An Index Classification of M-Matrices

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ABSTRACT

An M-matrix as defined by Ostrowski [5] is a matrix that can be split into $A = sI - B$, where $s > 0$, $B > 0$, with $s > r(B)$, the spectral radius of $B$. Following Plemmons [6], we develop a classification of all M-matrices. We consider $v$, the index of zero for $A$, i.e., the smallest nonnegative integer $n$ such that the null spaces of $A^n$ and $A^{n+1}$ coincide. We characterize this index in terms of convergence properties of powers of $s^{-1}B$. We develop additional characterizations in terms of nonnegativity of the Drazin inverse of $A$ on the range of $A^n$, extending (as conjectured by Poole and Boullion [7]) the well-known property that $A^{-1} > 0$ whenever $A$ is nonsingular.

1. INTRODUCTION

Let $A \in R^{S \times S}$. We write $A \geq 0$ ($A > 0$) if $A_{ij} \geq 0$ ($A_{ij} > 0$) for all $i,j = 1, \ldots, S$. We write $A > 0$ if $A > 0$ and $A \neq 0$. Similar notation applies to vectors. Also, let $r(A)$ denote the spectral radius of $A$.

A real matrix $A \in R^{S \times S}$ is called an M-matrix if $A$ can be split into $A = sI - B$, where $s > 0$, $B > 0$ with $s > r(B)$. Such matrices were introduced by Ostrowski [5] and arise in investigations concerning the convergence of iterative processes for systems of linear and nonlinear equations and in the study of nonnegative solutions to such systems. These investigations have various applications in problems in economics and linear programming. An extensive list of references to studies of M-matrices can be found in [7] and [6].

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Nonsingular $M$-matrices form a proper subclass of the set of monotone matrices, i.e., the class of matrices $A$ for which $A^{-1} > 0$. Monotone matrices are characterized by the condition that $x > 0$ whenever $Ax > 0$. We next summarize some well-known useful characterizations of singular $M$-matrices (e.g., [2] or [6, Lemma 1]).

**Proposition 1.** Let $A = (A_{ij})$ have the property that $A_{ij} \leq 0$ for all $i \neq j$. Then the following statements are equivalent:

1. $A = sI - B$ for some $B > 0$ and $s > r(B)$, i.e., $A$ is a nonsingular $M$-matrix.\(^1\)
2. The real part of each eigenvalue of $A$ is positive.
3. All principal minors of $A$ are positive.
4. $A^{-1}$ exists, and $A^{-1} > 0$.
5. $Ax > 0$ implies $x > 0$.
6. $Ax \gg 0$ for some $x \gg 0$.

A standard generalization of these results to arbitrary $M$-matrices (which are not necessarily nonsingular) yields considerably weaker results, namely (e.g., [3] or [6, Lemma 2]):

**Proposition 2.** Let $A = (A_{ij})$ have the property that $A_{ij} < 0$ for all $i \neq j$. Then the following conditions are equivalent:

1. $A = sI - B$ for some $B > 0$ and $s > r(B)$; i.e., $A$ is an $M$-matrix.
2. The real part of each nonzero eigenvalue of $A$ is positive.
3. All principal minors of $A$ are nonnegative.

Recently Plemmons [6] obtained a strong generalization of parts (4) and (5) of Proposition 1 for a class of $M$-matrices which strictly contains the set of nonsingular $M$-matrices. Unfortunately, this class does not include the set of all singular $M$-matrices. In particular, it was shown in Plemmons [6, Theorem 3] that

**Proposition 3 (Plemmons).** Let $A = (A_{ij})$ have the property that $A_{ij} \leq 0$ for all $i \neq j$. Then the following conditions are equivalent:

1. $A = sI - B$ for some $B > 0$ and $s > r(B)$, where $(s^{-1}B)^N$ converges as $N \to \infty$.
2. The real part of each nonzero eigenvalue of $A$ is positive, and all the elementary divisors associated with a zero eigenvalue of $A$ are linear.

\(^1\)The equivalence of the nonsingularity of $A$ to assuming that $r(B) < s$ follows from the Perron-Frobenius theorem (e.g., [13, p. 46]).
(3) All the principal minors of $A$ are nonnegative, and all the elementary divisors associated with a zero eigenvalue of $A$ are linear.

(4) The group inverse\(^2\) $A^\#$ of $A$ exists, and $A^\# x \geq 0$ for every $x > 0$ where $x$ belongs to the range of $A$.

(5) If $Ax > 0$ and $x$ is in the range of $A$, then $x > 0$.

The purpose of this paper is to extend Plemmons's results by giving a strong generalization to Proposition 1 which holds for the class of all M-matrices. In particular, this will lead to a study of several properties of singular M-matrices. As pointed out by Plemmons [6, Sec. 3], such matrices play a key role in the study of rank deficient splittings and the convergence of iterative methods for solving singular linear equations.

In Sec. 2, we summarize a few spectral properties of square matrices and introduce some notational conventions. We then, in Sec. 3, develop a classification of M-matrices $A$ by considering $v_0(A)$, the index of zero at $A$, i.e., the least nonnegative integer $n$ such that the null spaces of $A^n$ and $A^{n+1}$ coincide. We then characterize this classification in terms of convergence properties of $s^{-N}B^N$ as $N \to \infty$, where $A = sI - B$ is a representation of $A$ with $B > 0$ and $s = \max A_{ii}$.\(^3\) This characterization is obtained by using the asymptotic expansion of partial sums of matrix powers developed in [11].

In Sec. 4, we obtain the promised strong generalization of Proposition 1 by characterizing, for every $k = 0, 1, \ldots$, the set of M-matrices with $v_0(A) < k$ within the class of square matrices with nonpositive off diagonal elements. In particular, we obtain an analogue of conditions (4)–(6) of Proposition 1. We show that a matrix $A$ with nonpositive off diagonal elements is an M-matrix with $v = v_0(A) < k$ if and only if the Drazin inverse\(^4\) of $A$, $A^D$, has the property that $A^D x \geq 0$ for every $x > 0$ where $x \in \text{range } A^\infty$, or equivalently $x > 0$ for all $x$ in range $A^\infty$ with $Ax > 0$. We thereby establish a conjecture of Poole and Boullion [7] that (4) in Proposition 1 can be generalized to nonnegativity of the Drazin inverse of a singular M-matrix. A slightly sharper characterization of M-matrices which are symmetric is given.

2. NOTATIONAL CONVENTIONS

The real line (complex field) will be denoted $R$ ($C$). Let $S$ be a fixed positive integer. Throughout this paper we consider $S \times S$ matrices, both real and complex. The null (the range) space of an $S \times S$ matrix $B$ will be denoted

\(^2\)The group inverse of a square matrix $A$ is defined in Sec. 2.

\(^3\)Throughout this paper, $\max A_{ii}$ denotes the maximum of $A_{ii}$, taken over the complete range of $i$ (usually $1 < i < S$).

\(^4\)See Sec. 2 for the definition of the Drazin inverse.
null $B$ (range $B$). If $B$ is real, it will be clear from the context whether we mean the real or the complex null (range) space.

We will next summarize a few spectral definitions. Let $B \in C^{s \times s}$, $\lambda \in C$ and $Q \equiv B - \lambda I$. The index of $\lambda$ for $B$, denoted $v_\lambda(B)$, is the smallest nonnegative integer $n$ such that $\text{null} Q^n = \text{null} Q^{n+1}$. It is well known that $v_\lambda(B)$ equals the smallest nonnegative integer $n$ for which $\text{range} Q^n = \text{range} Q^{n+1}$; thus,

$$\text{range} Q^k = \text{range} Q^r \iff k \geq r \equiv v_\lambda(B). \quad (2.1)$$

The coindex of $\lambda \neq 0$ for $B$, denoted $\tau_\lambda(B)$, is defined by

$$\tau_\lambda(B) \equiv \max \{ v_\xi(B) | |\xi| = |\lambda| \text{ and } \xi \neq \lambda \}.$$

The eigenprojection of $B$ at $\lambda$, written $E_\lambda(B)$, is the unique projection on null $Q^r$ along range $Q^r$, where $r \equiv v_\lambda(B)$. The existence and uniqueness of this projection are well known (e.g., [11, Lemma 3.1]). The spectrum of $B$ will be denoted $\sigma(B)$, and its spectral radius will be denoted $r(B)$, i.e.,

$$r(B) = \max \{|\lambda| | \lambda \in \sigma(B)\}.$$

We finally remark that $v_\lambda(B) < S$ and $\tau_\lambda(B) < S$. Also, $v_\lambda(B) = 0$, or equivalently $E_\lambda(B) = 0$, if and only if $\lambda \notin \sigma(B)$.

The Drazin inverse of a square matrix $A$, written $A^D$, is the unique solution of the three equations

$$A^{n+1}X = A^n, \quad (2.2)$$

$XAX = X \quad (2.3)$

and

$$AX =XA, \quad (2.4)$$

where $v = v_0(A)$ (see [1]). It was shown in [9] that $A^D = (A - E)^{-1}(I - E)$, where $E = E_0(A)$. If $v_0(A) < 1$, then (2.2) can be replaced by

$$AXA = A.$$

In this case the Drazin inverse is called the group inverse of $A$ and is usually denoted $A^\#$.

Finally, we say that the polynomial $\psi$ forms a polynomial asymptotic expansion of degree $k$ for the sequence $\{B_N\}_{N=0,1,...}$ if

$$\lim_{N \to \infty} \{ B_N - \psi(N) \} = 0 \quad (2.5)$$
and the degree of $\psi$ is $k$. We remark that if $k = -1$, then (2.5) means that $\lim_{N \to \infty} B_N = 0$.

3. THE INDEX CLASSIFICATION OF M-MATRICES

Let $A$ be an $M$-matrix, i.e., $A$ has a representation $A = s^* I - B^*$ with $s^* > 0$ and $B^* > 0$ such that $r(B^*) < s^*$. Let $A = sI - B$ be an arbitrary representation of $A$ with $s > 0$ and $B > 0$. It is well known (cf. [6]) that in this case $r(B) < s$ and $s > \max A_{ii} > \min A_{ii} > 0$. We will next study the index and conidex of $s$ for $B$ in such a representation. This result is well known (cf. [6]). We state it for completeness.

**Lemma 1.** Let $A$ be an $M$-matrix. Then for every representation of $A$ as $A = sI - B$ with $s > 0$ and $B > 0$,

1. $\nu_s(B) = \nu_0(A)$.
2. If $s > \max A_{ii}$, then $\tau_s(B) = 0$.

We will next apply the asymptotic expansion for matrix powers obtained by Rothblum [10, Theorem 3.8]. We remark that the development of this asymptotic expansion was based on a representation of partial sums of matrix powers developed by Rothblum and Veinott [11, Sec. 2].

**Theorem 1.** Let $A = sI - B$ be an $M$-matrix with $\nu \equiv \nu_0(A)$ and $E = E_0(A)$, where $B > 0$ and $s > \max A_{ii}$. Then the polynomial

$$
\psi(N) = \sum_{j=0}^{\nu-1} \left( \frac{N}{j} \right) (-s)^j A^j E
$$

forms a polynomial asymptotic expansion of degree $\nu - 1$ for $s^{-N}B^N$.

**Proof.** The proof follows directly by applying Theorem 3.8 of [10] with $n = 0$ to the matrix $s^{-1}B$, after observing that $E_1(s^{-1}B) = E_0(A)$, $s^{-1}B - I = -s^{-1}A$, $[E - (s^{-1}B - I)]^{-1}(I - E) = s(A - E)^{-1}(I - E) = sA^D$, $\nu_1(s^{-1}B) = \nu_0(A)$ and (by Lemma 1) $\tau_1(s^{-1}B) = \tau_s(B) = 0$.

We point out that in (3.1) one has regular convergence, whereas in [10, Theorem 3.8] one has Cesaro convergence of (possibly) high order. The reason we can replace Cesaro convergence by regular convergence is that $\tau_1(s^{-1}B) = 0$. Also note that [10, Theorem 3.8] enables one to obtain polynomial asymptotic expansions like (3.1) for partial sums of powers of $B$.

We are now ready for the classification of $M$-matrices.
DEFINITION. Let $k = 0, 1, \ldots$. We say that a matrix $A$ is an $M$-$k$-matrix if $A$ is an $M$-matrix and $v_0(A) < k$. The set of all $M$-$k$-matrices will be denoted $\mathcal{M}_k$.

Of course, $\mathcal{M}_k$ is increasing in $k$, and for all $k = s, s + 1, \ldots$, $\mathcal{M}_k$ is the set of all $M$-matrices.

It is easily seen that $\mathcal{M}_0$ is the set of nonsingular $M$-matrices which were characterized in Proposition 1. It is also clear from Theorem 1 that $\mathcal{M}_1$ is precisely the set of $M$-matrices which were studied by Plemmons [6] and were characterized in Proposition 3.

The next theorem gives a characterization of $M$-$k$-matrices in the set of $M$-matrices. This characterization follows directly from Theorem 1. We remark that the equivalence of (1) and (3) in the following theorem for the case $k = 1$ was established by Plemmons [6, Theorem 1].

**Theorem 2.** Let $A = sI - B$ be an $M$-matrix where $B > 0$, $s > 0$ and $s > \max A_{ii}$. Then for $k = 0, 1, \ldots$ the following statements are equivalent:

1. $A \in \mathcal{M}_k$.
2. $s^{-N}B^N$ has a polynomial asymptotic expansion of degree less than $k$.
3. $N^{-k+1}s^{-N}B^N$ converges as $N \to \infty$.
4. $N^{-k+1}s^{-N}B^N$ is bounded.
5. $\lim_{N \to \infty} N^{-k}s^{-N}B^N = 0$.

**Proof.** The proof for $k = 1, 2, \ldots$ follows directly from Theorem 1 and the fact that $A^{r^{-1}}E_0(A) = 0$, where $v = v_0(A)$. The case $k = 0$ is well known and is left to the reader.

4. Characterizations of $M$-$k$-Matrices in the Set of Matrices Having Nonpositive Off Diagonal Elements

The purpose of this section is to establish some characterizations of matrices in $\mathcal{M}_k$ within the set of real matrices having nonpositive off diagonal elements. We will show how Proposition 1 can be extended to arbitrary (not necessarily nonsingular) $M$-matrices. Such an extension was obtained by Plemmons (cf. [6, Proposition 3]) for matrices in $\mathcal{M}_1$. We will first develop a condition which generalizes, to arbitrary $M$-matrices, condition (4) of Proposition 1.

We first introduce the concept that a matrix is nonnegative on a subspace.
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DEFINITION. We say that a matrix \( T \in \mathbb{R}^{s \times s} \) is nonnegative on the subspace \( M \subseteq \mathbb{R}^s \) if \( Tx > 0 \) whenever \( x > 0 \) and \( x \in M \).

If \( M = \mathbb{R}^s \), then the matrix \( T \) in the above definition is nonnegative on \( M \) if and only if \( T > 0 \).

It is well known that a matrix \( A \) is nonsingular and \( A^{-1} > 0 \) if and only if \( x > 0 \) whenever \( Ax > 0 \). This result was generalized by Plemmons [6, Lemma 8], who showed that the group inverse of \( A \) exists (i.e., \( \nu_0(A) < 1 \)) and is nonnegative on range \( A \) if and only if \( x > 0 \) whenever \( Ax > 0 \) and \( x \in \text{range } A \).

We next generalize this result.

**Lemma 2.** Let \( A \) be a square matrix with \( \nu = \nu_0(A) \), and let \( k = 0, 1, \ldots \). Then the following are equivalent:

1. \( \nu < k \) and \( A^D \) is nonnegative on \( \text{range } A^r = \text{range } A^k \).
2. \( x > 0 \) whenever \( Ax > 0 \) and \( x \in \text{range } A^k \).

**Proof.** We first show that (1) implies (2). Assume that (1) is satisfied and \( x \in \text{range } A^k \) where \( Ax > 0 \). Since \( \nu < k \), it follows from (2.1) that for some vector \( t \), \( x = A^r t \). Since \( A^{r+1} t = Ax > 0 \), it follows from the nonnegativity of \( A^D \) on \( \text{range } A^r \) and from (2.1), (2.2) and (2.4) that \( 0 < A^D A^{r+1} t = A^r t = x \), completing the proof that (1) implies (2).

Next assume that (2) is satisfied. We first show that \( \nu < k \). If \( A^{k+1} x = 0 \), then \( A^k x > 0 \) and \( A(-A^k x) > 0 \). By (2), this implies that \( A^k x > 0 \) and \( A(-A^k x) > 0 \), showing that \( A^k x = 0 \). The definition of the index now implies that \( \nu < k \), and therefore, by (2.1), \( \text{range } A^k = \text{range } A^r \). To see that \( A^D \) is nonnegative on \( \text{range } A^r \), let \( x > 0 \), where \( x = A^r t \) for some vector \( t \). Then, by (2.1), (2.2) and (2.4), \( A A^D A^r t = A^D A^{r+1} t = A^r t = x > 0 \). Thus, by (2), it follows that \( A^D x = A^D A^r t > 0 \), completing the proof of Lemma 2.

**Remarks.**

(1) We next illustrate that if \( A = (A_{ij}) \) with \( A_{ij} < 0 \) for all \( i \neq j \), then it is possible that \( A^D \) be nonnegative on a bigger subspace than \( \text{range } A^r \), where \( \nu = \nu_0(A) \). Let

\[
A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then \( A^D = A \), where this matrix is nonnegative on \( R^2 \), though \( \nu_0(A) = 1 \) and \( \text{range } A \neq R^2 \). We next show that if \( A \) is a singular M-matrix, then \( A^D \) is always nonnegative on a larger set than \( \text{range } A^r \), where \( \nu = \nu_0(A) \). Let \( A \) be a singular M-matrix; then \( \text{null } A^r \) is not a degenerate subspace. By Rothblum
[8, Theorem 3.1] this subspace is spanned by a set of semipositive vectors, but we know that $A D x = (A - E)^{-1} (I - E) x = 0$ for every $x \in \text{null} A^r$, where $E = E_0(A)$, and that $\text{null} A^r \cap \text{range} A^r = \{0\}$.

(2) We point out that $v_0(A) = 0$ if and only if $A$ is nonsingular and $v_0(A) \leq 1$ if and only if the group inverse of $A$ exists. Thus, if $k = 0$ in Lemma 2, one gets the known result that a matrix $A$ is nonsingular with $A^{-1} > 0$ if and only if $x > 0$ whenever $Ax > 0$. Moreover, if $k = 1$ one gets the conclusion of Lemma 8 of Plemmons [6].

We will next give a characterization of $M$-k-matrices, $k = 0, 1, \ldots$, within the set of matrices with nonpositive off diagonal elements analogous to (4) in Proposition 1. This was done for $k = 1$ by Plemmons [6, Theorem 2].

**Theorem 3.** Let $A = (A_{ij})$ with $A_{ij} \leq 0$ for all $i \neq j$. Then $A \in \mathcal{G}_{k}$ if and only if $r \equiv v_0(A) < k$ and $A D$ is nonnegative on $\text{range} A^r = \text{range} A^k$.

**Proof.** Let $A = sI - B$, where $B > 0$, $s > 0$ and $s > \max A_{ii}$. By (2.1), it follows that $\text{range} A^r = \text{range} A^k$ for all integers $k \geq r$. Let $A \in \mathcal{G}_k$. For $x \in \text{range} A^r$, $E_0(A)x = 0$. It follows from Theorem 3.8 in [10], and the fact (Lemma 1) that $\tau_0(A) = 0$, that

$$
\lim_{N \to \infty} \sum_{i=0}^{N} s^{-i}B^i x = sA D x. \tag{4.1}
$$

Since $B^i x > 0$ for all $x > 0$, it follows that $A D x > 0$ for all $0 \leq x \in \text{range} A^r$, completing the proof that $A D$ is nonnegative on $\text{range} A^r$. Next assume that $r < k$ and that $A D$ is nonnegative on $\text{range} A^r$. In order to show that $A \in \mathcal{G}_k$, it suffices to show that $r \equiv r(B) < s$. By the Perron-Frobenius theorem (e.g., [13, p. 46]), there exists a vector $z > 0$, $z \neq 0$, for which $Bz = rz$. If $r > s$, then $Az = (s - r)z$, which implies that $z = A^r (s - r)^{-1} z$. We see that $z \in \text{range} A^r$, and therefore $E_0(A)z = 0$. Moreover, since $z > 0$, it follows from the nonnegativity of $A$ on $\text{range} A^r$ that $A D z > 0$. Next observe that by [11, Sec. 2] and arguments similar to those used in the proof of Theorem 1, it follows that for $N = 0, 1, \ldots$

$$
\sum_{i=0}^{N} s^{-i}B^i z = sA D z - s^{-N}B^{N-1}A D z \leq sA D z,
$$

where the last inequality follows from the nonnegativity of $B$ and the fact
that $A^D z > 0$. This is a contradiction, since

$$\sum_{i=0}^{N} s^{-i} B^i z = \sum_{i=0}^{N} s^{-i} r^i z \geq (N+1) z;$$

thus proving that $r(B) < s$. This completes the proof of Theorem 3.

Before generalizing condition (6) of Proposition 1 to arbitrary $M$-matrices, we introduce some additional notation. For a finite set $J$, let $|J|$ denote the cardinality of $J$. Let $A \in R^{S \times S}$, $x \in R^S$ and $J \subseteq \{1, \ldots, S\}$. The $|J| \times |J|$ submatrix of $A$ corresponding to the indices in $J$ will be denoted $A_J$. Similarly, $x_J$ will stand for the corresponding subvector of $x$.

**Theorem 4.** Let $A = (A_{ij})$ have the property that $A_{ii} \leq 0$ for all $i \neq j$, and let $k = 0, 1, \ldots$. Then the following conditions are equivalent:

1. $A \in \mathcal{M}_k$.
2. There exist $J, N \subseteq \{1, \ldots, S\}$, where $J \cap N = \emptyset$ and $J \cup N = \{1, \ldots, S\}$, and vectors $x \in R^S$ and $y \in R^{|N|}$ such that $x_J > 0$, $x_N = 0$, $y > 0$, $A^k x = 0$ and $A_N y \geq 0$.
3. There exist $J, N \subseteq \{1, \ldots, S\}$, where $J \cap N = \emptyset$ and $J \cup N = \{1, \ldots, S\}$, and vectors $x \in R^S$ and $y \in R^{|N|}$ such that $x_J > 0$, $x_N = 0$, $y > 0$, $A^k x = 0$, $A_N y \geq 0$ and for $i = 0, 1, \ldots, A^i x > 0$.

**Proof.** The direction (3)⇒(2) is trivial. The direction (1)⇒(3) follows directly from Theorem 3.1 in [8]. To prove that (2)⇒(1), observe that (2) implies that if $N \neq \emptyset$ then $A_N$ is a singular $M$-matrix. This completes the proof for the case $J = \emptyset$. If $J \neq \emptyset$, let $A = sI - B$ with $s \geq 0$ and $B > 0$. We will show, by contradiction, that $r(B) < s$. If $r(B) > s$, then Theorem 3.1 of [8] and the fact that $r(B_N) < s$ imply the existence of a row vector $z \in R^n$, where $z > 0$, $z_J > 0$ and $z[ r(B) I - B ] = 0$. Now,

$$0 = z A^k x = z [ (s - r(B) I + r(B) I - B) ]^k x$$

$$= \sum_{i=0}^{k} \binom{k}{i} [ s - r(B) ]^i z [ r(B) I - B ]^{k-i} x$$

$$= [ s - r(B) ]^k x.$$
Since \(x > 0, z > 0, x_j > 0\) and \(z_j > 0\), it follows that \(zx > 0\), and therefore \(s = r(B)\), a contradiction.

If \(k = 0\) in Theorem 4, then part (1) of Theorem 4 states that \(v_0(A) = 0\), or equivalently that \(A\) is nonsingular. Also note that when \(k = 0\), the set \(J\) in conditions (2) and (3) of Theorem 4 has to be empty, i.e., these conditions state that \(Ax \gg 0\) for some \(x \gg 0\). This shows that Theorem 4 generalizes the equivalence of conditions (1) and (6) of Theorem 1.

We are now ready to summarize our results to obtain the promised generalization of Proposition 1. For every \(k = 0, 1, \ldots\), we give a number of characterizations of \(M-k\)-matrices within the set of real matrices with nonpositive off diagonal elements. We extend Theorem 3 of Plemmons [6], who considered only \(M-1\)-matrices.

**Theorem 5.** Let \(A = (A_{ij})\) have the property that \(A_{ij} < 0\) for all \(i \neq j\), and let \(k = 0, 1, \ldots\). Then the following statements are equivalent:

1. \(A \in \mathbb{R}_k\).
2. \(A = sI - B\) for some \(B > 0\) and \(s > 0\) where \(s^{-N}B^N\) has a polynomial asymptotic expansion of degree less than \(k\).
3. The real part of each nonzero eigenvalue of \(A\) is positive, and \(v_0(A) \leq k\).
4. All the principal minors of \(A\) are nonnegative, and \(v_0(A) \leq k\).
5. \(A\) is nonnegative on \(\text{range} A^k\).
6. \(Ax > 0\) and \(x \in \text{range} A^k\) implies that \(x > 0\).
7. (6a) There exist \(J, N \subseteq \{1, \ldots, S\}\), where \(J \cap N = \emptyset\) and \(J \cup N = \{1, \ldots, S\}\), and vectors \(x \in R^S\) and \(y \in R^{N^{\mathbb{N}}}\) such that \(x_j > 0, x_N = 0, y > 0, A_x^k = 0\) and \(A_{ij}y > 0\).
8. (6b) There exists \(J, N \subseteq \{1, \ldots, S\}\), where \(J \cap N = \emptyset\) and \(J \cup N = \{1, \ldots, S\}\), and vectors \(x \in R^S\) and \(y \in R^{N^{\mathbb{N}}}\) such that \(x_j > 0, x_N = 0, y > 0, A_x = 0\), \(A_{ij}y > 0\), and for \(i = 0, 1, \ldots\), \(A_{ij}^k > 0\).

**Proof.** The equivalence of (1), (2), (3), (4), (5), (6a) and (6b) follows directly from Proposition 2, Lemma 2 and Theorems 2, 3 and 4. The equivalence of (0) and (1) follows from Theorem 2 and the fact that if \(s^{-N}B^N\) has a polynomial expansion of degree \(k - 1\) (for an arbitrary matrix \(B\)), then \(\lim_{N \to \infty} N^{-k}s^{-N}B^N = 0\), which implies (e.g., [11, Theorem 3.1]) that \(r(B) \leq s\).

**Remark.** The equivalence of (0) and (5) in Theorem 4 says that if \(A = (A_{ij})\) with \(A_{ij} < 0\) for all \(i \neq j\), then for all \(c > 0, c \in \text{range} A\), and \(k = 0, 1, \ldots\), every solution in \(\text{range} A^k\) to the system of linear equations \(Ax = c\) is
nonnegative if and only if \( A \in \mathcal{M}_k \). In this case there exists exactly one solution \( x = A^Dc \) in \( \text{range } A^r \), and \( x \succ 0 \).

An immediate corollary of the equivalence of (0) and (4) in Theorem 4 is a characterization of \( M \)-matrices, within the set of matrices with nonpositive off diagonal elements, in terms of nonnegativity of the Drazin inverse. The existence of such a result was conjectured by Poole and Boullion [7, p. 422]. Namely:

**Corollary 1.** Let \( A = (A_{ij}) \) have the property that \( A_{ij} \leq 0 \) for all \( i \neq j \). Then \( A \) is an \( M \)-matrix if and only if \( A^D \) is nonnegative on \( \cap_{n=0}^{\infty} \text{range } A^n = \text{range } A^r \), where \( \nu = \nu_0(A) \).

When \( A \) is symmetric, some of the equivalent conditions in Theorem 5 can be sharpened in the following way:

**Theorem 6.** Let \( A = (A_{ij}) \) have the property that \( A_{ij} \leq 0 \) for all \( i \neq j \). If \( A \) is symmetric then the following statements are equivalent:

1. \( A \) is an \( M \)-matrix.
2. \( A \in \mathcal{M}_k \).
3. \( A \) is positive semidefinite (i.e., all the eigenvalues of \( A \) are nonnegative).
4. \( Ax \succeq 0 \) for some \( x \succeq 0 \).

**Proof.** Since \( A \) is symmetric, it is well known (e.g., [4, p. 266]) that \( \nu_\lambda(A) \) equals zero or one for every \( \lambda \in \mathbb{C} \). So \( \nu_0(A) \leq 1 \), and the equivalence of (0a) and (0b) follows immediately. The equivalence of (2) and (0a) follows from corresponding equivalence in Theorem 5 and the fact that all the eigenvalues of a symmetric matrix are real. The fact that (6) implies (0a) (even without the symmetry assumption) is well known (e.g., [12] or [7, p. 422]). Finally, assume that (6) holds. Since \( A \) is symmetric, it follows that \( A \) is block diagonal with indecomposable square matrices on the diagonal. Condition (0a) now follows directly from [7, Theorem 3.4].

**Remark.** The equivalence of (0a), (0b) and (2) generalizes Remark (a) of Plemmons [6], who pointed out that if (0a) holds, then (0b) and (2) are equivalent.

**REFERENCES**


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