On positive entire solutions of indefinite semilinear elliptic equations

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1. Introduction

In this paper, we study the elliptic equation

\[ \Delta u + K(x)u^p + \mu f(x) = 0, \quad (1.1) \]

where \( n \geq 3, \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) is the Laplace operator, \( p > 1, \mu \geq 0 \) is a parameter, and \( f \) as well as \( K \) is a given locally Hölder continuous function in \( \mathbb{R}^n \setminus \{0\} \). By an entire solution of Eq. (1.1), we mean a positive weak solution of (1.1) in \( \mathbb{R}^n \) satisfying (1.1) pointwise in \( \mathbb{R}^n \setminus \{0\} \).

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Entire solutions are said to have fast decay, if $|x|^{n-2}u(x)$ is bounded near $\infty$, or to have slow decay in the opposite case. Recent studies in [1–3] have shown the existence of a continuum of entire solutions with slow decay when non-negative $K(x)$ behaves like $|x|^l$ near $\infty$ for some $l \geq -2$ and $uf$ satisfies proper smallness conditions.

The purpose of this paper is to establish the existence without the sign condition on $K$ in any compact region. Our approach begins by considering the homogeneous equation

$$\Delta u + K(x)u^p = 0. \quad (1.2)$$

When $p > \frac{n+2}{n-2}$ for $l > -2$ and $K(x)$ behaves like $|x|^l$ at $\infty$, slowly decaying solutions of (1.2) may have the asymptotic behavior

$$\lim_{|x| \to \infty} |x|^m u(x) = L, \quad (1.3)$$

where $m = \frac{l+2}{p-1}$ and

$$L = L(n, p, l, c) = \left[ m(n-2-m)/c \right]^{\frac{1}{p-1}}. \quad (1.4)$$

Namely, the nonlinearity containing self-similarity at $\infty$ usually leads to the same asymptotic behavior. The natural next step is to check the second asymptotic expansion. Several works in [1,2,4–8, 10–12,15,17] paid special attention to analyzing the asymptotic behavior. As an important byproduct, separation phenomena of solutions with slow decay occur when $p \geq p_c(n, l)$, where

$$p_c = p_c(n, l) = \begin{cases} 
\frac{(n-2)^2 - 2(l+2)(n+1)+2(l+2)\sqrt{(n+l)^2-(n-2)^2}}{(n-2)(n+10-4l)} & \text{if } n > 10 + 4l, \\
\infty & \text{if } n \leq 10 + 4l.
\end{cases}$$

In particular, the asymptotic behavior of solutions can be described in detail by using the two positive real numbers,

$$\lambda_1 = \lambda_1(n, p, l) = \frac{(n - 2 - 2m) - \sqrt{(n - 2 - 2m)^2 - 4(l + 2)(n - 2 - m)}}{2}$$

and $\lambda_2 = \lambda_2(n, p, l) = n - 2 - 2m - \lambda_1$.

Moreover, when the sign of $K$ is non-negative, the viewpoint turns out to be useful in establishing the existence of infinitely many solutions, even a continuum of solutions.

The main results of the paper address this issue without the sign condition of $K$ in any compact region. The first result is on the case $p \geq p_c$.

**Theorem 1.1.** Let $p \geq p_c(n, l)$ with $l > -2$. Assume that $K$ satisfies

\begin{align*}
(K1) \quad & K(x) = O(|x|^\sigma) \text{ at } x = 0 \text{ for some } \sigma > -2, \text{ and} \\
(K2) \quad & |x|^{-1}K(x) = c + d|x|^{-\nu} + O(|x|^{-\lambda_1 (\log |x|)^{-\theta}}) \text{ near } |x| = \infty
\end{align*}

for some $c > 0$, $d \geq 0$, $\nu > 0$ and $\theta > 1$. Then (1.2) possesses a continuum $C$ of positive entire solutions with the asymptotic behavior (1.3). Moreover, there exists an infinite subset $S \subset C$ such that $S$ is at least countable and any two in $S$ do not intersect. In the radial case, there exists $\alpha^* > 0$ such that $C = S = \{ u_\alpha \mid u_\alpha(0) = \alpha \in (0, \alpha^*) \}$.

In Theorem 1.1, we suspect that (K2) can be replaced with $|x|^{-1}K(x) = g(|x|) + O(|x|^{-\lambda_1 (\log |x|)^{-\theta}})$ near $|x| = \infty$ for any decreasing function $g$ converging $c$ at $\infty$. 


When \( K(x) \) behaves like \( c|x|^{-2} \) near \( \infty \) for some \( c > 0 \), positive entire solutions with slow decay may have the logarithmic decay

\[
\lim_{|x| \to \infty} (\log |x|)^{\frac{1}{p-1}} u(x) = L.
\]

(1.5)

where

\[
L = L(n, p, -2, c) = \left[ \frac{n-2}{(p-1)c} \right]^{\frac{1}{p-1}}.
\]

(1.6)

See [13] for the asymptotic behavior. For this case, we establish the following

**Theorem 1.2.** Let \( p > 1 \). Assume that \( K \) satisfies (K1) and

\[
|\alpha|^2 K(\alpha) = c + O\left(\left[\log |\alpha|\right]^{-\theta}\right),
\]

near \( |\alpha| = \infty \) for some constants \( c > 0 \) and \( \theta > 1 \). Then (1.2) possesses a continuum \( C \) of positive entire solutions with the asymptotic behavior (1.5). Moreover, there exists an infinite subset \( \mathcal{S} \subset C \) such that any two in \( \mathcal{S} \) do not intersect. In the radial case, there exists \( \alpha^* > 0 \) such that \( C = \mathcal{S} = \{u_\alpha | u_\alpha(0) = \alpha \in (0, \alpha^*)\} \).

In fact, Theorems 1.1 and 1.2 are known provided that \( K \) is non-negative. See [1,2,6] and [3]. In order to remove the sign condition of \( K \) on compact region, we study the behavior of solutions in compact region and find a way to control it.

With aid of Theorems 1.1 and 1.2, we analyze the effect of the inhomogeneous term in (1.1) and conclude the following assertion.

**Theorem 1.3.** Let \( p \geq p_c(n, l) \) with \( l > -2 \). In addition to the hypotheses of Theorem 1.1, assume that \( f \) satisfies

(\text{F1}) \( f(x) = O(|x|^\tau) \) at \( x = 0 \) for some \( \tau > -2 \),

(\text{F2}) \( -(1 + |x|^{mp}) f(x) \leq K(x) \),

(\text{F3}) \( f(x) = O(|x|^{-\Theta |\log |x|| - \theta}) \) for a constant \( \theta > 1 \).

Then, there exists \( \mu_+ > 0 \) such that for every \( \mu \in [0, \mu_+] \), (1.1) possesses a continuum of positive entire solutions satisfying (1.3). In the radial case, there exists an interval \( (\alpha_\mu, \beta_\mu) \neq \emptyset \) such that for any \( 0 \leq \alpha_\mu < \alpha < \beta < \beta_\mu \), \( 0 < u_\alpha < u_\beta \) in \( \mathbb{R}^n \) where \( u_\eta \) is a radial solution with \( u_\eta(0) = \eta > 0 \).

On the other hand, when \( K \) behaves like \( |\alpha|^{-2} \) near \( \infty \), the result is the following

**Theorem 1.4.** Let \( p > 1 \). In addition to the hypotheses of Theorem 1.2, assume that \( f \) satisfies

(\text{F3}) \( f(x) = O(|x|^\tau) \) at \( x = 0 \) for some \( \tau > -2 \),

(\text{F4}) \( -(\log(2 + |x|))^{\frac{p}{p-1}} f(x) \leq \min_{|z|=|x|} K(z) \), and

(\text{F5}) near \( |x| = \infty \), \( f(x) = O(|x|^{-2}(\log |x|)^{-\theta}) \) for some constant \( \theta > \frac{2p-1}{p-1} \).

Then, there exists \( \mu_+ > 0 \) such that for every \( \mu \in [0, \mu_+] \), (1.1) possesses a continuum of positive entire solutions satisfying (1.5). In the radial case, there exists an interval \( (\alpha_\mu, \beta_\mu) \neq \emptyset \) such that for any \( 0 \leq \alpha_\mu < \alpha < \beta < \beta_\mu \), \( 0 < u_\alpha < u_\beta \) in \( \mathbb{R}^n \) where \( u_\eta \) is a radial solution with \( u_\eta(0) = \eta > 0 \).

We regard (1.2) on compact region as a regular perturbation of harmonic equation when the uniform norm of \( Ku^p + \mu f(x) \) is small enough, and moreover, perceive the assumptions of previous
theorems as monotone properties of $K$ and $f$ near $\infty$. The monotonicity preserves the partial separation which means the separation of solutions in a special class of entire solutions. Each solution in the class is characterized by the second term of the asymptotic expansion at $\infty$. In conclusion, the results of the paper follow from the combination of these two perspectives.

This paper is organized as follows. We review known facts on radial solutions of homogeneous equations in Section 2. In Section 3, we consider local positive radial solutions without the sign condition of $K$. In Section 4, we explain a sufficient condition for the partial separation and derive the existence of singular solutions for the inhomogeneous equations. In Section 5, we establish Theorems 1.1 and 1.2 for the homogeneous equation. In Section 6, we prove Theorems 1.3 and 1.4 and establish the partial separation for the inhomogeneous equation.

2. Preliminaries

In this section, we review known facts. For radially symmetric $K$, a radial solution of (1.2) satisfies the equation

$$u_{rr} + \frac{n-1}{r} u_r + K(r) u^p = 0,$$

(2.1)

where $u(x) = u(|x|)$ and $r = |x|$. It is easy to see that (2.1) with $u(0) = \alpha > 0$, has a unique positive solution $u \in C^2((0, \epsilon)) \cap C([0, \epsilon))$ for small $\epsilon > 0$ under the following condition:

$$(K) \begin{cases} K(r) \text{ is continuous on } (0, \infty), \\ K(r) \geq 0 \text{ and } K(r) \neq 0 \text{ on } (0, \infty), \\ \int_0^\infty r K(r) \, dr < \infty. \end{cases}$$

See Propositions 4.1 and 4.2 in [16]. Let $u_\alpha(r)$ denote the unique local solution with $u_\alpha(0) = \alpha > 0$. As the simplest example satisfying (K), the Lane–Emden equation,

$$\Delta u + c |x|^l u^p = 0$$

in $\mathbb{R}^n$ for $c > 0$ and $l > -2$, admits positive radial solutions with slow decay if and only if $p > \frac{n+2+2l}{n-2}$ (see [9,13,14]). More precisely, the solutions have the asymptotic behavior,

$$\lim_{r \to \infty} r^m u(r) = L,$$

(2.2)

where $m = \frac{14+2}{p-1}$ and $L = L(n, p, l, c)$ is given by (1.4). Furthermore, $W(t) := r^m u - L$ with $t = \log r$, satisfies a second-order equation whose linear part has the characteristic polynomial

$$P(z) := z^2 + (n - 2 - 2m)z + c(p-1)L^{p-1}.$$ 

Observe that $P(z)$ has the two negative real roots, $-\lambda_2 \leq -\lambda_1 < 0$, if and only if $n > 10 + 4l$ and $p > p_c(n, l)$. Hence, it is natural to expect that the exponent $p_c$ is critical in verifying the separation of solutions. It turns out by subsequent works in [4,5,15,17] that if $r^{-1}K(r)$ is non-increasing over $(0, \infty)$, then (2.1) with $p \geq p_c$ has the structure of Type SS: (2.1) possesses a slowly decaying solution $u_\alpha$ for each $\alpha > 0$ (i.e., $u_\alpha(r) > 0$ on $[0, \infty)$ and $r^{n-2}u_\alpha(r) \to \infty$ as $r \to \infty$) and any two of them do not intersect.

**Theorem 2.1.** Let $p > \frac{n+2+2l}{n-2}$ with $l > -2$. Assume (K) and $r^{-1}K(r)$ is non-increasing in $r \in (0, \infty)$.

(i) For $p_c(n, l) > p > \frac{n+2+2l}{n-2}$, if $r^{-1}K(r) \to c > 0$ as $r \to \infty$, then two solutions $u_\alpha$ and $u_\beta$ of (2.1) intersect infinitely many times.
(ii) For $p \geq p_c(n, l)$, (2.1) has the structure of Type SS, and there is a singular solution $U(r)$ such that every positive solution $u_\alpha$ of (2.1) satisfies

$$u_\alpha(r) < U(r) \leq \frac{L(n, p, l, 1)}{|r^2 K(r)|^{\frac{1}{p-1}}}$$

(2.3)

with the convention of $L/0 = \infty$, and $u_\alpha \to U$ as $\alpha \to \infty$. Moreover, $r^m u_\alpha(r)$ is strictly increasing as $r$ increases.

In order to specify the exact asymptotic behavior, we may assume the integrability

$$\int_1^\infty |r^{-l} K(r) - (c + dr^{-v})| r^{p-1} dr < \infty$$

(2.4)

for some $d \geq 0$ and $v > 0$ as the following theorem in [1] shows.

**Theorem 2.2.** Let $p \geq p_c(n, l)$ with $l > -2$. Assume (K), $r^{-1} K(r) \leq c p$ near $\infty$ and (2.4) for some $c > 0$, $d \geq 0$ and $v > 0$. Then, there exists $\alpha^* \in (0, \infty]$ with the property that (2.1) for each $0 < \alpha < \alpha^*$ has an entire solution $u_\alpha$ satisfying (2.2), $u_\beta > u_\alpha > 0$ for $0 < \alpha < \beta < \alpha^*$ and the limit

$$\Phi(\beta, \alpha) := \lim_{r \to \infty} \begin{cases} r^{m+\lambda_1} (u_\beta(r) - u_\alpha(r)) & \text{if } p > p_c, \\ r^{m+\lambda_1} (\log r)^{-1} (u_\beta(r) - u_\alpha(r)) & \text{if } p = p_c, \end{cases}$$

(2.5)

is a continuous and strictly increasing function in $\beta \in (0, \infty)$. In addition, if $r^{-1} K(r)$ is non-increasing in $r \in (0, \infty)$, then $\alpha^* = \infty$.

For each $0 < \alpha < \alpha^*$, (2.1) has a super-solution $u_\alpha^+$ such that $u_\alpha^+ > u_\alpha$ and

$$u_\alpha^+(r) - u_\alpha(r) = O(r^{m-\lambda_2}) \text{ at } \infty.$$ 

On the other hand, when $K(r) = cr^{-2}$ near $\infty$, slowly decaying radial solutions have a logarithmic decay (see [13]). That is the behavior

$$\lim_{r \to \infty} (\log r)^{\frac{1}{p-1}} u(r) = L,$$

(2.6)

where $L = L(n, p, -2, c)$ is given by (1.6). Since (2.4) is reduced to

$$\int_1^\infty |r^2 K(r) - c| r^{-1} dr < \infty$$

(2.7)

as $l \to -2$, we look for solutions satisfying (2.6) under (2.7). A crucial observation in [3] is that proper tools for $l = -2$ are two weights $(\log r)^{\frac{1}{p-1}}$ and $r^{m-2}(\log r)^{-\frac{d}{p-1}}$ which are compared with $r^{m+\lambda_1}$ and $r^{m+\lambda_2}$ for $l > -2$ with respect to their roles in describing the asymptotic behavior.

**Theorem 2.3.** Let $p > 1$. Assume that $K$ satisfies (K) and (2.7) for some $c > 0$, $r^2 K(r) \leq c p$ near $\infty$. Then, there exists $\alpha^* \in (0, \infty]$ with the property that (2.1) for each $0 < \alpha < \alpha^*$ has an entire solution $u_\alpha$ satisfying (2.6), $u_\beta > u_\alpha > 0$ for $0 < \alpha < \beta < \alpha^*$ and the limit

$$\Phi(\beta, \alpha) := \lim_{r \to \infty} (\log r)^{\frac{\mu}{p-1}} (u_\beta(r) - u_\alpha(r))$$

(2.8)

is a continuous and strictly increasing function in $\beta \in (0, \alpha^*)$. 

For each $0 < \alpha < \alpha^*$, (2.1) has a super-solution $u^+\alpha$ such that $u^+\alpha > u_\alpha$ and
\[
u^+\alpha(r) - u_\alpha(r) = O\left(r^{-n} (\log r)^{p-1}\right) \text{ at } \infty.
\]

Here, we use the same notation $\Phi$ in Theorems 2.2 and 2.3 to give emphasis on their similar roles.

We may take $\alpha^* = \infty$ in Theorem 2.3 if under the circumstances of Theorem 2.1(ii), $r^2 K(r) \geq c$ near $\infty$. If not, the existence of $\Phi$ over $(0, \infty)$ requires the stronger integrability than (2.7),
\[
\int_0^\infty \left|r^2 K(r) - c\right| r^{-1} (\log r)^\varepsilon \, dr < \infty
\]
for some $0 < \varepsilon < \frac{1}{2}$ provided that $u_\alpha$ satisfies (2.6) for every $\alpha > 0$. See Theorem 4.5 in [3].

3. Local positive radial solutions

In this section, we consider local existence of radial solutions for the inhomogeneous equation
\[
u_r + \frac{n-1}{r} u_r + K(r) u^p(r) + \mu f(r) = 0,
\]
where $K$ and $f$ are continuous on $(0, R)$ for some $R > 0$ and
\[
0 < \int_0^R \left|K(r)\right| \, dr < \infty, \quad 0 < \int_0^R |f(r)| \, dr < \infty.
\]

Let $u_\alpha(r)$ denote the unique local solution with $u_\alpha(0) = \alpha > 0$ where it exists and belongs to $C^2((0,\varepsilon)) \cap C([0, \varepsilon))$ for small $\varepsilon > 0$. We now give the proof on the existence of local solutions for any $p > 1$.

**Theorem 3.1.** Let $R > 0$ and $1 < \xi < 2$. Assume that continuous functions $K$ and $f$ on $(0, R)$ satisfy (3.2).

(i) There exists $\tilde{\alpha} > 0$ such that for each $0 < \alpha < \tilde{\alpha}$, (3.1) with $\mu = 0$ has a positive radial solution $u_\alpha$ on $(0, R)$ and $(2 - \xi)\alpha \leq u_\alpha(r) \leq \xi \alpha$ on $[0, R]$.

(ii) There exists $\tilde{\mu} > 0$ such that for each $0 < \mu < \tilde{\mu}$ there exist $\alpha_2 > \alpha_1 > 0$ such that for each $\alpha_1 < \alpha < \alpha_2$, (3.1) has a positive radial solution $u_\alpha$ on $(0, R)$ and $(2 - \xi)\alpha \leq u_\alpha(r) \leq \xi \alpha$ on $[0, R]$.

**Proof.** For given $\alpha > 0$ and $1 < \xi < 2$, setting a space
\[
S_R := \{u \in C([0, R]): 0 \leq u \leq \xi \alpha\},
\]
we consider a nonlinear operator $T$ from $S_R$ to $C([0, R])$ by
\[
T(u)(r) := \alpha - T_1(u)(r),
\]
where
\[
T_1(u)(r) := \int_0^r \frac{1}{s^{n-1}} \int_0^s t^{n-1} (K(t) u^p(t) + \mu f(t)) \, dt \, ds, \quad r \in [0, R].
\]
It is easy to check that
\[
\|T_1(u)\| \leq \frac{1}{n-2} \int_0^R t((\xi^p_K)^{\alpha} t + \mu|f(t)|) \, dt. \tag{3.4}
\]

In order to have \(T(S_R) \subset S_R\), we need the inequality
\[
\frac{1}{n-2} \int_0^R t((\xi^p_K)^{\alpha} t + \mu|f(t)|) \, dt \leq (\xi - 1)\alpha. \tag{3.5}
\]

We may regard (3.5) as the inequality of the form
\[
A\alpha^p + B\mu \leq C\alpha. \tag{3.6}
\]

If \(\mu = 0\), (3.6) holds for \(0 < \alpha \leq (\frac{C}{A})^{\frac{1}{p-1}}\). On the other hand, if
\[
0 < \mu < \frac{p-1}{BA^{\frac{1}{p-1}}} \left(\frac{C}{p}\right)^{\frac{1}{p-1}},
\]
there exist \(0 < \alpha_1(\mu) < \alpha_2(\mu) < (\frac{C}{A})^{\frac{1}{p-1}}\) such that (3.6) holds if and only if \(\alpha_1 \leq \alpha \leq \alpha_2\). Combining (3.3) and (3.4), we have \((2 - \xi)\alpha \leq T(u)(r) \leq \alpha\) and \(T(S_R) \subset S_R\). If \(\alpha > 0\) is sufficiently small, we may choose \(0 < \delta < 1\) such that
\[
\|T(u_2) - T(u_1)\| \leq \frac{1}{n-2} \int_0^R t((\xi^p_K)^{\alpha} t + \mu|f(t)|) \, dt \|u_2 - u_1\| \leq \delta \|u_2 - u_1\|.
\]

Since \(\alpha_1(\mu) \to 0\) as \(\mu \to 0\), we may take small \(\alpha > 0\) satisfying (3.5) even for the inhomogeneous equation. Hence, \(T\) is a contraction mapping in \(S_R\) and thus \(T\) has a unique fixed point \(\tilde{u}_\alpha\). In other words, \(u_\alpha\) satisfies
\[
\tilde{u}_\alpha(r) = \alpha - \int_0^r \int_0^s \left(\frac{t}{s}\right)^{n-1} (K(t)u_\alpha^p(t) + \mu f(t)) \, dt \, ds.
\]

Then, it is easy to see that \(\tilde{u}_\alpha\) is also a positive solution of (3.1) on \((0, R)\) with \(\tilde{u}_\alpha(0) = \alpha\). Hence, we have \(\tilde{u}_\alpha = u_\alpha\), which completes the proof. \(\Box\)

**Remark.** For \(r > 0\), we have
\[
r^{n-1}u_\alpha'(r) = - \int_0^r s^{n-1} (K(s)u_\alpha^p(s) + \mu f(s)) \, ds,
\]
\[
|r^{n-1}u_\alpha'(r)| \leq r^{n-2} \int_0^r s((\xi^p_K)^{\alpha} t + \mu|f(s)|) \, ds.
\]
which under (3.2) implies \( \lim_{r \to 0} ru'_\alpha(r) = 0 \). If, in addition,
\[
\lim_{r \to 0} r^{n-1} \int_0^r s^{n-1} K(s) \, ds = \lim_{r \to 0} r^{n-1} \int_0^r s^{n-1} f(s) \, ds = 0,
\]
then \( u'_\alpha(0) = 0 \).

Let \( \mu = 0 \). For given \( \alpha > 0 \), set \( R(\alpha) \) be the supremum of \( R > 0 \), where \( u_\alpha \) exists and remains positive in \( B_R \). We call the range \( (0, R(\alpha)) \) the maximal existence interval of \( u_\alpha \). It follows from (3.6) that \( R(\alpha) \to \infty \) as \( \alpha \to 0 \). Indeed, (3.6) holds for \( \alpha > 0 \) sufficiently small, even if \( A \) in (3.6) is very large. We state the fact separately in the following lemma.

**Lemma 3.2.** Let \( \mu = 0 \). Then, \( R(\alpha) \to \infty \) as \( \alpha \to 0 \).

### 4. Separation of solutions

In this section, we consider separation of solutions of (3.1). In [7], this issue for inhomogeneous equations was studied first. The main point in deriving separation of solutions for (3.1) is the existence of two separated solutions of the homogeneous equation. See Theorem 2.1 in [7]. By \( \bar{u}_\alpha \) with \( \bar{u}_\alpha(0) = \alpha > 0 \), we denote the solution of the equation
\[
ur_\alpha'' + \frac{n-1}{r} ur_\alpha + \bar{K}(r) u^p(r) = 0,
\]
(4.1)
where \( \bar{K} \) satisfies (K). For the sake of completeness, we give the proof.

**Lemma 4.1.** Assume that (3.2) holds and \( K \leq \bar{K} \), and moreover, for some \( \xi > \beta > 0 \) there exist two entire solutions \( \bar{u}_\xi, \bar{u}_\beta \) of (4.1) satisfying \( \bar{u}_\beta(0) = \beta, \bar{u}_\xi(0) = \xi \) and \( 0 < \bar{u}_\beta < \bar{u}_\xi \). If, for \( 0 < \alpha < \eta < \beta \), \( u_\alpha \) and \( u_\eta \) are the local solutions of (3.1) satisfying \( 0 < u_\eta \leq \bar{u}_\beta \) on \( (0, R_\eta) \) for some \( R_\eta > 0 \), then \( u_\alpha < u_\eta \) as long as \( u_\alpha \) remains positive in \( (0, R_\eta) \).

**Proof.** Suppose that \( u_\eta \) meets \( u_\alpha \) at some \( 0 < R < R_\eta \) and \( w_1 := u_\eta - u_\alpha \) is positive in \( [0, R) \). Then, \( w_1 \) satisfies
\[
\begin{aligned}
\Delta w_1 + k_1 w_1 &= 0 \quad \text{in } B_R, \\
w_1 &> 0 \quad \text{in } B_R \text{ and } w_1|_{\partial B_R} = 0,
\end{aligned}
\]
where
\[
k_1 := K \frac{u_\eta^p - u_\alpha^p}{u_\eta - u_\alpha} \leq p \bar{K} u_\eta^{p-1}
\]
in \( B_R \). We note \( w_1'(R) \leq 0 \). On the other hand, we have \( w_2 := \bar{u}_\xi - \bar{u}_\beta > 0 \) in \( [0, \infty) \) and \( w_2 \) satisfies
\[
\Delta w_2 + k_2 w_2 = 0
\]
in \( \mathbb{R}^n \), where
\[
k_2 := K \frac{\bar{u}_\xi^p - \bar{u}_\beta^p}{\bar{u}_\xi - \bar{u}_\beta} > p \bar{K} \bar{u}_\beta^{p-1}.
\]
It follows from Green's identity and (3.2) that
\[
\omega_n R^{n-1} w'_1(R) w_2(R) = \int_{B_R} (w_2 \Delta w_1 - w_1 \Delta w_2) \\
\geq \int_{B_R} (k_2 - k_1) w_1 w_2 > 0,
\]
where \( \omega_n \) denotes the surface area of the unit sphere in \( \mathbb{R}^n \). We reach a contradiction, \( w'_1(R) > 0 \). Hence, \( u_\alpha \) cannot touch \( u_\eta \) in \((0, R_\eta)\). \( \square \)

In general, Lemma 4.1 leads to partial separation, that is, any two solutions in a special set do not intersect. The whole separation needs stronger conditions.

When (4.1) has the structure of Type SS, it follows from Theorem 2.1 in [7] that if (3.1) with \( f \geq 0 \) has a positive entire solution \( u_\alpha \), then \( u_\alpha \leq \bar{u}_\alpha \) and \( u_\alpha < u_\beta < u_\gamma \leq \bar{u}_\gamma \) for any \( \alpha < \beta < \gamma \). The separation of solutions for (3.1) was considered explicitly in [8]. See Theorem 1.1 in [8].

We now turn our attention to the problem of existence of singular solutions. The monotonicity of entire solutions in initial data is useful in verifying the existence as the following theorem shows.

**Theorem 4.2.** Let \( p \geq p_\ast(n, l) \) with \( l > -2 \). Assume that \( K \) satisfies (K) and \( r^{-l} K(r) \) is non-increasing on \((0, \infty)\) while \( f \) is continuous on \((0, \infty)\), \( f \geq 0 \) and \( rf(r) \) is integrable near 0 and \( r^{\frac{p+1}{p-1}} f(r) \) is bounded. Then there exists \( \bar{\mu} > 0 \) with the property that for each \( 0 < \mu < \bar{\mu} \), there exists \( \alpha_\mu > 0 \) such that \( \alpha_\mu \) is increasing in \( \mu \in (0, \bar{\mu}) \) and (3.1) has a positive entire solution \( u_{\mu, \alpha} \) with \( u_{\mu, \alpha}(0) = \alpha \) if and only if \( \alpha \geq \alpha_\mu \), while (3.1) has no positive entire solution for \( \mu > \bar{\mu} \). Moreover, any two solutions of (3.1) do not intersect each other, and for each \( 0 < \mu < \bar{\mu} \) there exists a singular solution \( U_\mu \) which is the monotone upper limit of entire solutions as \( \alpha \uparrow \infty \) and satisfies
\[
U_{\mu, \alpha}(r) < U_\mu(r) \leq \frac{L(n, p, l, 1)}{[r^{2}K(r)]^{\frac{1}{p-1}}}. \tag{4.2}
\]

Furthermore, \( U_\mu \) is monotonically decreasing as \( \mu \) increases to \( \bar{\mu} \).

**Proof.** Let
\[
W := \frac{\eta}{(1 + r^2)^{\frac{2p+1}{2(p-1)}}}
\]
for \( \eta > 0 \) small. Then
\[
F := - (\Delta W + KW^p) = \frac{\eta}{(1 + r^2)^{\frac{2p+1}{2(p-1)}}} \left[ L^{p-1} + \frac{(2p+l)(2+l)}{(p-1)^2(1+r^2)} - \eta^{p-1}(1 + r^2)^{\frac{2l}{2(p-1)}} K \right],
\]
is positive in \( \mathbb{R}^n \) for \( \eta > 0 \) small enough, where \( L = L(n, p, l, 1) \). It is easy to see that \( r^{\frac{2p+1}{p-1}} F \) converges a positive constant as \( r \to \infty \). By applying super- and sub-solutions method to (3.1), we conclude the existence of solutions for \( \mu > 0 \) small. Since \( f \geq 0, \neq 0 \), \( \alpha_\mu \) is positive and increasing in \( \mu > 0 \). The existence of \( \bar{\mu} \) follows from similar arguments as in Proposition 3.3 in [7]. The arguments of Theorem 2.1 in [7] show the separation of solutions. Moreover, we have \( u_{\mu_1, \alpha} \geq u_{\mu_2, \alpha} \) for \( 0 < \mu_1 \leq
\( \mu_2 < \tilde{\mu} \) if \( \alpha > \alpha_{\mu_2} \). Hence, we have \( U_{\mu_1} \geq U_{\mu_2} \). Since \( u_{\mu, \alpha} \leq u_\alpha \) where \( u_\alpha \) is the solution of (2.1), (4.2) follows from (2.3).

Combining (4.2) and the fact that \( r^{-1} K(r) \) is non-increasing, we have

\[
-u'_{\mu, \alpha}(r) = \frac{1}{r^{n-1}} \int_0^r (K(s)u_{\mu, \alpha}(s) + \mu f(s))s^{n-1} \, ds
\]

\[
\leq \frac{L^p}{r^{n-1}} \int_0^r s^{n-1-\frac{2p}{p+1}} K(s) \frac{1}{r^{p-1}} \, ds + \frac{1}{r^{n-1}} \int_0^r \mu f(s)s^{n-1} \, ds
\]

\[
\leq \frac{L^p}{r^{n-1}} r^{\frac{1}{p+1}} K(r) \frac{1}{r^{p-1}} \int_0^r s^{n-1-\frac{2p}{p+1}} - \frac{1}{r^{p-1}} \, ds + \frac{\mu}{r} \int_0^r s f(s) \, ds
\]

\[
= \frac{(p-1)L^p}{[(n-2)p-(n+1)][r^{p+1}K(r)]^{\frac{1}{p+1}}} + \frac{\mu}{r} \int_0^r s f(s) \, ds.
\]

Hence, \( u_{\mu, \alpha}' \) is uniformly bounded on any compact subset of \((0, \infty)\) in \( \alpha \) and consequently, \( \{u_{\mu, \alpha}\} \) is equicontinuous on any compact subset. Since \( u_{\mu, \alpha} \) is monotonically increasing, it follows from the Arzelà–Ascoli Theorem that \( u_\mu(r) := \lim_{\alpha \to \infty} u_{\mu, \alpha}(r) \) is well defined and continuous on \((0, \infty)\). Let \( B_{R, \rho} = \{ r < R = |x| < R \} \). Consider the following boundary problem

\[
\Delta u + K(r)U_\mu^p + \mu f(r) = 0, \quad u|_{\partial B_{R, \rho}} = U_\mu.
\]

For each \( \alpha > \alpha_\mu \), by the maximum principle, we have \( u - u_{\mu, \alpha} > 0 \) and thus, \( u - U_\mu \geq 0 \) in \( B_{R, \rho} \).

Letting \( \phi_\epsilon = \epsilon e^r \), we have \( \Delta(u - u_{\mu, \alpha} + \phi_\epsilon) > 0 \) in \( B_{R, \rho} \) for any fixed \( R, \rho \) and \( \epsilon \) if \( \alpha \) is large enough. Letting \( \alpha \to \infty \) and then \( \epsilon \to 0 \), we have \( u - U_\mu \leq 0 \). Hence, \( u = U_\mu \) in \( B_{R, \rho} \) and \( u = U_\mu \) on \((0, \infty)\).

Therefore, \( U_\mu \) is a singular solution of (3.1) and the proof of Theorem 4.2 is complete. \( \square \)

An interesting question in Theorem 4.2 is to identify the limits of \( u_{\mu, \alpha, \mu} \) and \( U_\mu \) as \( \mu \to \tilde{\mu} \).

5. Homogeneous equation

In this section, we establish the existence of a continuum of positive entire solutions of (2.1) by employing similar arguments to those devised in Section 3 of [6]. Our situation is that \( K \) behaves like \( r^l \) near \( \infty \) for \( l \geq -2 \), but \( K \) may change sign.

We first consider the case \( l > -2 \). When \( K(r) = k(r) \) satisfies the hypotheses of Theorem 2.1(ii) and \( r^{-l} k(r) = c + dr^{-\nu} \) near \( \infty \) for some \( c > 0 \), \( d \geq 0 \), \( \nu > 0 \), we denote the solution of (2.1) with \( u(0) = \alpha > 0 \) by \( \tilde{u}_\alpha \) for this special \( \kappa \). It follows from Theorem 2.2 that for each \( \alpha > 0 \), there exists a positive radial super-solution \( \tilde{u}_\alpha \geq \tilde{u}_\alpha \) of the equation \( \Delta u + \kappa(|x|)u^p = 0 \) satisfying \( F_\alpha(r) := \tilde{u}_\alpha^p(r) - \tilde{u}_\alpha^p(r) = O(r^{-m^{-2}}) \) at \( \infty \) and

\[
\Delta F_\alpha \leq -\kappa(|x|) (\tilde{u}_\alpha^p) - \tilde{u}_\alpha^p \leq -p\kappa(|x|) \tilde{u}_\alpha^{p-1} F_\alpha.
\]

We are now in a position to prove the radial part of Theorem 1.1.
Proposition 5.1. Let \( p \geq p_c(n, l) \) with \( l > -2 \) and \( \kappa(r) = cr^l + dr^{l-v} \) for some \( c > 0, d \geq 0, v > 0 \). Assume that \( K \) is continuous on \((0, \infty)\) and

\[
\int_0^r |K(r)| \, dr < \infty. \tag{5.2}
\]

If \( K \geq 0 \) on \((R_+, \infty)\) for some \( R_+ > 0 \) and satisfies

\[
\int_{R_+}^\infty r^{-1}(K(r) - \kappa(r))_+ r^{-1-\lambda_1} \, dr < \infty \tag{5.3}
\]

and either \( K \leq p\kappa \) on \((R_+, \infty)\),

\[
\int_{R_+}^\infty r^{-1}(K(r) - \kappa(r))_- r^{-1-\lambda_1} \, dr < \infty \tag{5.4}
\]

or

\[
\int_{R_+}^\infty r^{-1}(K(r) - \kappa(r))_+ r^{-1+m-\lambda_1} \, dr < \infty \tag{5.5}
\]

where \( k_\pm = \max(\pm k, 0) \), then there exists a positive constant \( \alpha^* = \alpha^*(p, K) \) such that for each \( \alpha \in (0, \alpha^*) \), (2.1) possesses a positive radial solution \( u_\alpha \) with \( u_\alpha(0) = \alpha \) satisfying (2.2) and any two of them do not intersect.

Proof. By Theorem 3.1(i), for all \( \gamma > 0 \) small, there exists a unique local positive solution \( u_{\gamma} \) of (2.1). We first claim that for given \( \beta > 0 \) small, there exists \( 0 < \tilde{\gamma} = \tilde{\gamma}(\beta) < \beta \) such that for every \( 0 < \gamma < \tilde{\gamma} \), \( u_{\gamma} < \tilde{u}_\beta \) in \( B(R_{\gamma}) \) whenever \( u_{\gamma} > 0 \) in \( B(R_{\tilde{\gamma}}) \) for some \( R_{\tilde{\gamma}} > 0 \).

Suppose for contradiction that for any \( 0 < \gamma < \beta \), there exists \( 0 < \tilde{\gamma} < \gamma \) such that \( u_{\tilde{\gamma}} > 0 \) in \( B(R_{\tilde{\gamma}}) \), \( w_{\tilde{\gamma}}(r) := \tilde{u}_\beta(r) - u_{\tilde{\gamma}}(r) > 0 \) on \([0, R_{\tilde{\gamma}}]\) but \( w_{\tilde{\gamma}}(R_{\tilde{\gamma}}) = 0 \) for some \( R_{\tilde{\gamma}} > 0 \). Then, \( w_{\tilde{\gamma}} \) satisfies

\[
\Delta w_{\tilde{\gamma}} = -\kappa \tilde{u}_\beta^p + K u_{\tilde{\gamma}}^p
\]

in \( B(R_{\tilde{\gamma}}) \). To utilize (5.1), we impose on \( \kappa \) the extra conditions in the above. Fix \( \alpha > \beta \). Applying Green’s identity, we have

\[
0 \leq \int_{B(R_{\tilde{\gamma}})} (w_{\tilde{\gamma}} \Delta F_\alpha - F_\alpha \Delta w_{\tilde{\gamma}})
\]

\[
\leq \int_{B(R_{\tilde{\gamma}})} \left\{ -p\kappa w_{\tilde{\gamma}} \tilde{u}_\beta^{p-1} F_\alpha + \kappa \tilde{u}_\beta^p F_\alpha - K u_{\tilde{\gamma}}^p F_\alpha \right\}
\]

\[
\leq \int_{B(R_{\tilde{\gamma}})} \left\{ -p\kappa w_{\tilde{\gamma}} \tilde{u}_\beta^{p-1} F_\alpha + p\kappa w_{\tilde{\gamma}} \tilde{u}_\beta^{p-1} F_\alpha + (\kappa - K) u_{\tilde{\gamma}}^p F_\alpha \right\}
\]
and
\[
p \int_{B(R_\gamma)} \kappa w_{R_\gamma} [\bar{u}_{R_\gamma}^{p-1} - \bar{u}_{R_\beta}^{p-1}] F_{R_\alpha} \leq \int_{B(R_\gamma)} (\kappa - K) u_{R_\gamma}^{p-1} F_{R_\alpha}.
\]

It follows from Theorem 3.1(i) that for any $\tilde{\gamma} > 0$ small, $\frac{1}{2} \gamma \leq u_{R_\gamma} \leq \frac{3}{2} \gamma$ on $[0, R_+]$. Hence, we may assume that $R_\gamma > R_+$ and $w_{R_\gamma} \geq \frac{1}{2} \bar{u}_\beta(R_+)$ in $B(R_+)$, since $\bar{u}_\beta > 0$ in $\mathbb{R}^n$. Then, for small $\gamma > 0$ and thus, for small $0 < \gamma \leq \gamma$, we have
\[
\frac{p}{2} \bar{u}_\beta(R_+) \int_{B(R_\gamma)} \kappa [\bar{u}_{R_\gamma}^{p-1} - \bar{u}_\beta^{p-1}] F_{R_\alpha} \leq \int_{B(R_\gamma)} (\kappa - K) u_{R_\gamma}^{p-1} F_{R_\alpha}
\leq \int_{B(R_\gamma)} (K - \kappa) \bar{u}_\beta^{p-1} F_{R_\alpha}.
\]

(5.6)

Since $u_{R_\gamma}$ satisfies, for $R_+ < r < R_\gamma$,
\[
u_R(r) = \tilde{\gamma} - \frac{1}{n-2} \int_0^r \left[ 1 - \left( \frac{t}{r} \right)^{n-2} \right] K(t) u_{R_\gamma}^{p-1}(t) dt
\leq \tilde{\gamma} - \frac{1}{n-2} \int_0^{R_+} \left[ 1 - \left( \frac{t}{R_+} \right)^{n-2} \right] K(t) u_{R_\gamma}^{p-1}(t) dt
= u_{R_\gamma}(R_+)
\]

and
\[
u_{R_+}(r) \leq \tilde{\gamma} + \frac{1}{n-2} \left( \frac{3}{2} \gamma \right) \int_0^{R_+} |K(t)| dt,
\]

we have $u_{R_\gamma} = O(\tilde{\gamma})$ in $B(R_\gamma)$. Then, from (2.2), (5.3) and the Dominated Convergence Theorem, the right-hand side of (5.6) goes to 0 as $\tilde{\gamma} \to 0$ while the left-hand side is a fixed positive constant, a contradiction. Therefore, there exists $0 < \gamma < 0$ such that for all $0 < \gamma \leq \gamma$, $0 < u_{R_\gamma} < \bar{u}_\beta$ in $B(R_\gamma)$. Here, we regard $R_\gamma$ as $R(\gamma)$, where $(0, R(\gamma))$ is the maximal existence interval of $u_{\gamma}$. Then, from Lemma 3.2, we see that $R_\gamma \to \infty$ as $\gamma \to 0$.

**Case 1.** Consider the case that $K \leq pk$ on $(R_+, \infty)$. Then, there exists $0 < \gamma_1 \leq \gamma$ such that for all $0 < \gamma < \gamma_1$, $R_\gamma \geq R_+$ and $\frac{1}{2} \gamma \leq u_{R_\gamma}(r) \leq \frac{3}{2} \gamma$ on $[0, R_+]$. Let $J_\beta$ be the set of $0 < \gamma < \gamma_1$ satisfying
\[
p \int_{B(R_\gamma)} \kappa \bar{u}_\beta^{p-1} F_{\beta} > \int_{B(R_\gamma)} K u_{R_\gamma}^{p-1} F_{\beta}.
\]

(5.7)

Then, $J_\beta$ contains an interval, say, $(0, \gamma_2]$. Suppose that $R_\gamma < \infty$ for some $0 < \gamma < \gamma_2$. From Green's identity, it follows that
\[0 \leq \int_{B(R_\gamma)} (u_\gamma \Delta F_\beta - F_\beta \Delta u_\gamma)\]
\[\leq \int_{B(R_\gamma)} \left[ -p\kappa u_\gamma \bar{u}_\beta^{p-1} + Ku_\gamma^p \right] F_\beta\]
\[\leq \int_{B(R_\gamma)} \left[ -p\kappa u_\gamma \bar{u}_\beta^{p-1} + Ku_\gamma^p \right] F_\beta + \int_{B(R_\gamma) \setminus B(R_\gamma)} (K - p\kappa)u_\gamma \bar{u}_\beta^{p-1} F_\beta.\]

Then,
\[
\frac{\gamma p}{2} \int_{B(R_\gamma)} \kappa \bar{u}_\beta^{p-1} F_\beta \leq p \int_{B(R_\gamma)} \kappa u_\gamma \bar{u}_\beta^{p-1} F_\beta \leq \int_{B(R_\gamma)} Ku_\gamma^p F_\beta.
\]

Thus,
\[
\frac{p}{3} \int_{B(R_\gamma)} \kappa \bar{u}_\beta^{p-1} F_\beta \leq \int_{B(R_\gamma)} Ku_\gamma^p F_\beta,
\]

a contradiction. Therefore, \(R_\gamma = \infty\) for all \(0 < \gamma < \gamma_2\), which implies that for every \(0 < \gamma < \gamma_2\), \(u_\gamma\) is an entire solution and thus, \(0 < u_\gamma < \bar{u}_\beta\) in \(\mathbb{R}^n\).

Fix \(0 < \gamma < \gamma_2\). Next, we claim that there exists \(0 < \delta < \gamma\) such that \(\bar{u}_\delta < u_\gamma\) in \(\mathbb{R}^n\) and thus, for every \(0 < \epsilon < \delta\), \(0 < \bar{u}_\epsilon < u_\gamma\) in \(\mathbb{R}^n\).

Suppose that there exist \(\epsilon_j > 0\) going to 0 and \(r_{\epsilon_j} > 0\) going to \(\infty\) as \(j \to \infty\) such that for each \(j \geq 1\), \(0 < \epsilon_j < \gamma\), \(\bar{w}_{\epsilon_j} = u_\gamma - \bar{u}_{\epsilon_j} > 0\) in \(B(r_{\epsilon_j})\) and \(\bar{w}_{\epsilon_j}(r_{\epsilon_j}) = 0\). By Green’s identity,

\[0 \leq \int_{B(r_{\epsilon_j})} (\bar{w}_{\epsilon_j} \Delta F_\beta - F_\beta \Delta \bar{w}_{\epsilon_j})\]
\[\leq \int_{B(r_{\epsilon_j})} \left\{ -p\kappa \bar{w}_{\epsilon_j} \bar{u}_\beta^{p-1} F_\beta + Ku_\gamma^p F_\beta - \kappa \bar{u}_{\epsilon_j}^p F_\beta \right\}\]

and

\[0 \leq \int_{B(r_{\epsilon_j})} \left\{ p\kappa \bar{w}_{\epsilon_j} \bar{u}_\beta^{p-1} F_\beta - \kappa (u_\gamma^p - \bar{u}_{\epsilon_j}^p) F_\beta \right\}\]
\[\leq \int_{B(r_{\epsilon_j})} (K - \kappa)u_\gamma^p F_\beta\]
\[\leq \int_{B(r_{\epsilon_j})} (K - \kappa) \bar{u}_\beta^p F_\beta. \quad (5.8)\]
Since the integrand of (5.8) is positive, it follows by Fatou’s Lemma and the Dominated Convergence Theorem with (5.3) and (5.4) that
\[ 0 \leq \int_{\mathbb{R}^n} [p\kappa u_\gamma \bar{u}_\beta^{p-1} F_\beta - \kappa u_\gamma^{p} F_\beta] \leq \int_{\mathbb{R}^n} (K - \kappa) u_\gamma^{p} F_\beta < \infty. \]

Hence,
\[ \int_{\mathbb{R}^n} (p\kappa \bar{u}_\beta^{p-1} - Ku_\gamma^{p-1}) u_\gamma F_\beta \leq 0, \]
and thus,
\[ \int_{B(R_+)} (p\kappa \bar{u}_\beta^{p-1} - Ku_\gamma^{p-1}) u_\gamma F_\beta \leq 0. \]

Therefore,
\[ \frac{p}{3} \int_{B(R_+)} \kappa \bar{u}_\beta^{p-1} F_\beta \leq \int_{B(R_+)} Ku_\gamma^{p-1} F_\beta, \]
which contradicts (5.7).

**Case 2.** For \( \beta > 0 \), let \( I_\beta \) be the set of \( 0 < \gamma < \bar{\gamma}(\beta) \) satisfying
\[ \frac{p}{6} \int_{B(R_+)} \kappa [\bar{u}_\beta^{p-1} - u_\gamma^{p-1}] F_\beta > \int_{B(R_\gamma)} (K - \kappa) u_\gamma^{p-1} F_\beta. \]

Then, \( I_\beta \supset (0, \gamma_\beta) \) for some \( \gamma_\beta > 0 \) since from (2.2) and (5.5), the right-hand side goes to 0 as \( \gamma \to 0 \) by the Dominated Convergence Theorem while the left-hand side is bounded below a positive constant which is irrelevant to \( \gamma \) when \( \gamma > 0 \) is small.

It follows from Theorem 3.1(i) that there exists \( 0 < \hat{\gamma} \leq \gamma_\beta \) such that for all \( 0 < \gamma < \hat{\gamma} \), \( R_\gamma > R_+ \) and \( \frac{1}{2} \gamma \leq u_\gamma \leq \frac{3}{4} \gamma \) on \( [0, R_+]. \)

We now claim that for small \( 0 < \gamma < \hat{\gamma} \) so that \( \frac{1}{2} \gamma \leq u_\gamma \leq \frac{3}{4} \gamma \) for \( 0 \leq r \leq R_+ \), there exists \( 0 < \eta < \gamma \) such that \( u_\gamma > \bar{u}_\eta \) in \( \mathbb{R}^n \). Suppose that there exists \( 0 < \gamma_1 < \hat{\gamma} \) such that for each \( 0 < \eta < \gamma_1 \), there exists \( r_\eta > 0 \) satisfying \( \hat{\gamma}_\eta(r) = u_{\hat{\gamma}_1}(r) - \bar{u}_\eta(r) > 0 \) in \( [0, r_\eta] \) and \( \hat{\gamma}_\eta(r_\eta) = 0 \). From Green’s identity,
\[ 0 \leq \int_{B(r_\eta)} (\hat{\gamma}_\eta \Delta F_\beta - F_\beta \Delta \hat{\gamma}_\eta) \]
\[ \leq \int_{B(r_\eta)} \left\{ -p\kappa \hat{\gamma}_\eta \bar{u}_\beta^{p-1} F_\beta + K u_\gamma^{p} F_\beta - \kappa \bar{u}_\beta^{p} F_\beta \right\} \]
and
any fixed $R$ increasing in $\alpha$ solutions satisfying (2.6) which are indexed by $\alpha > 0$, we may assume that $r_\eta > R_+$ and $\widetilde{\omega}_\eta(r) \geq \frac{1}{2} \tilde{\gamma}_1 - \tilde{\omega}_\eta(r) \geq \frac{1}{4} \tilde{\gamma}_1$ in $B(R_+)$ if $\eta > 0$ is small enough. Then, we have

\[
\int_{B(R_+)} \kappa \left[ \tilde{u}_\beta^{p-1} - u_{\tilde{\gamma}_1}^{p-1} \right] d\beta \leq \int_{B(R_+)} \left[ \kappa \left( u_{\tilde{\gamma}_1}^{p-1} - \tilde{u}_\beta^{p-1} \right) \right] d\beta
\]

\[
\leq \int_{B(R_+)} (K - \kappa)_{+} u_{\tilde{\gamma}_1}^{p-1} d\beta.
\]

Since $\tilde{u}_\eta$ is monotonically decreasing to 0 as $\eta$ decreases to 0 so that $\tilde{u}_\eta \to 0$ uniformly on $[0, R]$ for any fixed $R > 0$, we may assume that $r_\eta > R_+$ and $\tilde{\omega}_\eta(r) \geq \frac{1}{2} \tilde{\gamma}_1 - \tilde{\omega}_\eta(r) \geq \frac{1}{4} \tilde{\gamma}_1$ in $B(R_+)$ if $\eta > 0$ is small enough. Then, we have

\[
\frac{p}{6} \int_{B(R_+)} \kappa \left[ \tilde{u}_\beta^{p-1} - u_{\tilde{\gamma}_1}^{p-1} \right] d\beta \leq \int_{B(R_+)} (K - \kappa)_{+} u_{\tilde{\gamma}_1}^{p-1} d\beta,
\]

which is impossible because $\tilde{\gamma}_1 \in I_\beta$.

Repeating the preceding arguments, we find a decreasing sequence $\{u_{\tilde{\gamma}_i}\}$ of positive solutions of (2.1) such that there exists a positive decreasing sequence $\{\alpha_i\}$ going to 0 as $i \to 0$ satisfying $u_{\tilde{\gamma}_i} > u_{\alpha_{i+1}}$ in $\mathbb{R}^n$ for each $i \geq 1$. By virtue of Lemma 4.1, we observe the partial separation for small initial data. \(\square\)

We now consider the case $l = -2$. Let $K$ in (2.1) be a continuous positive radial function $\kappa$ entailing $\kappa(r) = cr^{-2}$ near $\infty$. We observe from Theorem 2.3 that (2.1) has a family $\{\tilde{u}_\alpha\}$ of positive radial solutions satisfying (2.6) which are indexed by $\tilde{u}_\alpha(0) = \alpha \in (0, \alpha^*)$ for some $\alpha^* > 0$ and $\tilde{u}_\alpha$ is strictly increasing in $\alpha \in (0, \alpha^*)$. Moreover, for each $\alpha \in (0, \alpha^*)$, there exists a super-solution $\tilde{u}^\alpha_\alpha$ such that $\tilde{u}^\alpha_\alpha > \tilde{u}_\alpha$ and as $r \to \infty$,

\[
\tilde{u}^\alpha_\alpha(r) - \tilde{u}_\alpha(r) = O(r^{2-n} \log r)^{\frac{p}{p-1}},
\]

which plays an important role in establishing the following assertion.

**Proposition 5.2.** Let $p > 1$. Assume that $K$ is continuous on $(0, \infty)$ and satisfies (5.2). If $K \geq 0$ on $(R_+, \infty)$ for some $R_+ > 1$ and satisfies

\[
\int_{R_+} (r^2 K(r) - c)_- r^{-1} dr < \infty
\]

and either $r^2 K(r) \leq cp$ on $(R_+, \infty)$,

\[
\int_{R_+} (r^2 K(r) - c)_+ r^{-1} dr < \infty
\]

or

\[
\int_{R_+} (r^2 K(r) - c)_+ r^{-1} (\log r)^{\frac{1}{p-1}} dr < \infty
\]

for some $c > 0$, then there exists a positive constant $\alpha^* = \alpha^*(p, K)$ such that (2.1) possesses a family $\{u_\alpha\}$, $u_\alpha(0) = \alpha \in (0, \alpha^*)$, of positive radial solutions satisfying (2.6) among which any two do not intersect.

The proof is similar to the proof of Proposition 5.1.
6. Inhomogeneous equation

On the asymptotic behavior of solutions under (2.4), we recall Lemma 4.1 in [6] for \( d = 0 \) and Lemma 3.5 in [2] for some \( d > 0 \), \( 0 < \nu \leq \lambda_1 \).

**Lemma 6.1.** Let \( p \geq p_c \). Assume that \( K_1 \) and \( K_2 \) satisfy (2.4) respectively for some \( d \geq 0 \), \( \nu > 0 \) if \( u_1 < u_2 \) are positive solutions satisfying (2.2) of (2.1) with \( K = K_1 \) and \( K_2 \) near \( \infty \) respectively such that

\[
\varphi(\beta, \alpha)(r) := \begin{cases} 
  r^{m+\lambda_1}(u_2(r) - u_1(r)) & \text{if } p > p_c, \\
  r^{m+\lambda_1}(\log r)^{-1}(u_2(r) - u_1(r)) & \text{if } p = p_c
\end{cases}
\]

is bounded at \( \infty \), then \( \varphi \) converges as \( r \to \infty \).

Hence, for fixed \( K \), we may define

\[
\Phi(\beta, \alpha) := \lim_{r \to \infty} \varphi(\beta, \alpha)(r),
\]

where

\[
\varphi(\beta, \alpha)(r) := \begin{cases} 
  r^{m+\lambda_1}(u_\beta(r) - u_\alpha(r)) & \text{if } p > p_c, \\
  r^{m+\lambda_1}(\log r)^{-1}(u_\beta(r) - u_\alpha(r)) & \text{if } p = p_c
\end{cases}
\]

Furthermore, the continuity of \( \Phi \) follows from Proposition 4.2 in [6] for \( d = 0 \) and Proposition 3.4 in [2] for \( d > 0 \), \( 0 < \nu \leq \lambda_1 \).

**Proposition 6.2.** Let \( p \geq p_c \). Suppose the assumptions of Proposition 5.1. Then, for fixed \( 0 < \beta \leq \alpha^* \), \( \Phi(\beta, \alpha) := \lim_{r \to \infty} \varphi(\beta, \alpha)(r) \) is continuous in \( \alpha \in (0, \alpha^*] \). Moreover, \( \Phi(\beta, \alpha) \to \infty \) as \( \alpha \to 0 \).

In order to establish a continuum of solutions of (1.1), we construct super- and sub-solutions of (2.1) which are characterized by the second term of the asymptotic behavior at \( \infty \). Here, we use the same notations \( \bar{u}_\alpha, \bar{\kappa} \) which are defined just before Proposition 5.1.

**Theorem 6.3.** Let \( p \geq p_c(n, l) \) with \( l > -2 \). Assume that \( K \) and \( f \) satisfy (K1) and (f1) respectively. Suppose there exist radial functions \( H^\pm \) such that

(i) \( H^\pm(r) \geq 0 \), \( H^\pm(r) \in C((0, \infty)) \) and \( \int_0^\infty rH^\pm(r) \, dr < \infty \);

(ii) \( \max(\pm f(\lambda), 0) \leq (1 + |x|^{mp})^{-1}H^\pm(|x|) \);

(iii) \( H^- \leq K^- \) on \( (R_+, \infty) \) for some \( R_+ > 0 \) and

\[
\int_{R_+^\infty} r^{-l}(K^- - H^- - \kappa) - r^{-1-\lambda_1} \, dr < \infty;
\]

(iv) either \( H^+(r) = O(\kappa(r)) \), \( K^+(r) < (\leq) \kappa \) on \( (R_+, \infty) \) (in case \( H^+ \equiv 0 \)),

\[
\int_{R_+^\infty} r^{-l}(K^+ + H^+ - \kappa) + r^{-1-\lambda_1} \, dr < \infty.
\]
or
\[
\int_{R_+}^\infty r^{-l}(K^+ + H^+ - \kappa)_+^{-1+m+\lambda_1} dr < \infty
\]

for some \( c > 0 \), where \( K^-(r) := \inf_{|x|=r} K(x) \), \( K^+(r) := \sup_{|x|=r} K(x) \), \( \lambda_1 = \lambda_1(n, p, l) \). Then, there exists \( \mu_\alpha > 0 \) such that for every \( \mu \in [0, \mu_\alpha) \), (1.1) has a continuum of positive entire solutions with (1.3).

**Proof.** We take the barrier method to construct entire solutions, and thus consider the two problems
\[
v'' + \frac{n-1}{r} v' + (K^\pm \pm H^\pm) v^p = 0 \quad \text{in } (0, \infty), \quad v(0) = \alpha > 0. \tag{6.1}
\]

By \( v_\alpha^\pm \), denote the solutions respectively. Here, we may assume \( K^+ + H^+ \leq pk \) on \((R_+, \infty)\) in the first case of (iv) by taking \( \mu > 0 \) small, and consider only the case that \( K^- - H^- \neq \kappa \neq K^+ + H^+ \) and \( f \neq 0 \) because the other cases can be handled similarly. It follows from the proof of Proposition 5.1 that there exists \( \alpha^* > 0 \) such that for each \( \alpha \in (0, \alpha^*) \), there exist positive entire solutions \( v_\alpha^\pm \) of (6.1) respectively which are increasing as \( \alpha \) increases and below \( \bar{\alpha}_\theta \) for some \( \theta > \alpha^* \). Moreover, for given \( \alpha \in (0, \alpha^*) \), there exist \( 0 < \eta \leq \gamma < \xi < \alpha \) such that
\[
\bar{u}_\eta < v_{\gamma}^- < \bar{u}_\xi < v_\alpha^+ \quad \text{in } \mathbb{R}^n.
\]

Define
\[
\gamma_\alpha = \sup \{ \beta \in (\eta, \alpha): v^-_\beta < v_\alpha^+ \text{ in } \mathbb{R}^n \}.
\]

Obviously, \( v_{\gamma_\alpha}^- \leq v_\alpha^+ \). Then, the strong maximum principle implies that \( v_{\gamma_\alpha}^- < v_\alpha^+ \) in \( \mathbb{R}^n \). By Lemma 6.1, we may set
\[
\Phi^-_\alpha(\gamma_\alpha) := \lim_{r \to \infty} \frac{r^{m+\lambda_1}}{\log r} (v_\alpha^+(r) - v_{\gamma_\alpha}^-(r))
\]

if \( p > p_c \), and
\[
\Phi^+_\alpha(\gamma_\alpha) := \lim_{r \to \infty} \frac{r^{m+\lambda_1}}{\log r} (v_\alpha^+(r) - v_{\gamma_\alpha}^-(r))
\]

if \( p = p_c \). Similarly, we use the notations \( \Phi^\pm(\beta, \alpha) \) defined by \( v_\beta^\pm \) and \( v_\alpha^\pm \). Then, it follows from Proposition 6.2 that \( \Phi^\pm(\beta, \alpha) = 0 \). Indeed, if \( \Phi^\pm(\beta, \gamma_\alpha) > 0 \), then \( v_{\gamma_\alpha}^- < v_\beta^+ \) near \( \infty \). Hence, the continuity of \( \Phi^-(\cdot, \gamma_\alpha) \) implies that there exist \( R > 0 \) and \( \epsilon > 0 \) such that \( 0 < \beta - \gamma_\alpha < \epsilon \) and \( \beta < \alpha \), then \( v_{\beta}^+(r) < v_{\gamma_\alpha}^-(r) \) for \( r \in [R, \infty) \). Since \( v_{\beta}^+ \) is monotonically decreasing to \( v_{\gamma_\alpha}^- \) as \( \beta \) decreases to \( \gamma_\alpha \) and \( v_{\beta}^+ \to v_{\gamma_\alpha}^- \) uniformly on \([0, R] \), there exists \( \gamma_\alpha < \gamma_1 < \beta \) such that \( v_{\gamma_1}^- < v_{\gamma_\alpha}^+ \) in \( \mathbb{R}^n \) which contradicts the definition of \( \gamma_\alpha \).

Fix \( \alpha_1 \in (0, \alpha^*) \). From the proof of Proposition 5.1, there exist \( 0 < \eta_1 < \gamma_{\alpha_1} \) and \( 0 < \eta_2 < \alpha_2 < \frac{\eta_1}{2} \) such that
\[
\bar{u}_{\eta_2} < v_{\gamma_{\alpha_2}}^- < v_{\alpha_2}^+ < \bar{u}_{\eta_1} < v_{\gamma_{\alpha_1}}^- \quad \text{in } \mathbb{R}^n.
\]

Since by Theorem 2.2, \( \Phi(\alpha, \frac{\eta_1}{2}) \) is strictly increasing as \( \alpha \) increases from \( \frac{\eta_1}{2} \) to \( \eta_1 \), we have
\[
\Phi^+(\alpha_1, \alpha_2) = \Phi^-(\gamma_{\alpha_1}, \gamma_{\alpha_2}) \geq \Phi\left(\eta_1, \frac{\eta_1}{2}\right) > 0. \tag{6.2}
\]
By the continuity of $\Phi^+$,
\[
\Phi^+(\alpha_2, \alpha_1, \alpha_2) = [0, \Phi^+(\alpha_1, \alpha_2)].
\]  
(6.3)

We apply (ii) to find $\mu^\pm$ satisfying
\[
\mu^+ f_+ \leq H^+ (v^+_{\alpha_2})^p, \quad \mu^- f_- \leq H^- (v^-_{\gamma_2})^p.
\]
For each $0 \leq \mu \leq \min\{\mu^+, \mu^\pm\}$, we conclude by the barrier method that for every $\alpha \in [\alpha_2, \alpha_1]$, (1.1) possesses a positive entire solution $u_\alpha$ satisfying
\[
v^-_{\gamma_2} < u_\alpha < v^+_{\alpha_2} \text{ in } \mathbb{R}^n,
\]
and (1.3). Consequently, every $u_\alpha$ is characterized by the asymptotic behavior
\[
\lim_{|x| \to \infty} |x|^{m+\lambda_1} (u_\alpha(x) - v^+_{\alpha_2}(|x|)) = \Phi^+(\alpha, \alpha_2)
\]
if $p > p_c$ and
\[
\lim_{|x| \to \infty} \frac{|x|^{m+\lambda_1}}{\log |x|} (u_\alpha(x) - v^+_{\alpha_2}(|x|)) = \Phi^+(\alpha, \alpha_2)
\]
if $p = p_c$. By (6.2) and (6.3), we conclude the existence of a continuum of positive entire solutions of (1.1). □

The existence of a continuum of solutions in Theorem 1.3 follows from Theorem 6.3 by taking
\[
H^{\pm}(|x|) = (1 + |x|^{mp}) F^{\pm}(|x|),
\]
where $F^{\pm}(r) := \max_{|x|=r} f_{\pm}(x)$. The first case in (iv) is applied to derive the existence of a continuum. Under conditions (K2) and (f3), the integral conditions in (iii) and (iv) are satisfied.

We now consider (3.1) in the radial case. We adopt the same notation $u_\alpha$ for the solution of (3.1) with $u_\alpha(0) = \alpha > 0$. For inhomogeneous term $f$, we need the following compatibility conditions.

(fR0) $f(r)$ is continuous on $(0, \infty)$ and $\int_0^r |f(r)| \, dr < \infty$.
(fR1) $-(1 + r^{mp}) f(r) \leq K(r)$.
(fR2) $f(r) = O(r^{-(\lambda_1 + mp)}(\log r)^{-\delta})$ for a constant $\delta > 1$.

Under these conditions, we have the partial separation of solutions for (3.1).

**Theorem 6.4.** In addition to the hypotheses of Theorem 2.2, assume (fR0–2). Then, there exists $\mu^* > 0$ with the property that for fixed $0 < \mu < \mu^*$, there exists an interval $I_\mu = (\alpha_\mu, \beta_\mu)$, $0 < \alpha_\mu < \beta_\mu \leq \infty$, such that for each $\xi \in I_\mu$, (3.1) has an entire solution $u_\xi$ satisfying (2.2). Moreover, any two solutions among them do not intersect, and for each $\alpha \in I_\mu$, the limit $\Phi(\beta, \alpha)$ defined by (2.5) is a continuous and strictly increasing function in $\beta \in I_\mu$.

The arguments in the proof of Theorem 2.2 also work in the proof of the asymptotic behavior in Theorem 6.4 since the inhomogeneous term disappears in the difference function of two solutions of (3.1). Then, the partial separation follows from Lemma 4.1.

In order to prove Theorem 1.4, we need Lemma 4.2 in [3] on the asymptotic behavior of solutions with logarithmic decay.
Lemma 6.5. Assume the hypotheses of Proposition 5.2. Then there exists $\alpha^* > 0$ such that

$$D(\alpha) := \lim_{r \to \infty} \left[ (\log r)^{\frac{p}{p-1}} u_\alpha(r) - L \log r + \frac{pL}{(p-1)^2(n-2)} \log(\log r) \right]$$

is continuous on $(0, \alpha^*)$, where $L = L(n, p, -2, c)$ is given by (1.6).

We may consider the difference function

$$\varphi(u_2, u_1)(r) := (\log r)^{\frac{p}{p-1}} (u_2(r) - u_1(r)),$$

when $D$ is defined for any two solutions $u_1, u_2$. Moreover, we have the strictness of $D$ for fixed equation (2.1) as the following theorem shows. (See Theorem 4.4 in [3].)

Theorem 6.6. Assume $\alpha^* > 0$ is the supremum below which separation of solutions happens, and (2.7) holds for some $c > 0$. If $D(\alpha)$ is defined on $(0, \alpha^*)$, then $D(\alpha)$ is a continuous and strictly increasing function in $\alpha \in (0, \alpha^*)$.

Note that $\Phi(\beta, \alpha) = D(\beta) - D(\alpha)$ where $\Phi(\beta, \alpha)$ is defined by (2.8).

As comparison functions, we utilize the special solutions $\bar{u}_\alpha$ defined just before Proposition 5.2. With aid of Lemma 6.5 and Theorem 6.6, we establish the following

Theorem 6.7. Let $p > 1$. Assume that $K$ and $f$ satisfy (K1) and (f1) respectively. Suppose there exist radial functions $H^{\pm}(r)$ such that

(i) $H^{\pm}(0) \geq 0, H^{\pm}(r) \in C((0, \infty))$, and $\int_0^1 rH^{\pm}(r) dr < \infty$;

(ii) $\max(\pm f(x), 0) \leq [\log(2 + |x|)]^{-\frac{p}{p-1}} H^{\pm}(|x|)$;

(iii) $H^- \leq K^- \in C((0, \infty))$ for some $R^+ > 1$ and

$$\int_{R^+}^{\infty} [r^2(K^- - H^-) - c]_+ r^{-1} dr < \infty;$$

(iv) either $H^+(r) = O(r^{-2}), K^+(r) < (\leq) cpr^{-2}$ on $(R^+, \infty)$ (in case $H^+ \equiv 0$),

$$\int_{R^+}^{\infty} [r^2(K^+ + H^+) - c]_+ r^{-1} dr < \infty,$$

or

$$\int_{R^+}^{\infty} [r^2(K^+ + H^+) - c]_+ r^{-1}(\log r)^{\frac{1}{p-1}} dr < \infty,$$

for some $c > 0$. Then, there exists $\mu_* > 0$ such that for every $\mu \in [0, \mu_*), (1.1)$ possesses a continuum of positive entire solutions with (1.5).

For the radial case, we need the following compatibility condition.

(R3) $-(\log(2 + r))^{\frac{p}{p-1}} f(r) \leq K(r)$ and

(R4) $f(r) = O(r^{-2} \log(r)^{-\delta})$ for a constant $\delta > \frac{2p-1}{p-1}$. 


By the arguments in the proof of Theorem 6.4, we conclude the partial separation for the borderline problem.

**Theorem 6.8.** In addition to the hypotheses of Theorem 2.3, assume \((fR0), (fR3–4)\). Then, there exists \(\mu^* > 0\) with the property that for fixed \(0 < \mu < \mu^*\), there exists an interval \(I_\mu = (\alpha_\mu, \beta_\mu), 0 < \alpha_\mu < \beta_\mu \leq \infty\), such that for each \(\xi \in I_\mu\), (3.1) has an entire solution \(u_\xi\) satisfying (2.6). Moreover, any two solutions among them do not intersect, and the limit \(\Phi(\beta, \alpha)\) defined by (2.8) is a continuous and strictly increasing function in \(\beta \in I_\mu\).

**References**