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New Runge–Kutta Methods For Initial Value Problems

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Abstract. Runge-Kutta methods for solving initial value problems of the form $y' = f(x,y)$ can be reassessed by geometric mean (rather than arithmetic mean) averaging of the functional values in the integration interval. Initially a low order accuracy formula is obtained but by recomparing the Taylor series expansions in terms of the functional derivatives, new weighting parameters can be obtained to yield new Runge-Kutta formulae of 3rd and 4th order.

1. INTRODUCTION

The third order classical Runge-Kutta formula based on arithmetic means is well known and given by,

$$\left.\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(y_n + ha_1k_1) \\ k_3 &= f(y_n + ha_2k_1 + ha_3k_2) \end{aligned}\right\} \tag{1.1}$$

and

$$y_{n+1} = y_n + h(\omega_1k_1 + \omega_2k_2 + \omega_3k_3). \tag{1.2}$$

Kutta's third order rule, for example uses,

$$a_1 = \frac{1}{2},\ a_2 = -1,\ a_3 = 2,\ \omega_1 = \frac{1}{6},\ \omega_2 = \frac{2}{3},\ \omega_3 = \frac{1}{6},$$

i.e., equation (1.2) is written as,

$$y_{n+1} = u_n + \frac{h}{6}(k_1 + 4k_2 + k_3). \tag{1.3}$$

It is also possible to obtain a formula of the form,

$$y_{n+1} = y_n + \frac{h}{4}(k_1 + 2k_2 + k_3), \tag{1.4}$$

and make an adjustment of the parameters a_i, $i = 1,2,3$ to attain third order accuracy. Now equation (1.4) can be written also as,

$$y_{n+1} = y_n + \frac{h}{2}(\frac{k_1 + k_2}{2} + \frac{k_2 + k_3}{2}) \tag{1.5}$$

then by substituting the arithmetic means of $k_i, i = 1,2,3$ in (1.5) with their geometric means we obtain a new formula of the form,

$$y_{n+1} = y_n + \frac{h}{2}(\sqrt{k_1k_2} + \sqrt{k_2k_3}), \tag{1.6}$$

and adjust the parameters a_i, $i = 1,2,3$ so that equation (1.6) will have the highest accuracy possible.

Using the REDUCE symbolic computer program, we obtain the following three equations of condition,

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$$h^2 f f_y : \qquad -24a_1 - 12a_2 - 12a_3 + 24 = 0 \tag{1.7}$$

$$h^2 f f_y^2 : \qquad 6a_1^2 - 6a_1a_2 - 18a_1a_3 + 3a_2^2 + 6a_2a_3 + 3a_3^2 + 8 = 0 \tag{1.8}$$

$$h^2 f^2 f_{yy} : \qquad -12a_1^2 - 6a_2^2 - 12a_2a_3 - 6a_3^2 + 8 = 0 \tag{1.9}$$

Solving these three equations simultaneously we obtain the values,

$$a_1 = \frac{2}{3}, a_2 = -\frac{1}{2}, a_3 = \frac{7}{6}.$$

Thus, the new method can be written as follows,

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{2h}{3}, y_n + \frac{2}{3}hk_1) \\ k_3 &= f(x_n + \frac{2h}{3}, y_n - \frac{1}{2}hk_1 + \frac{7}{6}hk_2) \end{aligned} \tag{1.10}$$

and

$$y_{n+1} = y_n + \frac{h}{2}(\sqrt{k_1k_2} + \sqrt{k_2k_3}).$$

The truncation error in the formula is found to be,

$$\text{LTE} = \frac{3h^4}{64o}[f^3 f_{yyy} - 2f^2 f_y f_{yy} + 7 f f_y^3], \tag{1.11}$$

Numerical Example

By solving $y' = -y, y(0) = 1$ in $0 \leq x \leq 1$, using h = 0.1, we obtain the following results for the 3rd order formula.

cpu time	x	Exact Solution	Num. Solution	Error
2688	0.10	$0.9048374E+00$	$0.9048347E+00$	$0.2711474E-05$
2160	0.20	$0.8187308E+00$	$0.8187258E+00$	$0.4916619E-05$
2262	0.30	$0.7408182E+00$	$0.7408116E+00$	$0.6664981E-05$
2174	0.40	$0.6703200E+00$	$0.6703120E+00$	$0.8047910E-05$
2054	0.50	$0.6065307E+00$	$0.6065216E+00$	$0.9098352E-05$
2123	0.60	$0.5488116E+00$	$0.5488018E+00$	$0.9880673E-05$
2129	0.70	$0.4965853E+00$	$0.4965479E+00$	$0.1042790E-04$
2539	0.80	$0.4493290E+00$	$0.4493182E+00$	$0.1078387E-04$
2567	0.90	$0.4065697E+00$	$0.4065587E+00$	$0.1097874E-04$
2305	1.00	$0.3678794E+00$	$0.3678684E+00$	$0.1103946E-04$

TABLE 1

2. A Fourth Order Runge-Kutta Method

A new fourth order Runge-Kutta method for the initial value problem which is based on the geometric mean was developed by Evans and Sanugi [1986]. The method is of the form,

$$\left.\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(y_n + ha_1k_1) \\ k_3 &= f(y_n + h(a_2k_1 + a_3k_2)) \\ k_4 &= f(y_n + h(a_4k_1 + a_5k_2 + a_6k_3)) \end{aligned}\right\} \tag{2.1}$$

$$y_{n+1} = y_n + \frac{h}{3}\left(\sqrt{k_1k_2} + \sqrt{k_2k_3} + \sqrt{k_3k_4}\right) \tag{2.2}$$

In particular by setting $a_2+a_3 = \frac{1}{2}$, $a_4+a_5+a_6 = 1$ and by comparing the RHS of (2.2) with the Taylor series expansion for $y(x_{n+1})$, the following six equations of conditions were obtained,

$$h^2ff_y: \quad -192a_1 + 96 = 0 \tag{2.3a}$$

$$h^3ff_y^2: \quad 108 - 48a_6 - 24a_1 - 96a_1a_5 - 192a_1a_3 + 48a_1^2 = 0 \tag{2.3b}$$

$$h^3f^2f_{yy}: \quad 24 - 96a_1^2 = 0 \tag{2.3c}$$

$$\begin{aligned} h^4ff_y^3: \quad & 18 + 12a_6 + 3a_1 + 24a_1a_5 - 96a_1a_3a_6 + \\ & +6a_1^2 - 48a_1^2a_3 - 24a_1^3 = 0 \end{aligned} \tag{2.3d}$$

$$\begin{aligned} h^4f^2f_yf_{yy}; \quad & 108 - 60a_6 - 6a_1 - 96a_1a_5 - 96a_1a_3 - \\ & -12a_1^2 - 48a_1^2a_5 - 96a_1^2a_3 + 48a_1^3 = 0 \end{aligned} \tag{2.3e}$$

and

$$h^4f^3f_{yyy}: \quad 4 - 32a_1^3 = 0 \tag{2.3f}$$

by solving these equations simultaneously the six parameters are given by,

$$\begin{aligned} a_1 &= \frac{1}{2}, \quad a_2 = -\frac{1}{16}, \quad a_3 = \frac{9}{16} \\ a_4 &= -\frac{1}{8}, \quad a_5 = \frac{5}{24}, \quad a_6 = \frac{11}{12}. \end{aligned} \tag{2.4}$$

Thus the new method is written as follows:

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{h}{2},\ y_n + \frac{h}{2}k_1) \\ k_3 &= f(x_n + \frac{h}{2},\ y_n + \frac{h}{16}[-k_1 + 9k_2]) \\ k_4 &= f(x_n + h,\ y_n + \frac{h}{24}[-3k_1 + 5k_2 + 22k_2]) \\ y_{n+1} &= y_n + \frac{h}{3}(\sqrt{k_1k_2} + \sqrt{k_2k_3} + \sqrt{k_3k_4}), \end{aligned} \tag{2.5}$$

with error term,

$$\text{LTE} = \frac{h^5}{184320}[64f^4 f_{yyyy} - 208f^3 f_y f_{yyy} - 3744f^3 f_{yy}^2 - \\ -1974f^2 f_y^2 f_{yy} + 3561 f f_y^4] \tag{2.6}$$

which confirms the fourth order accuracy of the method as shown in Table 2.
h=0.1000

Standard formula

cpu t	x	Exact Solution	Num. Solution	Error
2768	0.10	$0.9048374E+00$	$0.9048375E+00$	$-0.8537769E-07$
2571	0.20	$0.8187308E+00$	$0.8187309E+00$	$-0.1447658E-06$
2473	0.30	$0.7408184E+00$	$0.7408184E+00$	$-0.2046044E-06$
2594	0.40	$0.6703200E+00$	$0.6703203E+00$	$-0.2399892E-06$
2123	0.50	$0.6065307E+00$	$0.6065309E+00$	$-0.2755226E-06$
2392	0.60	$0.5488116E+00$	$0.5488119E+00$	$-0.2975112E-06$
2901	0.70	$0.4965853E+00$	$0.4965856E+00$	$-0.3166163E-06$
2532	0.80	$0.4493290E+00$	$0.4493293E+00$	$-0.3270209E-06$
2183	0.90	$0.4065697E+00$	$0.4065700E+00$	$-0.3314901E-06$
2220	1.00	$0.3678794E+00$	$0.3678798E+00$	$-0.3315419E-06$

RK-New Formula

4327	0.10	$0.9048374E+00$	$0.9048376E+00$	$-0.1993220E-06$
3651	0.20	$0.8187308E+00$	$0.8187311E+00$	$-0.3509680E-06$
3758	0.30	$0.7408182E+00$	$0.7408187E+00$	$-0.4844737E-06$
3321	0.40	$0.6703200E+00$	$0.6703206E+00$	$-0.5776375E-06$
3361	0.50	$0.6065307E+00$	$0.6065313E+00$	$-0.6574187E-06$
3095	0.60	$0.5488116E+00$	$0.5488123E+00$	$-0.7121759E-06$
2756	0.70	$0.4965833E+00$	$0.4965861E+00$	$-0.7543546E-06$
2725	0.80	$0.4493290E+00$	$0.4493297E+00$	$-0.7796861E-06$
2746	0.90	$0.4065697E+00$	$0.4065705E+00$	$-0.7922771E-06$
2825	1.00	$0.3678794E+00$	$0.3668802E+00$	$-0.7948056E-06$

TABLE 2

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