Lattice methods for multiple integration *

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Abstract: This paper reviews the use of lattice methods for the approximate integration of smooth periodic functions over the unit cube in any number of dimensions.

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1. Introduction

The objective of this paper is the numerical integration of smooth periodic functions in s dimensions. That is, we consider

$$I(f) = \int_0^1 \ldots \int_0^1 f(x_1, \ldots, x_s) \, \mathrm{d}x_1 \ldots \mathrm{d}x_s = \int_{U^s} f(x) \, \mathrm{d}x, \tag{1.1}$$

where U^s is the s-dimensional unit cube (which for later convenience we take to be open on part of the boundary),

$$U^s = \{ x \in \mathbb{R}^s : 0 \leqslant x_i < 1, 1 \leqslant i \leqslant s \}, \tag{1.2}$$

and f is periodic with period 1 with respect to each coordinate separately, or equivalently

$$f(x + j) = f(x), \qquad j \in \mathbb{Z}^s, \quad x \in \mathbb{R}^s. \tag{1.3}$$

In one dimension the problem reduces to

$$I(f) = \int_0^1 f(x) \, \mathrm{d}x,$$

where f is periodic with period 1. For that problem the favoured method of numerical integration (see, for example, [4, p. 106]) is the trapezoidal rule, or what is equivalent (because of the periodicity of f), the rectangle rule

$$I_N(f) = \frac{1}{N} \sum_{j=0}^{N-1} f\left(\frac{j}{N}\right). \tag{1.4}$$

The justification may be expressed in terms of the Euler–Maclaurin expansion, as in [4], or in terms of a Fourier series argument (see Section 2).

What is the s-dimensional analogue of the 1-dimensional rectangle rule? There is no unique answer, but probably the most obvious generalization is the product-rectangle rule (hereafter called simply the rectangle rule),

$$I_N(f) = \frac{1}{N} \sum_{j_1=0}^{n-1} \cdots \sum_{j_s=0}^{n-1} f\left(\frac{j_1}{n}, \ldots, \frac{j_s}{n}\right), \tag{1.5}$$

in which the total number of points N is given by $N = n^s$. The rectangle rule suffers badly from the 'curse of dimensionality', in that for fixed n the total number of points N rises very rapidly indeed as s increases.

A more interesting generalization is the number-theoretic 'good-lattice' method of Korobov [10] and others [2,7,8,12,17]. (Useful reviews are in [5,6,12,16,18].) In this method one first chooses the total number of points N (with N often taken to be prime, or the product of two primes), and chooses an integer vector $p \in \mathbb{Z}^s$. (The choice of p will be discussed later.) Then $I(f)$ is approximated by

$$I_N(f) = \frac{1}{N} \sum_{j=0}^{N-1} f\left(\left\{j\frac{p}{N}\right\}\right), \tag{1.6}$$

where the braces about a vector indicate that the fractional part of each component is to be taken (with the fractional part lying in $[0, 1)$). For example, for $s = 2$ we obtain, by choosing $N = 5$ and $p = (1, 2)$,

$$I_5(f) = \tfrac{1}{5}\left[f(0, 0) + f(\tfrac{1}{5}, \tfrac{2}{5}) + f(\tfrac{2}{5}, \tfrac{4}{5}) + f(\tfrac{3}{5}, \tfrac{1}{5}) + f(\tfrac{4}{5}, \tfrac{3}{5})\right].$$

As here, we shall always assume that at least one component of p is relatively prime with N, so that the quadrature rule (1.6) uses N distinct points in U^s; in practice this is usually achieved by choosing $p_1 = 1$. the one-dimensional rectangle rule (1.4) is recovered by setting $s = 1$ and $p_1 = 1$.

In this paper we shall call a rule of the form (1.6) a 'single-generator rule', because all the quadrature points are generated by the single vector p/N—a fact that makes the rule remarkably easy to code.

Much of the number-theoretic literature referred to above is concerned with proving that in a certain sense, to be made precise later, there exist 'good lattices', that is good choices of N and p for use in (1.6). These arguments are based on the following simple expression for the error in (1.6): if f has the absolutely convergent Fourier series representation

$$f(x) = \sum_{m \in \mathbb{Z}^s} a(m)\, e^{2\pi i m \cdot x}, \tag{1.7}$$

where $m \cdot x = m_1 x_1 + \cdots + m_s x_s$, then, as is easily shown, the error in (1.6) is

$$I_N(f) - I(f) = \sum_{m \cdot p \equiv 0 (\mathrm{mod}\ N)}{}' a(m), \tag{1.8}$$

where the prime indicates that the $m = 0$ term is to be omitted from the sum. Note that the values of m *not* satisfying the equation $m \cdot p \equiv 0 \pmod{N}$ contribute nothing to the error, the

corresponding Fourier components being integrated exactly by the rule (1.6). Thus the villains of the piece are the non-zero values of m that satisfy the diophantine equation: for them the rule (1.6) gives the wrong answer $I_N(e^{2\pi i m \cdot x}) = 1$, because each term in the sum in (1.6) now equals 1.

The point of view taken here, as in a recent paper of Sloan and Kachoyan [13], is that the rectangle rule (1.5) and the single-generator rule (1.6) are both special cases of a much more general family of 'lattice methods', of the form

$$I_N(f) = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j), \tag{1.9}$$

where $x_1, \ldots, x_N$ are the points of a suitable periodic lattice that lie in U^s. (Precise definitions are given in the next section.) The important point is that every lattice method yields an error expression analogous to (1.8); the precise result, quoted from [13], is stated as Theorem 1. Some of the arguments that have been used in the number-theoretic literature can then be extended to lattice methods in general, and a sketch of that development is given in Section 3. None of this would be of much interest, however, unless the theory leads to the development of interesting lattice methods different from those known already. At the end of the paper some first attempts in this direction, taken from [14], are reported.

2. Lattice methods

A lattice, in the sense required here, is an infinite set S of points in $\mathbb{R}^s$ with the following three properties:
(1) $x, x' \in S \Rightarrow x \pm x' \in S$;
(2) S contains s linearly independent points;
(3) $\exists$ a sphere centred at O that contains no point of S other than O itself.
Of particular interest to us are lattices that have the same periodicity property as f:

Definition [13]. A lattice S is a *multiple-integration lattice* if it contains $\mathbb{Z}^s$ as a sub-lattice.

For each multiple-integration lattice there exists a corresponding lattice method:

Definition [13]. A *lattice method* is a quadrature rule of the form (1.9) in which the points $x_0, \ldots, x_{N-1}$ are all the points of a multiple-integration lattice S that lie in U^s.

Both (1.5) and (1.6) are examples of lattice methods. For the former the corresponding lattice is the rectangular lattice

$$S = \left\{ \left(\frac{j_1}{n}, \ldots, \frac{j_s}{n} \right) : j_i \in \mathbb{Z}, 1 \leqslant i \leqslant s \right\}, \tag{2.1}$$

and for the latter it is

$$S = \left\{ j\frac{p}{N} + k : j \in \mathbb{Z}, \, k \in \mathbb{Z}^s \right\}. \tag{2.2}$$

The following result, proved by group-theoretical arguments in [13], is a generalization of (1.8).

Theorem 1 [13]. *Let I_N be a lattice method, corresponding to the multiple-integration lattice S, and let f have the absolutely convergent Fourier series representation (1.7). Then*

$$I_N(f) - I(f) = \sum_{m \in S^{\perp}}{}' \, a(m),$$

where $S^{\perp}$ is the 'dual' of S.

The 'dual' of a lattice is a concept of geometric number theory [1], which also plays an important role in coding theory [15], X-ray diffraction [9], solid-state physics [9], and now multiple integration.

Definition. The *dual* of the lattice S is

$$S^{\perp} = \{ m \in \mathbb{R}^s : m \cdot x \in \mathbb{Z}, \, \forall x \in S \}. \tag{2.3}$$

The dual of a lattice is a lattice in its own right [1]. The dual of a multiple-integration lattice is easily seen to be a subset of $\mathbb{Z}^s$. Further properties of lattices and duals are given in [1,15] and [13].

Example 1. For the one-dimensional rectangle rule (1.4), the dual of the corresponding lattice is

$$S^{\perp} = N\mathbb{Z}. \tag{2.4}$$

Thus, from Theorem 1, we have

$$I_N(f) - I(f) = \sum_{m \equiv 0 (\mathrm{mod}\ N)}{}' \, a(m).$$

Example 2. For the s-dimensional rectangle rule (1.5), the dual of the corresponding lattice (2.1) is

$$S^{\perp} = (n\mathbb{Z})^s. \tag{2.5}$$

Thus

$$I_N(f) - I(f) = \sum_{\substack{m \in \mathbb{Z}^s \\ m_i \equiv 0 (\mathrm{mod}\ N), \\ 1 \leqslant i \leqslant s}}{}' \, a(m).$$

Example 3. For the single generator rule (1.6) the dual of the corresponding lattice (2.2) is easily seen to be

$$S^{\perp} = \{ m \in \mathbb{Z}^s : m \cdot p \equiv 0 (\mathrm{mod}\ N) \}. \tag{2.6}$$

Thus the error expression (1.8) is recovered.

In general, the construction of the dual of a lattice is an easy task once so-called 'generators' of the lattice are known. (A linearly independent subset $\{a_1, \ldots, a_s\}$ of the lattice S is a set of generators if S consists of the integer linear combinations of $a_1, \ldots, a_s$.) For further details see [15] and [13].

3. Good lattices

Theorem 1 makes it possible to compare theoretically the performance of different lattice methods, once we decide on a suitable class of test functions f. Following Korobov, we define, for each $\alpha > 1$ and each $c > 0$, a class $E_s^\alpha(c)$:

$$E_s^\alpha(c) = \left\{ f : |a(m)| \leqslant \frac{c}{(\overline{m}_1 \ldots \overline{m}_s)^\alpha}, \; m \in \mathbb{Z}^s \right\},$$ (3.1)

where $a(m)$ is the mth Fourier coefficient of f (see (1.7)), and

$$\overline{m} = \begin{cases} m, & \text{if } |m| > 0, \\ 1, & \text{if } m = 0. \end{cases}$$ (3.2)

Then the following result is an immediate consequence of Theorem 1.

Corollary. *If* $f \in E_s^\alpha(c)$, *where* $\alpha > 1$ *and* $c > 0$, *and if* I_N *is a lattice method corresponding to the lattice S, then*

$$|I_N(f) - I(f)| \leqslant c \sum_{m \in S^\perp}{}' \frac{1}{(\overline{m}_1 \ldots \overline{m}_s)^\alpha}.$$

The error bound in the Corollary may be written as

$$|I_N(f) - I(f)| \leqslant c P_\alpha, \quad f \in E_s^\alpha(c),$$ (3.3)

where

$$P_\alpha = \sum_{m \in S^\perp}{}' \frac{1}{(\overline{m}_1 \ldots \overline{m}_s)^\alpha}.$$ (3.4)

This quantity may be used, for fixed α (e.g. $\alpha = 2$), as a quantitative measure of the performance of different lattices—the smaller the value of P_α the better is the lattice, in the sense of giving a better bound in (3.3).

The quantity P_α is the integration error for the particular function $f_\alpha \in E_s^\alpha(1)$ defined by

$$f_\alpha(x) = \sum_{m \in \mathbb{Z}^s} \frac{1}{(\overline{m}_1 \ldots \overline{m}_s)^\alpha} e^{2\pi i m \cdot x},$$ (3.5)

since Theorem 1 gives

$$P_\alpha = I_N(f_\alpha) - I(f_\alpha).$$ (3.6)

The function f_α is a 'worst' function in $E_s^\alpha(1)$, in that for $f = f_\alpha$ and $c = 1$ the inequality (3.3)

becomes an equality. Note that f_α is a product of functions of a single variable,

$$f_\alpha(x) = F_\alpha(x_1) \dots F_\alpha(x_s),$$

where

$$F_\alpha(x) = \sum_{m \in \mathbb{Z}} \frac{1}{\bar{m}^\alpha} e^{2\pi i m x}. \tag{3.7}$$

For $\alpha = 2, 4, \dots$ the function F_α can be obtained explicitly; in particular, it is easily verified that

$$F_2(x) = 1 + 2\pi^2(x^2 - x + \tfrac{1}{6}), \qquad 0 \leqslant x < 1,$$

$$F_4(x) = 1 + \tfrac{1}{45}\pi^4(1 - 30x^2(1 - x)^2), \quad 0 \leqslant x < 1,$$

together with, for all α,

$$F_\alpha(x) = F_\alpha(x + 1), \quad x \in \mathbb{R}.$$

In the number-theoretic literature, a particular choice of N and p in the formula (1.6) is said to give a good lattice, or some such phrase, if it gives sufficiently good behaviour for P_α as $N \to \infty$. For example, Korobov's rather technical definition of a sequence of 'optimal coefficients' (for details see, for example, [5]) has the property that for such a sequence there exist $c_\alpha > 0$ and $\beta > 0$, with β depending only on s, and c_α only on α and s, such that

$$P_\alpha \leqslant c_\alpha \frac{(\log N)^{\alpha\beta}}{N^\alpha}. \tag{3.8}$$

It has been a significant achievement of the number-theorists to show that such sequences exist. However, the proofs are generally not constructive. For example, in the simplest case Korobov restricts N to be prime, takes p to be of the form $p = (1, a, a^2, \dots, a^{s-1})$, for $1 \leqslant a \leqslant N - 1$, and shows that averaging over the allowed values of a already yields the rate of convergence (3.8) for suitable c_α and β; thus the best values of a must do as well or better. To find good coefficients in practice, one usually fixes N, and finds p by minimising P_α or a similar quantity by a more or less exhaustive search.

Zaremba's criterion of a good lattice is expressed slightly differently, in terms of an integer ρ:

$$\rho = \min(\bar{m}_1 \dots \bar{m}_s), \quad \text{subject to } \boldsymbol{m} \cdot \boldsymbol{p} \equiv 0 (\bmod N), \quad \boldsymbol{m} \neq \boldsymbol{0}.$$

According to Zaremba [18], a particular choice of N and p gives a good lattice if

$$\rho > (s - 1)! N / (2 \log N)^{s-1}. \tag{3.9}$$

It is natural to extend the definition of ρ to the case of an arbitrary lattice S; thus we define

$$\rho = \min(\bar{m}_1 \dots \bar{m}_s), \quad \text{subject to } \boldsymbol{m} \in S^\perp, \quad \boldsymbol{m} \neq \boldsymbol{0}. \tag{3.10}$$

Intuitively, one expects that large values of ρ will correspond to small values of P_α, since the largest terms in (3.4) are equal to $1/\rho^\alpha$. Zaremba and others have made this notion precise for the case of single-generator rules by estimating the sum P_α in terms of ρ, the sum now being over the non-zero solutions of $\boldsymbol{m} \cdot \boldsymbol{p} \equiv 0 (\bmod N)$. It turns out that the key point in several of those arguments is that the solutions of the diophantine equation form a lattice. That fact allows some of the arguments to be extended to the dual of an arbitrary multiple-integration lattice S. In that

way the following result, which generalizes a theorem of Hua and Wang [8] for the case of single-generator rules, has been proved in [13]:

Theorem 2 [13]. *Let S be a multiple-integration lattice. Then for* $\alpha > 1$

$$P_\alpha \leqslant d(s, \alpha)\rho^{-\alpha}(1 + \log \rho)^{s-1},$$

where $d(s, \alpha)$ *depends only on s and* α.

The numerical results in subsequent sections show that the size of ρ is a good, but not perfect, guide to the size of P_α. (The intuitive argument above misses one important aspect: there may be a number of terms in (3.4) of the same size $1/\rho^\alpha$, and that number can be markedly different for different kinds of lattice.)

4. Examples

We begin with some two-dimensional examples. In Fig. 1 we show the quadrature points for a good single-generator rule, with $N = 89$ and $p = (1, 55)$, due to Zaremba [17]. That this *is* a good lattice (using the words here in a general sense) is apparent from the dual lattice, shown in Fig. 2: roughly speaking, there are no non-zero points in the dual very close to the origin. From the point of view of the error bound (3.3) the worst points in the dual are those at $\pm(34, 1)$ (which happen to be off the picture). Accordingly, ρ, defined by (3.10), has the value 34. (It is no accident that the numbers 34, 55 and 89 are consecutive Fibonacci numbers. For details of the Fibonacci construction for $s = 2$, see Zaremba [17]. There seems to be no known analogue of this construction for $s > 2$.) The values of P_2 and P_4 obtained with the quadrature points in Fig. 1 are given in the first line of Table 1, and may be considered satisfactorily small.

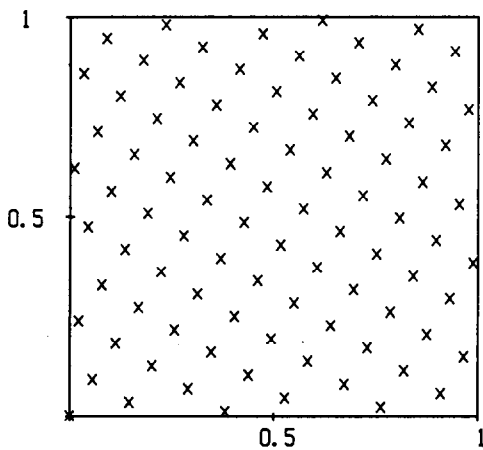The quadrature points for a less good single-generator rule with the same number of points are

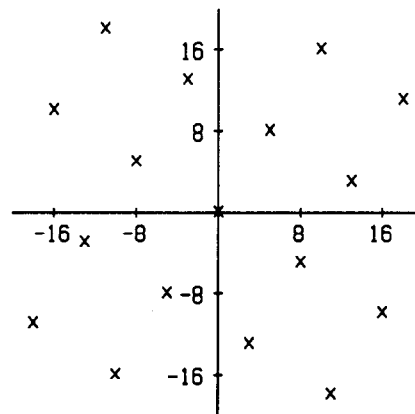Fig. 1. Quadrature points for a good single-generator rule with $N = 89$ and $p = (1, 55)$, due to Zaremba [17].
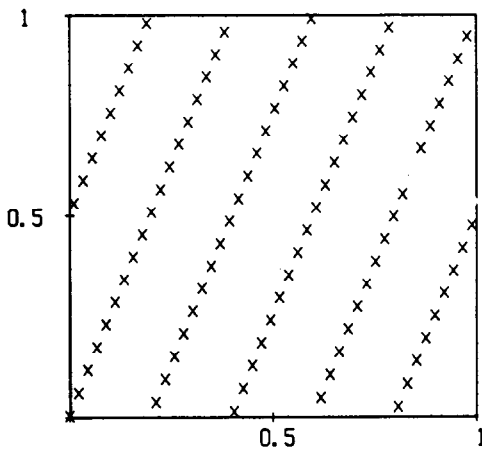
Fig. 2. The dual of the lattice in Fig. 1.

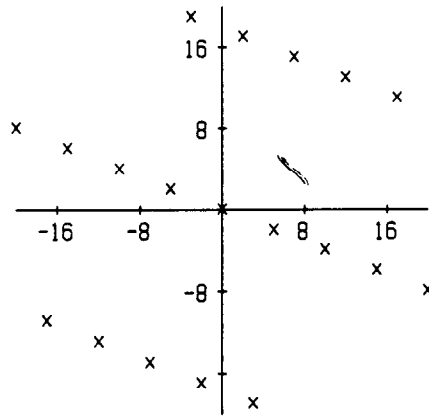Fig. 3. Quadrature points for the single-generator rule with $N = 89$ and $p = (1, 47)$.

Fig. 4. The dual of the lattice in Fig. 3.

shown in Fig. 3—in this case the vector p has been rather arbitrarily taken as $(1,47)$. This time the dual lattice, shown in Fig. 4, has non-zero points closer (in the relevant sense) to the origin: the worst points are at $\pm(5, -2)$, giving $\rho = 10$, which is markedly worse than before. It is therefore no surprise that the corresponding values of P_2 and P_4 in Table 1 are much larger than those for the first lattice.

We also show in Table 1 the result for two rectangular lattices, with $N = 16 = 4^2$ and $N = 64 = 8^2$ respectively. The quadrature points for the first of these are shown in Fig. 5, and the corresponding dual is shown in Fig. 6. In any number of dimensions the worst points in the rectangular dual lattice (2.5) are those at $\pm(n, 0, \ldots, 0)$ and similar points on the other axes, thus the value of ρ is

$$\rho = n = N^{1/s}. \tag{4.1}$$

In particular, $\rho = 4$ and 8 respectively for the two rectangular lattices in Table 1. The latter is a very poor value compared with the value $\rho = 34$ obtained with the good single-generator rule with a comparable number of points. In a sense that poor value of ρ 'explains' the relatively poor values of P_2 and P_4 obtained with the 64-point rectangle rule.

The two remaining entries in Table 1 are explained in the next section.

Table 1.
Results for $s = 2$.

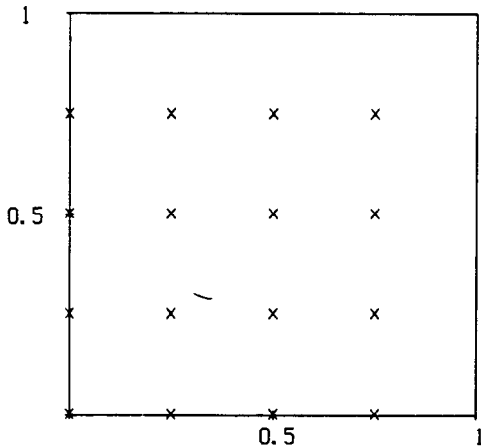| N	lattice	ρ	P_2	P_4
89	$p = (1, 55)$	34	0.016	0.000008
89	$p = (1, 47)$	10	0.032	0.0002
16	rectangular	4	0.45	0.017
64	rectangular	8	0.11	0.001
32	W_{42}	8	0.13	0.0011
64	W_{44}	16	0.04	0.00010

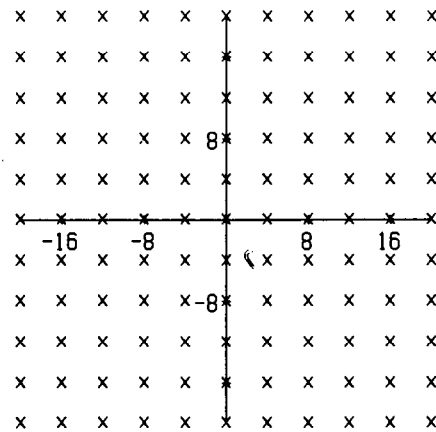Fig. 5. Quadrature points for the rectangle rule with $N = 16 = 4^2$ points.

Fig. 6. The dual of the lattice in Fig. 5.

5. Other lattices

Lattice methods are not in short supply, since many well-understood lattices (see, for example, the tables of Sloane [15]) can be transformed into multiple-integration lattices by a suitable scaling.

Here we follow a different approach, as in [13,14], and ask if the 'worst' points in the dual of the rectangular lattice can be removed by the judicious addition of further quadrature points—the addition of quadrature points adds points to the lattice S, and correspondingly depletes the dual lattice $S^\perp$.

We recall that the worst points in the rectangular dual, from the point of view of the sum (3.4), are the innermost points on each axis. These can all be removed by adding to the rectangular lattice (2.1) one point at the centre of each of the small cubes of side $1/n$: that is, we replace the rectangular lattice (2.1) by the 'body-centred cubic' lattice [13]

$$S = \left\{ \left(\frac{j_1}{n}, \ldots, \frac{j_s}{n} \right) + \frac{k}{2n}(1, \ldots, 1) : j_1, \ldots, j_s, \ k \in \mathbb{Z} \right\}.$$

The total number of points in the corresponding lattice method is $N = 2n^s$; and as the worst points in the dual are now $\pm(2n, 0, \ldots, 0)$ (and similar points on the other axes), we have

$$\rho = 2n = 2^{1-1/s} N^{1/s} \tag{5.1}$$

which is certainly better than (4.1).

We may go further in the same direction, following [14], and introduce the lattice W_{nr}, $1 \leq r \leq n$, defined by

$$S = \left\{ \left(\frac{j_1}{n}, \ldots, \frac{j_s}{n} \right) + \frac{k}{rn}(1, \ldots, 1) : j_1, \ldots, j_s, \ k \in \mathbb{Z} \right\}. \tag{5.2}$$

Table 2
Results for $s = 6$.

N	lattice	ρ	P_2	P_4
2129	$p = (1, 41, \ldots)$	4	2.0	0.019
2187	W_{33}	9	1.5	0.007
15019	$p = (1, 8743, \ldots)$	8	0.2	0.0007
16384	W_{44}	16	0.3	0.0007
71053	$p = (1, 18010, \ldots)$	18	0.033	0.00003
78125	W_{55}	25	0.11	0.0001

The total number of points in the corresponding lattice method is $N = rn^s$, and the value of ρ can be shown to be [14]

$$\rho = rn = r^{1-1/s}N^{1/s}. \tag{5.3}$$

The cases $r = 1$ and 2 give the rectangle and body-centred cubic rules respectively. Another case of particular interest is $r = n$: for the lattice W_{nn} we have the result

$$\rho = n^2 = N^{2/(s+1)}, \tag{5.4}$$

which is markedly better than the result (4.1) for the rectangular lattice.

Numerical results for the two-dimensional lattices W_{42} and W_{44} are shown in Table 1. The quadrature points for the latter are shown in Fig. 7, and the corresponding dual lattice is shown in Fig. 8. Note that W_{44}, with $N = 64$, has $\rho = 16$, a quite respectable value, and correspondingly reasonable values for P_2 and P_4— certainly much better values than those for the 64-point rectangle rule.

We turn now to more serious calculations and higher dimensions. In Table 2 (for $s = 6$) we show values of ρ, P_2 and P_4 for the lattices W_{33}, W_{44} and W_{55}, and also for three single-generator
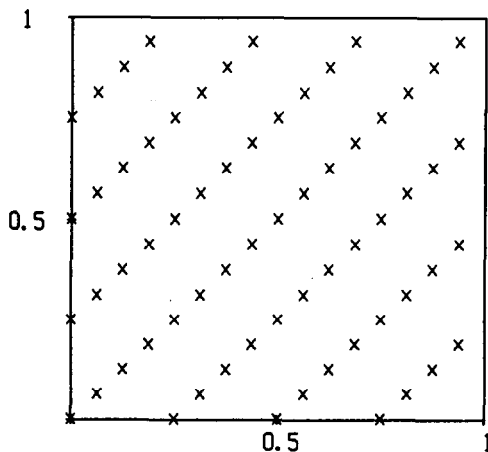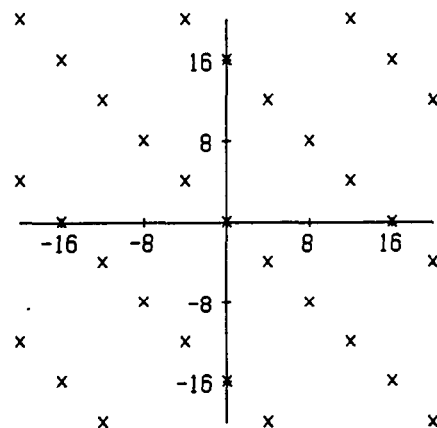

Fig. 7. Quadrature points for W_{44}.

Fig. 8. The dual of the lattice W_{44}.

Table 3
Results for $s = 10$.

N	lattice	ρ	P_4
155093	$p = (1, 90485, \ldots)$	4	0.069
177147	W_{33}	9	0.020

'good lattices' with comparable values of N. (The coefficients and the values of ρ for the latter are taken from the tables of Maisonneuve [11].)

The results in Table 2 for the largest values of N show that a well chosen single-generator rule with a number of points comparable to W_{55} yields better results than W_{55}, at least as judged by P_2 or P_4. On the other hand, for the two smallest values of N the better results are given by W_{33}; and for the two cases in the middle the result is a tie. The better performance of the good single-generator rules for large enough N is, of course, inevitable, given (3.8). However, the results also remind us that for smaller values of N the asymptotic order estimates are not necessarily a reliable guide.

The point is seen even more strikingly in Table 3: there we see that for $s = 10$ the largest single-generator rule in the tables of [11] yields a worse value of P_4 than W_{33} does. Thus the higher order of convergence of the single-generator rules may seem, in this case, rather academic.

6. Discussion

Among the important aspects of lattice methods that we cannot do justice to here are:

(1) Methods for periodizing non-periodic functions. (For a discussion in the context of single-generator rules, see [18,8].)

(2) Extrapolation. (Whereas the convergence of the number-theoretic single-generator rules is erratic, a sequence of rectangle rules, or similarly of the rules W_{nn}, may allow Richardson extrapolation for suitable functions f.)

(3) Error estimation. (The method of Cranley and Patterson [3] can be extended to any lattice method. Alternatively, for regular sequences such as $\{W_{nn}\}$ extrapolation arguments may be useful for estimating the error if s is not too large.)

Table 4
Approximate integrals of the function (6.1).

q	N	lattice	$I_N(f)$
$6/\pi^2$	71053	$p = (1, \ldots)$	1.0026
	78125	W_{55}	1.048
1	71053	$p = (1, \ldots)$	1.033
	78125	W_{55}	1.112
$15/\pi^2$	71053	$p = (1, \ldots)$	1.306
	78125	W_{54}	1.247

(4) Symmetric integrands. (For an integrand with many of the symmetries of the cube, a symmetric rule such as the body-centred cubic rule may become attractive, because of the great reduction that can result in the number of quadrature points, especially in high dimensions. By contrast, the single-generator rules usually have as their *only* symmetry inversion in the centre of the cube, so that the reduction in the number of quadrature points due to symmetry is in this case at most a factor of one half.)

I want to conclude on a note of caution. The use of the numbers P_2 and P_4 to compare different lattices, as we have done in Sections 4 and 5, and which is at the very heart of the notion of good lattice in the number-theoretic literature, can be misleading. We recall that the use of these quantities is based on the error bound (3.3), and that P_α is actually the integration error for a 'worst' function f_α in the class $E_s^\alpha(1)$. The trouble is that the *relative* performance of different lattices can change significantly under apparently quite small changes in the function f.

To help make the point, I show in Table 4 some $s = 6$ results, for the function

$$f^{(q)}(x) = \prod_{j=1}^{6}\left[1 + q\sum_{m\in Z}'\frac{1}{m^2}e^{2\pi imx_j}\right] = \prod_{j=1}^{6}\left[1 + q2\pi^2\left(x_j^2 - x_j + \tfrac{1}{6}\right)\right], \tag{6.1}$$

for three different values of the parameter q. Note that for $q = 1$ the function is just f_2, so we recover results already seen in Table 2. The first value of q is smaller than 1, and so $f^{(q)}$ has some of its Fourier coefficients reduced compared to f^2. (The reduction is by a factor q^l, where $l = l(m)$ is the number of non-zero components of m.) The change is seen to favour the single-generator rule more than it favours W_{55}. Conversely, the third value of q is greater than 1, and this time the change is seen to cause relatively less harm to W_{55} than to the single-generator rule. The different behaviour is understandable in terms of the different nature of the dual lattices in the two cases. (The 'worst' points in the dual of W_{nn} have only one or two non-zero components, whereas for the well chosen single-generator rules the 'worst' points in the dual may have many non-zero components). However, it bedevils any attempt to use a single number to characterise the performance of a lattice method.

In spite of this difficulty of interpretation, the results for W_{nn} in Section 5 do suggest that these lattices might have a role to play for suitable classes of functions f. More importantly, they may encourage the search for other and better lattices. It seems clear that an essential tool for the design of such lattices is the dual lattice: for, in the end, a lattice is only as good as its dual.

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