha \) is a fts with fuzzy topology \( \mathcal{T}_\alpha \). Let \( B \in \mathcal{T}_\alpha \), then by Definition 2.3, \( P_\alpha^{-1}[B] \) is a fuzzy set in \( X \). The family of fuzzy sets \( \mathcal{F} = \{P_\alpha^{-1}[B] \mid B \in \mathcal{T}_\alpha, \alpha \in I\} \) is now used to generate a fuzzy topology \( \mathcal{F} \) for \( X \) in the following manner: Let \( \mathcal{B} \) be the family of all finite intersections of members of \( \mathcal{F} \). Let \( \mathcal{F} \) be the family of all unions of members of \( \mathcal{B} \). It is clear that \( \mathcal{F} \) is indeed a fuzzy topology for \( X \), with \( \mathcal{B} \) as a base and \( \mathcal{F} \) a subbase.

**Definition 6.1.** Given a family of fts \( \{(X_\alpha, \mathcal{T}_\alpha)\}, \alpha \in I \), the fuzzy topology \( \mathcal{F} \) defined as above is called the product fuzzy topology for \( X = \prod_{\alpha \in I} X_\alpha \) and \( (X, \mathcal{F}) \) is called the product fts.

**Theorem 6.1.** Let \( \{(X_\alpha, \mathcal{T}_\alpha)\}, \alpha \in I \), be a countable family of \( C_1 \) fts's. Then the product fts \( (X, \mathcal{F}) \) is \( C_1 \).

**Proof.** Let \( p \) be a fuzzy point in \( X \). Then its projection \( p_\alpha = P_\alpha[p] \) is again a fuzzy point in \( X_\alpha \). By assumption, \( p_\alpha \) has a countable local base \( \mathcal{B}_\alpha = \{B_{\alpha,i}\}, i = 1, 2, \ldots \) For each \( \alpha \), let \( \mathcal{V}_\alpha \) be the family of open fuzzy sets in \( X \) defined by \( \mathcal{V}_\alpha = \{P_\alpha^{-1}[B_{\alpha,i}] \mid B_{\alpha,i} \in \mathcal{B}_\alpha\}. \) Let \( K \) be any finite subset of \( I \). Let \( \mathcal{U}_k \) be the family of open fuzzy sets in \( X \) defined by

\[
\mathcal{U}_k = \left\{ \bigcap_{\alpha \in K} V_\alpha \mid V_\alpha \in \mathcal{V}_\alpha \right\}.
\]

Finally, let \( \mathcal{B}_p \) be the family of all \( \mathcal{U}_k \)'s, where \( K \) ranges over all finite subsets of \( I \). Note that \( \mathcal{B}_p \) is a countable family of open fuzzy sets in \( X \) since the set of all finite subsets of a countable set is countable. All members of \( \mathcal{B}_p \) contain \( p \). We shall next show that \( \mathcal{B}_p \) is a local base of \( p \). To see this, let \( A \) be a member of \( \mathcal{F} \) such that \( p \in A \). Then by definition of the product fuzzy topology \( \mathcal{F} \), and by Theorem 3.2, there exists a base member of \( \mathcal{F} \) of the form

\[
B = \bigcap_{j=1}^n P_{\alpha_j}^{-1}[A_j],
\]

where \( A_j \in \mathcal{F}_\alpha \), such that \( p \in B \subset A \). Since \( p_{\alpha_j} \in A_j \), it follows that there exists a member \( B_{\alpha_j,i_j} \) of \( \mathcal{B}_\alpha \) such that \( p_{\alpha_j} \in B_{\alpha_j,i_j} \subset A_j \). Do this for \( j = 1, 2, \ldots, n \). Then \( p \in B_p \subset B \), where

\[
B_p = \bigcap_{j=1}^n P_{\alpha_j}^{-1}[B_{\alpha_j,i_j}]
\]

is a member of \( \mathcal{B}_p \). Thus the product fts \( (X, \mathcal{F}) \) is \( C_1 \).
Next we shall show that there exist uncountable family of \( C_1 \) spaces, whose product is not \( C_1 \).

**Theorem 6.2.** Let \( \{(X_\alpha, \mathcal{T}_\alpha)\}, \alpha \in I, \) be an uncountable family of \( C_1 \) spaces such that:

(i) none is indiscrete, i.e., for each \( \alpha \in I, \) there exists \( U_\alpha \in \mathcal{T}_\alpha \) such that \( U_\alpha \neq \emptyset, X_\alpha; \)

(ii) for each \( \alpha \in I, \) there exists a fuzzy point \( p_\alpha \in U_\alpha \) such that

\[
P = \bigcap_{\alpha \in I} P^{-1}_{\alpha}[p_\alpha]
\]

is a fuzzy point in \( X; \) and

(iii) in each \( \text{fts}(X_\alpha, \mathcal{T}_\alpha), \) for any \( A \in \mathcal{T}_\alpha \) and \( A \neq \emptyset, \) there exists a point \( x \in X_\alpha \) such that \( \mu_A(x) = 1, \) where \( \mu_A \) is the membership function of \( A. \)

Then the product \( \text{fts}(X, \mathcal{T}) \) is not \( C_1. \)

Remark. Unlike general topology, given fuzzy points \( p_\alpha \) in \( X_\alpha, \alpha \in I, \)

\[
P = \bigcap_{\alpha \in I} P^{-1}_{\alpha}[p_\alpha]
\]

is not always a fuzzy point in \( X; \) it is either a fuzzy point or the empty fuzzy set \( \emptyset. \) For example, let \( I = (0, 1). \) In each \( X_\alpha, \) let \( p_\alpha \) be a fuzzy point with support \( x_\alpha \) and value \( \alpha. \) Then

\[
P = \bigcap_{\alpha \in I} P^{-1}_{\alpha}[p_\alpha] = \emptyset.
\]

**Proof.** Suppose \( (X, \mathcal{T}) \) is \( C_1. \) Then \( p \) has a countable local base \( B_p = \{B_i\}, \)

\( i = 1, 2, \ldots. \) For any \( B_i \in \mathcal{B}_p, \) by definition of product fuzzy topology, there exists a base member

\[
B_0 = \bigcap_{j=1}^{n} P_\alpha^{-1}A_{\alpha_j}, \quad A_{\alpha_j} \in \mathcal{F}_{\alpha_j},
\]

of \( \mathcal{T} \) such that \( p \in B_0 \subset B_i. \) By assumption (iii), \( P_\alpha[B_0] = X_\alpha \) for \( \alpha \neq \alpha_j, \)

\( j = 1, 2, \ldots, n. \) Indeed, let \( x_j \in X_{\alpha_j}, \) such that \( \mu_{A_{\alpha_j}}(x_j) = 1, j = 1, 2, \ldots, n. \)

Let \( x_\alpha \subset X_\alpha. \) Consider the subset

\[
S = \{x_1\} \times \cdots \times \{x_n\} \times \{x_\alpha\} \times \prod_{\beta \neq \alpha_j, \alpha, j=1,2,\ldots,n} X_\beta
\]

of \( X. \) Then \( \mu_{B_\alpha}(s) = 1, \) for all \( s \in S. \) Therefore \( \mu_{P_\alpha[B_0]}(x_\alpha) = 1. \) Since \( x_\alpha \) is arbitrarily chosen, it follows that \( P_\alpha[B_0] = X_\alpha. \) Consequently, \( P_\alpha[B_i] = X_\alpha \)
for $\alpha \neq \alpha_j, j = 1, 2, ..., n$. Do this for all $i$. We obtain a countable subset $K$ of $I$ such that for every $\alpha \in I - K$, $P_\alpha[B_j] = X_\alpha$ for all $i$. Since $I$ is uncountable, there exists $\alpha_0 \in I - K$. Hence $P_{\alpha_0}[B_j] = X_{\alpha_0}$ for all $i$. On the other hand, $p \in P_{\alpha_0}^{-1}[U_{\alpha_0}]$. By definition of local base, there exists $B_i \in \mathcal{B}_\alpha$ such that $p \in B_i \subset P_{\alpha_0}^{-1}[U_{\alpha_0}]$. Therefore, $P_{\alpha_0}[B_i] \subset U_{\alpha_0}$, a contradiction.

**Theorem 6.3.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}, \alpha \in I$, be a countable family of separable spaces. Then the product fts $(X, \mathcal{T})$ is also separable.

**Proof.** In $X_\alpha$, there exists a countable sequence of fuzzy points $\{p_{\alpha i}\}, i = 1, 2, ..., $ such that for each member $A_\alpha \in \mathcal{T}_\alpha$, there is a fuzzy point $p_{\alpha i} \in A_\alpha$. Let $\mathcal{U}_\alpha$ be the family of open fuzzy sets in $X$ defined by $\mathcal{U}_\alpha = \{P_\alpha^{-1}[p_{\alpha i}]\}, i = 1, 2, ...$. Let $K$ be a finite subset of $I$. Let $\mathcal{V}_K$ be the family of open fuzzy sets in $X$ defined by

$$
\mathcal{V}_K = \left\{ \bigcap_{\alpha \in K} V_\alpha \mid V_\alpha \in \mathcal{U}_\alpha \right\}.
$$

Finally, let $\mathcal{V}$ be the collection of all $\mathcal{V}_K$'s, with $K$ ranging over all possible finite subsets of $I$. $\mathcal{V}$ is a countable family of open fuzzy sets in $X$. Every member of $\mathcal{V}$ is of the form $\bigcap_{\alpha \in K} V_\alpha$ and since $K$ is finite, there always exists a fuzzy point, say, $p_\beta \in \bigcap_{\alpha \in K} V_\alpha$. Do this for all members of $\mathcal{V}$. Then $\mathcal{P} = \{p_\beta\}$ is a countable family of fuzzy points in $X$. Furthermore, for every member $A$ of $\mathcal{T}$, there exists an open fuzzy set

$$
B = \bigcap_{j=1}^n P_{\alpha j}^{-1}[A_j],
$$

where $A_j \in \mathcal{T}_{\alpha_j}$, such that $B \subset A$. By construction there exists a fuzzy point $p_\beta$ of $\mathcal{P}$ such that

$$
p_\beta \in \bigcap_{j=1}^n P_{\alpha j}^{-1}[p_{\alpha j}] \subset B \subset A.
$$

Therefore $(X, \mathcal{T})$ is separable.

**Theorem 6.4.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}, \alpha = 1, 2, ..., n$, be a finite family of locally compact fts's. Then the product fts $(X, \mathcal{T})$ is also locally compact.

**Proof.** Let $p$ be a fuzzy point in $X$. Let $p_\alpha = P_\alpha[p]$. By assumption, there exists a member $A_\alpha$ of $\mathcal{T}_\alpha$ such that $p_\alpha \in A_\alpha$ and $A_\alpha$ is compact. The fuzzy set $B = \bigcap_{\alpha=1}^n P^{-1}_\alpha[A_\alpha]$ is a member of $\mathcal{T}$ and $p = \bigcap_{\alpha=1}^n P^{-1}_\alpha[p_\alpha]$ is in it. We have only to show that $B$ is compact. As in [5], define the product $A_1 \times A_2$ of two fuzzy sets $A_1, A_2$ by

$$
\mu_{A_1 \times A_2}(x_1, x_2) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2)).
$$
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Therefore, \( B = A_1 \times A_2 \times \cdots \times A_n \). By exactly the same arguments as used in the proof of Theorem 3.4 of [5], one can show that \( B \) is compact. Therefore \( (X, \mathcal{T}) \) is locally compact.

Next, we shall discuss quotient fuzzy topology.

**Definition 6.2.** Let \( X \) be a space of points. Let \( R \) be an equivalence relation defined on \( X \). Let \( X/R \) be the usual quotient set, and let \( P \) be the usual projection from \( X \) onto \( X/R \).

If \((x, \mathcal{T})\) is a fts, one can define a fuzzy topology in \( X/R \) such that \( P \) is \( F \)-continuous as follows. Let \( \mathcal{U} \) be the family of fuzzy sets in \( X/R \) defined by \( \mathcal{U} = \{ B : P^{-1}[B] \in \mathcal{T} \} \). Then \( \mathcal{U} \) is a fuzzy topology, called the quotient topology for \( X/R \) and \((X/R, \mathcal{U})\) is called the quotient fts. We have the following results.

**Theorem 6.5.** (i) If \((X, \mathcal{T})\) is separable, then the quotient fts \((X/R, \mathcal{U})\) is separable.

(ii) If \((X, \mathcal{T})\) is \( C_1 \) and \( P \) is \( F \)-open, then the quotient fts \((X/R, \mathcal{U})\) is \( C_1 \).

(iii) If \((X, \mathcal{T})\) is locally compact, and \( P \) is \( F \)-open, then the quotient fts \((X/R, \mathcal{U})\) is locally compact.

**Proof.** (i) follows from Theorem 3.5. (ii) follows from Theorem 3.6. (iii) follows from Theorem 5.2.

**References**

2. L. A. Zadeh, A Fuzzy-Set-Theoretic Interpretation of Linguistic Hedges, ERL Memorandum M335, University of California, Berkeley, April 1972. (To appear in Information Sciences.)