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Soft set relations and functions

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ABSTRACT

The traditional soft set is a mapping from a parameter to the crisp subset of universe. Molodtsov introduced the theory of soft sets as a generalized tool for modeling complex systems involving uncertain or not clearly defined objects. In this paper the concepts of soft set relations are introduced as a sub soft set of the Cartesian product of the soft sets and many related concepts such as equivalent soft set relation, partition, composition, function etc. are discussed.

1. Introduction

Modern set theory formulated by George Cantor is fundamental for the whole of Mathematics. One issue associated with the notion of a set is the concept of vagueness. Mathematics requires that all mathematical notions including set must be exact. This vagueness or the representation of imperfect knowledge has been a problem for a long time for philosophers, logicians and mathematicians. However, recently it became a crucial issue for computer scientists, particularly in the area of artificial intelligence. To handle situations like this, many tools have been suggested. They include Fuzzy sets, Multi sets, Rough sets, Soft sets and many more.

Owing to the fact that many mathematical objects such as fuzzy sets, topological spaces, rough sets (see [1,2]) can be considered as particular types of soft sets, it is a very general tool to handle objects which are defined in terms of loose or general set of characteristics. A soft set can be considered as an approximate description of an object precisely consisting of two parts, namely predicate and approximate value set. Exact solutions to the mathematical models are needed in classical mathematics. If the model is so complicated that we cannot set an exact solution, we can derive an approximate solution and there are many methods for this. On the other hand, in soft set theory as the initial description of object itself is of an approximate nature, we need not have to introduce the concept of an exact solution.

Soft theory was initiated by the Russian researcher Molodtsov in 1999. Molodtsov proposed the soft set as a completely generic mathematical tool for modeling uncertainties. There is no limited condition to the description of objects; so researchers can choose the form of parameters they need, which greatly simplifies the decision-making process and make the process more efficient in the absence of partial information. There are many mathematical tools available for modeling complex systems such as probability theory, fuzzy set theory, interval mathematics etc. But there are inherent difficulties associated with each of these techniques. Probability theory is applicable only for a stochastically stable system. Interval mathematics is not sufficiently adaptable for problems with different uncertainties. Setting the membership function value is always been a problem in fuzzy set theory. Moreover all these techniques lack in the parameterization of the tools and

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hence they could not be applied successfully in tackling problems especially in areas like economic, environmental and social problem domains. Soft set theory is standing in a unique way in the sense that it is free from the above difficulties and has a wider scope for many applications in a multidimensional way.

Soft set theory has a rich potential for application in many directions, some of which are reported by Molodtsov [2] in his work. He successfully applied soft set theory in areas such as the smoothness of functions, game theory, operation research, Riemann integration and so on. Later Maji et al. [3] presented some new definitions on soft sets such as a subset, the complement of a soft set and discussed in detail the application of soft set theory in decision making problems [1]. Chen et al. [4] pointed out that the method of attribute reduction in rough set theory cannot simply transplant to parameter reduction in soft set theory, but the detailed process of parameter reduction in soft set theory was not described. Also an attempt was made by Kostek [5] to assess sound quality based on a soft set approach. Mushrif et al. [6] presented a novel method for the classification of natural textures using the notions of soft set theory.

Presently, work on the soft set theory is making progress rapidly. In the standard soft set theory, a situation may be complex in the real world because of the fuzzy nature of the parameters. With this point of view Yang et al. [7] expanded this theory to fuzzy soft set theory and discussed some immediate outcomes. To continue the investigation on fuzzy soft sets, Kharal and Ahmad [8] introduced the notion of a mapping on the classes of fuzzy soft sets which is a pivotal notion for the advanced development of any new area of mathematical sciences.

The algebraic nature of set theories dealing with uncertainties has been studied by some authors. Aktas and Çagman [9] compared soft sets to the related concepts of fuzzy sets and rough sets. They also defined the notion of soft groups and derived some properties. Feng et al. [10] dealt with the algebraic structure of semi-rings by applying soft set theory. The concept of a fuzzy soft group was introduced by Aygunoglu and Aygun [11]. Furthermore, Jun [12] introduced and investigated the notion of soft BCK/BCI-algebras. Jun and Park [13] discussed the applications of soft sets in an ideal theory of BCK/BCI-algebras.

This paper is an attempt to open up the theoretical aspects of soft sets by extending the notions of equivalence relations, composition of relations, partition and function to the framework of soft sets. In order to refresh the fundamental concepts of set theory we refer to [14–16]. The organization of paper is as follows: In Section 2 basic notions about soft sets is reviewed. Section 3 focuses on the Cartesian product and relation on soft sets. We also define induced relations from the universal set and the attribute set with examples. In Section 4 we discuss the equivalence relations and partitions on soft sets and results involving them are obtained. We present the composition of soft set relationships with examples and prove some theorems based on it. Section 5 gives the concept of a soft set function and explains the composition of functions with related results. The last section summarizes all the contributions and points out future research work.

2. Preliminaries and basic definitions

In this section we will collect the basic definitions and notations as introduced by Molodtsov [2] and Maji et al. [3].

Definition 2.1 (*[2]*)**.** Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U and $A \subset E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

Example 2.2. Consider the following.

U is the set of all students under consideration.

E is set of parameters. Each parameter is a word or sentence.

$$E = \{\text{brilliant, average, healthy}\}.$$

In this case to define a soft set means to point out brilliant students, average students and so on. Thus the soft set (F, A) describes different types of students.

We consider below the same example in more detail for our next discussion.

Suppose that there are six students in the universe U given by

$$U = \{x_1, x_2, x_3, x_4, x_5, x_6\} \quad \text{and} \quad E = \{e_1, e_2, e_3\}$$

where

e_1 stands for brilliant

e_2 stands for average

e_3 stands for healthy.

Suppose that

$$F(e_1) = \{x_1, x_2, x_5\},$$
$$F(e_2) = \{x_3, x_4, x_6\},$$
$$F(e_3) = \{x_1, x_4, x_5, x_6\}.$$

The soft set (F, E) is a parameterized family $\{F(e), i = 1, 2, 3\}$ of subsets of the set U and gives us a collection of approximate descriptions of an object. Here note that for each $e \in E$, $F(e)$ is a crisp set. Thus the soft set (F, A) is called standard soft set. In [7] Yang et al. define fuzzy soft set where $F(e)$ is a fuzzy subset of U for each parameter 'e'.

Definition 2.3 (*[3]*)**.** For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(i) $A \subset B$, and
(ii) $\forall \varepsilon \in A$, $F(\varepsilon)$ and $G(\varepsilon)$ are identical approximations.

We write $(F, A)\tilde{\subset}(G, B)$.

(F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A)\tilde{\supset}(G, B)$.

Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4 (*[3]*)**.** Let $E = \{e_1, e_2, e_3, \ldots, e_n\}$ be a set of parameters. The NOT set of E denoted by $\rceil E$ is defined by, $\rceil E = \{\rceil e_1, \rceil e_2, \rceil e_3, \ldots, \rceil e_n\}$, where $\rceil e_i = \text{not } e_i$; $\forall i$.

Definition 2.5 (*[3]*)**.** The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \rceil A)$ where $F^c : \rceil A \to p(U)$ is a mapping given by $F^c(\rceil\alpha) = U - F(\alpha), \forall \rceil\alpha \in \rceil A$.

Let us call F^c to be the soft complement function of F. Clearly $(F^c)^c$ is the same as F and $((F, A)^c)^c = (F, A)$.

Definition 2.6 (*[3]*)**.** A soft set (F, A) over U is said to be a NULL soft set denoted by Φ, if $\forall \varepsilon \in A$, $F(\varepsilon) = \phi$, (null-set).

A soft set (F, A) over U is said to be an absolute soft set denoted by $\bar{A}$, if $\forall \varepsilon \in A$, $F(\varepsilon) = U$. Clearly, $(\bar{A})^c = \Phi$, and $\Phi^c = \bar{A}$.

Proposition 2.7 (*[3]*)**.** *If A and B are two sets of parameters then we have the following*

1. $\rceil(\rceil A) = A$
2. $\rceil(A \cup B) = \rceil A \cup \rceil B$
3. $\rceil(A \cup B) = \rceil A \cap \rceil B$.

Definition 2.8 (*[3]*)**.** A union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = AUB$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A)\tilde{\cup}(G, B) = (H, C)$.

Definition 2.9 (*[17]*)**.** An intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cap B$, and $\forall e \in C$, $H(e) = F(e) \cap G(e)$.

We write $(F, A)\tilde{\cap}(G, B) = (H, C)$.

3. Cartesian product and relations

Definition 3.1. Let (F, A) and (G, B) be two soft sets over U, then the Cartesian product of (F, A) and (G, B) is defined as, $(F, A) \times (G, B) = (H, A \times B)$, where $H : A \times B \to P(U \times U)$ and $H(a, b) = F(a) \times G(b)$, where $(a, b) \in A \times B$

$$\text{i.e., } H(a, b) = \{(h_i, h_j);\ \text{where } h_i \in F(a) \text{ and } h_j \in G(b)\}.$$

The Cartesian product of three or more nonempty soft sets can be defined by generalizing the definition of the Cartesian product of two soft sets. The Cartesian product $(F_1, A) \times (F_2, A) \times \cdots \times (F_n, A)$ of the nonempty soft sets $(F_1, A), (F_2, A), \ldots, (F_n, A)$ is the soft set of all ordered n-tuple $(h_1, h_2, \ldots, h_n)$ where $h_i \in F_i(a)$.

Example 3.2. Consider the soft set (F, A) which describes the "cost of the houses" and the soft set (G, B) which describes the "attractiveness of the houses".

Suppose that $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}\}$,

$A = \{\text{very costly; costly; cheap}\}$ and

$B = \{\text{beautiful; in the green surroundings; cheap}\}$.

Let $F(\text{very costly}) = \{h_2, h_4, h_7, h_8\}$,

$F(\text{costly}) = \{h_1, h_3, h_5\}$,

$F(\text{cheap}) = \{h_6, h_9, h_10\}$, and

$G(\text{beautiful}) = \{h_2, h_3, h_7\}$,

$G(\text{in the green surroundings}) = \{h_6, h_5, h_8\}$,

$G(\text{cheap}) = \{h_6, h_9, h_10\}$.

Now $(F, A) \times (G, B) = (H, A \times B)$ where a typical element will look like

$$\begin{aligned} H(\text{very costly, beautiful}) &= \{h_2, h_4, h_7, h_8\} \times \{h_2, h_3, h_7\} \\ &= \{(h_2, h_2), (h_2, h_3), (h_2, h_7), (h_4, h_2), (h_4, h_3), (h_4, h_7), (h_7, h_2), \\ &\quad (h_7, h_3), (h_7, h_7), (h_8, h_2), (h_8, h_3), (h_8, h_7)\}. \end{aligned}$$

Definition 3.3. Let (F, A) and (G, B) be two soft sets over U, then a relation from (F, A) to (G, B) is a soft subset of $(F, A) \times (G, B)$.

In other words, a relation from (F, A) to (G, B) is of the form (H_1, S) where $S \subset A \times B$ and $H_1(a, b) = H(a, b), \forall (a, b) \in S$ where $(H, A \times B) = (F, A) \times (G, B)$ as defined in Definition 3.1. Any subset of $(F, A) \times (F, A)$ is called a relation on (F, A).

In an equivalent way, we can define the relation R on the soft set (F, A) in the parameterized form as follows:

If $(F, A) = \{F(a), F(b), \ldots\}$, then $F(a)RF(b)$ iff $F(a) \times F(b) \in R$.

Definition 3.4. Let R be a soft set relation from (F, A) to (G, B). Then the domain of R (dom R) is defined as the soft set (D, A_1) where

$$A_1 = \{a \in A : H(a, b) \in R \text{ for some } b \in B\} \quad \text{and} \quad D(a_1) = F(a_1) \quad \forall a_1 \in A_1.$$

The range of R (ran R) is defined as the soft set (RG, B_1), where $B_1 \subset B$ and $B_1 = \{b \in B : H(a, b) \in R \text{ for some } a \in A\}$ and $RG(b_1) = G(b_1)\ \forall b_1 \in B_1$.

Example 3.5. Let us consider an example to illustrate a relation on soft sets.

Let U denotes set of people in a social gathering.

i.e. $U = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\}$.

Let A denotes different job categories.

Take $A = \{\text{chartered accountant, doctors, engineers, teachers}\}$

i.e. $A = \{c, d, e, t\}$.

Let B denote the qualification of people.

Take $B = \{\text{B.Sc., B.Tech., MBBS, M.Sc.}\}$

i.e. $B = \{b_1, b_2, m_1, m_2\}$.

Then the soft set (F, A) is given by $\{F(c) = \{p_1, p_2\};\ F(d) = \{p_4, p_5\};\ F(e) = \{p_7, p_9\};\ F(t) = \{p_3, p_4, p_7\}\}$ and it describes people having different jobs and the soft set (G, B) is given by $\{G(b_1) = \{p_1, p_6, p_8, p_{10}\};\ G(b_2) = \{p_3, p_6, p_7, p_9\};\ G(m_1) = \{p_3, p_4, p_5, p_8\},\ G(m_2) = \{p_3, p_8\}\}$ and it represents the people qualified in various courses.

Define a relation R from (F, A) to (G, B) as follows:

$F(a)RG(b)$ iff $F(a) \subseteq G(b)$.

Then $R = \{F(d) \times G(m_1), F(e) \times G(b_2)\}$.

Then dom $R = (D, A_1)$ where $A_1 = \{\text{doctors, engineers}\}$ and

$D(a) = F(a)$ for every $a \in A_1$.

Similarly ran $R = (RG, B_1)$ where $B_1 = \{\text{B.Tech., MBBS}\}$ and

$RG(b) = (b)$ for every $b \in B_1$.

Induced relations from a universal set and the attribute set

In soft set we are dealing with two kinds of ordinary sets, universal set and the attribute set. We can think about relations defined on the universal set as well as the attribute set. Corresponding to these, we can induce some relations on the soft set.

Definition 3.6. Let (F, A) be a soft set defined on the universal set and R be a relation defined on U. (i.e., $R \subset U \times U$). Then the induced soft set relation R_U on (F, A) is defined as follows: $F(a)R_UF(b) \Longleftrightarrow uRv$ for every $u \in F(a)$ and $v \in F(b)$.

Example 3.7. Suppose that U is the set of students who have applied for a Ph.D. in an institution given by $U = \{s_1s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}\}$ and A denotes the nationality of students given as $A = \{\text{India, Pakistan, US, Canada}\}$

i.e. $A = \{i, p, u, c\}$.

Then the soft set (F, A) is to point out Indian students, Pakistani students and so on. Let R be a relation defined on U as s_iRs_j iff s_j and s_j come under the same continent.

Then the induced relation R_U on (F, A) is given by $\{F(i) \times F(p), F(p) \times F(i), F(i) \times F(i), F(p) \times F(p), F(u) \times F(c), F(c) \times F(u), F(u) \times F(u), F(c) \times F(c)\}$.

In a similar manner we can define the induced soft set relation corresponding to a relation in the attribute set also.

Definition 3.8. Let (F, A) be a soft set defined on the attribute set A and R be a relation defined on A (i.e. $R \subset A \times A$). Then the induced soft set relation R_A on (F, A) is defined as follows $F(a)R_AF(b) \Leftrightarrow aRb$.

Example 3.9. Consider the soft set (F, A) over U defined as follows:

Let U denote the set of all candidates attending for an interview

i.e. $U = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9\}$.

A describes the qualification of candidates

$A = \{\text{SSLC, B.Sc., M.Sc., M.Tech.}\}$

i.e. $A = \{s, b, m_1, m_2\}$.

Let R be a relation defined on A as aRb iff a and b are master's degree.

Then $R = \{m_1Rm_2, m_2Rm_1, m_1Rm_1, m_2Rm_2\}$.

Then the induced soft set relation R_A on (F, A) is $\{F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_1) \times F(m_1), F(m_2) \times F(m_2)\}$.

4. Equivalence relations and partitions on soft sets

Definition 4.1. Let R be a relation on (F, A), then

1. R is reflexive if $H_1(a, a) \in R, \forall a \in A$.
2. R is symmetric if $H_1(a, b) \in R \Rightarrow H_1(b, a) \in R, \forall (a, b) \in A \times A$.
3. R is transitive if $H_1(a, b) \in R, H_1(b, c) \in R \Rightarrow H_1(a, c) \in R, \forall a, b, c \in A$.

Definition 4.2. A soft set relation R on a soft set (F, A) is called an equivalence relation if it is reflexive, symmetric and transitive.

Example 4.3. Consider a soft set (F, A) over U where $U = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9\}, A = \{m_1, m_2\}$ and $F(m_1) = \{c_1, c_2, c_5, c_6\}, F(m_2) = \{c_3, c_4, c_7, c_8, c_9\}$. Consider a relation R defined on (F, A) as $\{F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_1) \times F(m_1), F(m_2) \times F(m_2)\}$.

This relation is a soft set equivalence relation.

Definition 4.4. Let (F, A) be a soft set, then equivalence class of $F(a)$ denoted by $[F(a)]$ is defined as $[F(a)] = \{F(b) : F(b)RF(a)\}$.

In the Example 4.3 we have $[F(m_1)] = \{F(m_1), F(m_2)\} = [F(m_2)]$.

Lemma 4.5. *Let R be an equivalence relation on a soft set (F, A). For any $F(a), F(b) \in (F, A)$, $F(a)RF(b)$ iff $[F(a)] = [F(b)]$.*

Proof. Suppose $[F(a)] = [F(b)]$. Since R is reflexive $F(b)RF(b)$.

Hence $F(b) \in [F(b)] = [F(a)]$ which gives $F(a)RF(b)$.

Conversely suppose $F(a)RF(b)$. Let $F(a^1) \in [F(a)]$. Then $F(a^1)RF(a)$. Using the transitive property of R this gives $F(a^1) \in [F(b)]$. Hence $[F(a)] \subseteq [F(b)]$. Using a similar argument $[F(b)] \subseteq [F(a)]$. Hence $[F(a)] = [F(b)]$. □

Definition 4.6. A collection of nonempty soft sub sets $P = \{(F_i, A_i), i \in I\}$ of a soft set (F, A) is called a partition of (F, A)

1. $(F, A) = \widetilde{\bigcup}_i (F_i, A_i)$ and
2. $A_i \cap A_j = \phi$. Whenever $i \neq j$.

Example 4.7. $U = \{h_l, h_2, h_3, h_4, h_5\}$ and $A = \{a_1, a_2, a_{3,} a_4\}$

$$(F, A) = \{F(a_1), F(a_2), F(a_3), F(a_4)\}$$
$$F(a_1) = \{h_1, h_2\}$$
$$F(a_2) = \{h_3\}$$
$$F(a_3) = \{h_3, h_4\}$$
$$F(a_4) = \{h_4, h_5\}$$
$$A_1 = \{a_1, a_2\}, \qquad A_2 = \{a_3, a_4\}$$
$$(F_1, A_1) = \{F_1(a_1), F_1(a_2)\}$$
$$(F_2, A_2) = \{F_2(a_3), F_2(a_4)\}.$$

So that F© $= F\ \forall i = 1, 2$

$$(F, A) = (F_1, A_1) \cup (F_2, A_2) \quad \text{and} \quad A_1 \cap A_2 = \phi.$$

Thus $\{(F_1, A_1), (F_2, A_2)\}$ is a soft set partition.

Remark 4.8. Members of the partition are called a block of (F, A).

Moreover corresponding to a partition $\{(F_i, A_i)\}$ of a soft set (F, A), we can define a soft set relation on (F, A) by $F(a)RF(b)$ iff $F(a)$ and $F(b)$ belong to the same block. Now we will prove that the relation defined in this manner is an equivalence relation.

Theorem 4.9. *Let $\{(F_i, A_i), i \in I\}$ be a partition of soft set (F, A) the soft set relation defined on (F, A) as $F(a)RF(b)$ iff $F(a)$ and $F(b)$ are the members of the same block is an equivalence relation.*

Proof. *Reflexive:* Let $F(a)$ be any element of (F, A). It is clear that $F(a)$ is in the same block itself. Hence $F(a)RF(a)$.
Symmetric: If $F(a)RF(b)$, then $F(a)$ and $F(b)$ are in the same block. Therefore $F(b)RF(a)$.
Transitive: If $F(a)RF(b)$, $F(b)RF(c)$ then $F(a)$, $F(b)$, $F(c)$ must lie in the same block. Therefore $F(a)RF(c)$. □

Remark 4.10. The equivalence soft relation defined in the above theorem is called an equivalence soft set relation determined by the partition P.

In the example given above the equivalence relation determined by the partition $P = \{(F, A_1), (F, A_2)\}$ is given by $R = \{F(a_1) \times F(a_1), F(a_2) \times F(a_2), F(a_3) \times F(a_3), F(a_4) \times F(a_4), F(a_1) \times F(a_2), F(a_2) \times F(a_1), F(a_3) \times F(a_4), F(a_4) \times F(a_3)\}$.

Theorem 4.11. *Corresponding to every equivalence relation defined on a soft set (F, A) there exists a partition on (F, A) and this partition precisely consists of the equivalence classes of R.*

Proof. Let $[F(a)]$ be equivalence class w.r.t. a relation R on (F, A). Let A_a denote all those elements in A corresponding to $[F(a)]$. i.e. $A_a = \{b \in A : F(b)RF(a)\}$. Thus we can denote $[F(a)]$ as (F, A_a). So we have to show that the collection $\{(F, A_a) : a \in A\}$ of such distinct sets forms a partition P of (F, A). In order to prove this we should prove

(1) $(F, A) = \widetilde{\bigcup}_{a \in A} (F, A_a)$.
(2) If A_a, A_b, are not identical then $A_a \cap A_b = \phi$.

Since R is reflexive $F(a)RF(a)\ \forall a \in A$ so that part (1) can prove easily.
Now for the second part,
Let $x \in A_a \cap A_b$. Then $F(x) \in (F, A_a)$ and $F(x) \in (F, A_b)$

$$\Rightarrow F(x)RF(a) \text{ and } F(x)RF(b).$$

Using the transitive property of R we have $F(A)RF(b)$. Now using the Lemma 4.5 we have $[F(a)] = [F(a)]$. This gives $A_a = A_b$. □

Remark 4.12. The partition constructed in the above theorem therefore consists of all equivalence classes of R and is called the quotient soft sets of (F, A) and is denoted by $(F, A)/R$.

Composition of soft set relations

Definition 4.13. Let (F, A), (G, B) and (H, C) be three soft sets. Let R be a soft set relation from (F, A) to (G, B) and S be a soft set relation from (G, B) to (H, C). Then a new soft set relation, the composition of R and S expressed as $S \circ R$ from (F, A) to (H, C) is defined as follows: If $F(a)$ is in (F, A) and $H(c)$ is in (H, C) then

$$F(a)S \circ R\, H(c) \text{ iff there is some } G(b) \text{ in } (G, B) \text{ such that } F(a)RG(b) \text{ and } G(b)RH(c).$$

Example 4.14. Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$ and $U = \{h_1, h_2, h_3, h_4, h_5\}$. Let R and S be soft set relation defined respectively from (F, A) to (G, B) and (G, B) to (H, C) as $R = \{F(a_1) \times G(b_1), F(a_2) \times G(b_2), F(a_3) \times G(b_2)\}$ and $S = \{G(b_1) \times H(c_1), G(b_2) \times H(c_2)\}$.

Then $S \circ R = \{F(a_1) \times H(c_1), F(a_2) \times H(c_1), F(a_3) \times H(c_2)\}$.

The following example shows that the composition is not commutative.

Let $A = B = C = \{e_1, e_2, e_3\}$

$$R = \{F(e_1) \times F(e_2), F(e_2) \times F(e_3), F(e_3) \times F(e_1)\}$$
$$S = \{F(e_1) \times F(e_3), F(e_2) \times F(e_2), F(e_3) \times F(e_3)\}$$
$$S \circ R = \{F(e_1) \times F(e_3), F(e_2) \times F(e_2), F(e_3) \times F(e_3)\}$$
$$R \circ S = \{F(e_1) \times F(e_1), F(e_2) \times F(e_3), F(e_3) \times F(e_1)\}.$$

Thus in general $S \circ R \neq RoS$.

Definition 4.15. The inverse of a soft set relation R denoted as R^{-1} is defined by $R^{-1} = \{(F(b) \times F(a)) : F(a)RF(b)\}$.

It is clear from the above definition that the inverse of R is defined by reversing the order of every pair belonging to R.

Example 4.16. Let $(F, A) = \{F(a), F(b), F(c)\}$ with a relation R defined as

$$R = \{F(a) \times F(a), F(b) \times F(c), F(c) \times F(a)\}$$
$$R^{-1} = \{F(a) \times F(a), F(c) \times F(b), F(a) \times F(c)\}.$$

Theorem 4.17. *Let R be soft set relation from (F, A) to (G, B) and S be a soft set relation from (G, B) to (H, C). Then $(S \circ R)^{-1} = R^{-1}oS^{-1}$.*

Proof. Clearly $(S \circ R)^{-1}$ is a soft set relation from (H, C) to (F, A). Now let $H(c)$ be any element in (H, C) and $F(a)$ be an element in (F, A). Then $H(c)(S \circ R)^{-1}F(a)$ if $F(a)S \circ RH(c)$. This by definition exists if there is some $G(b)$ in (G, B) such that $F(a)RG(b)$ and $G(b)RH(c)$. This is equivalent to $G(b)R^{-1}F(a)$ and $H(c)S^{-1}G(b)$. Then $H(c)R^{-1}oS^{-1}F(a)$. Hence $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. □

Definition 4.18. The identity relation I_{FA} on any soft set (F, A) is defined as follows:

$$F(a)I_{FA}F(b) \quad \text{iff} \quad a = b.$$

In the above example $I_{FA} = \{F(a) \times F(a), F(b) \times F(b), F(c) \times F(c)\}$.

5. Soft set functions

Definition 5.1. Let (F, A) and (G, B) be two non-empty soft sets. Then a soft set relation 'f' from (F, A) to (G, B) is called a soft set function if every element in domain has a unique element in the range. If $F(a)fG(b)$ then we write $f(F(a)) = G(b)$.

Example 5.2. Let $U = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$

$$A = \{a_1, a_2, a_3, a_4\}$$
$$B = \{b_1, b_2\}.$$

Consider the soft sets (F, A) and (G, B) defined by $F(a_1) = \{p_1, p_2, p_5\}$, $F(a_2) = \{p_2, p_3, p_4\}$, $F(a_3) = \{p_1, p_2\}$, $F(a_4) = \{p_5, p_6\}$ and $G(b_1) = \{p_1, p_2\}$, $G(b_2) = \{p_2, p_5, p_6\}$. Then a soft set function f from (F, A) to (G, B) is given by $f = \{F(a_1) \times G(b1), F(a_2) \times G(b_1), F(a_3) \times G(b_2), F(a_4) \times G(b_2)\}$.

Definition 5.3. A function f from (F, A) to (G, B) is called injective (one–one) if $F(a) \neq F(b)$ implies $f(F(a)) \neq f(F(b))$.

i.e. f is called injective if each element of the ran f appears exactly once in the function.

Definition 5.4. A function f from (F, A) to (G, B) is called surjective (onto) if ran $f = (G, B)$.

Definition 5.5. A function which is both injective and surjective is called a bijective function.

In the above example if we take $A = \{a_1, a_3\}$. Then the function f from (F, A) to (G, B) defined as $f = \{F(a)_1 \times G(b_1), F(a_3) \times G(b_2)\}$ is a bijective function.

Definition 5.6. A constant soft set function is a function in which every element of dom f has the same image.

Definition 5.7. Identity soft set function I on a soft set (F, A) is defined by the function $I : (F, A) \to (F, A)$ as $I(F(a)) = F(a)$ for every $F(a)$ in (F, A).

Theorem 5.8. *Let $f : (F, A) \to (G, B)$ be a soft set function and (F, A_1), (F, A_2) be soft subsets of (F, A). Then*

(i) $(F, A_1) \subseteq (F, A_2) \Rightarrow f(F, A_1) \subseteq f(F, A_2)$
(ii) $f[(F, A_1) \cup (F, A_2)] = f(F, A_1) \cup f(F, A_2)$
(iii) $f[(F, A_1) \cap (F, A_2)] \subseteq f(F, A_1) \cap f(F, A_2)$, *equality holds if f is one–one.*

Proof. (i) Let $G(b)f(F, A_1)$.

Then $G(b) = f(F(a))$ for some $F(a)$ in (F, A_1)
$= f(F(a))$ for some $F(a)$ in (F, A_2) as $(F, A_1)(F, A_2)$.

Therefore $G(b) \in f(F, A_2)$

$\therefore f(F, A_1) \subseteq f(F, A_2)$.

(ii) Let $G(b) \in f[(F, A_1) \cup (F, A_2)]$.

Then $G(b) = f(F(a))$ for some $F(a)$ in $(F, A_1) \cup (F, A_2)$

$= f(F(a))$ for $F(a) \in (F, A_1)$ or $F(a) \in (F, A_2)$

$\Rightarrow G(b) \in f(F, A_1)$ or $G(b) \in f(F, A_2)$

$\Rightarrow G(b) \in f(F, A_1) \cup f(F, A_2)$

$$\therefore f[(F, A_1) \cup (F, A_2)] \subseteq f(F, A_1) \cup f(F, A_2). \quad (1)$$

Now clearly $(F, A_1) \subseteq (F, A_1) \cup (F, A_2)$

$(F, A_2) \subseteq (F, A_1) \cup (F, A_2)$

$f(F, A_1) \subseteq f[(F, A_1) \cup (F, A_2)]$

$f(F, A_2) \subseteq f[(F, A_1) \cup (F, A_2)]$

$$\therefore f(F, A_1) \cup f(F, A_2) \subseteq f[(F, A_1) \cup (F, A_2)]. \quad (2)$$

From (1) and (2), we have $f[(F, A_1) \cup (F, A_2)] = f(F, A_1) \cup f(F, A_2)$.
(iv) Let $G(b) \in f[(F, A_1 \cap (F, A_2))]$.

Then $G(b) = f(F(a))$ for some $F(a)$ in $(F, A_1) \cap (F, A_2)$

$= f(F(a))$ for $F(a) \in (F, A_1)$ and $F(a) \in (F, A_2)$

$\Rightarrow G(b) \in f(F, A_1)$ and $G(b) \in f(F, A_2)$

$\Rightarrow G(b) \in f(F, A_1) \cap f(F, A_2)$

$\therefore f[(F, A_1) \cap (F, A_2)] \subseteq f(F, A_1) \cap f(F, A_2)$.

Conversely suppose $G(b) \in f(F, A_1) \cap f(F, A_2)$

$\Rightarrow G(b) \in f(F, A_1)$ and $G(b) \in f(F, A_2)$

$G(b) = f(F(a_1))$ for some $F(a_1) \in (F, A_1)$

$G(b) = f(F(a_2))$ for some $F(a_2) \in (F, A_2)$.

Now $f(F(a_1)) = f(F(a_2)) = G(b)$

$\Rightarrow F(a_1) = F(a_2)$ if f is one one

$\Rightarrow G(b) \in f[(F, A_1) \cap (F, A_2)]$.

Then $f[(F, A_1) \cap (F, A_2)] = f(F, A_1) \cap f(F, A_2)$ if f is one–one. □

Composition of soft set functions

Definition 5.9. Let $f : (F, A) \to (G, B)$, $g : (G, B) \to (H, C)$ be two soft set functions. Then $g \circ f : (F, A) \to (H, C)$ is also a soft set function defined by $(g \circ f)(F(a)) = g(f(F(a)))$.

Definition 5.10. Let f be an injective function from (F, A) to (G, B). Then the inverse relation f^{-1} is called the inverse function.

Theorem 5.11. *If $f : (F, A) \to (G, B)$ is bijective then Let $f^{-1} : (G, B) \to (F, A)$ is also a bijective function.*

Proof. Let $G(b_1) \neq G(b_2)$ for $G(b_1), G(b_2)$ in (G, B).
Let $f^{-1}(G(b_1)) = F(a_1)$ and $f^{-1}(G(b_2)) = F(a_2)$.
Then $f(F(a_1)) = G(b_1)$ and $f(F(a_2)) = G(b_2)$.
Thus $f(F(a_1)) \neq f(F(a_2))$

$$\Rightarrow F(a_1) \neq F(a_2) \text{ since } f \text{ is one–one}$$
$$\Rightarrow f^{-1}(G(b_1)) \neq f^{-1}(G(b_2)).$$

Hence f is one–one.

Now $F(a)$ be an element of (F, A). Since f is surjective there exists a unique element $G(b)$ in (G, B) such that $f(F(a)) = G(b)$
$\Rightarrow F(a) = f^{-1}(G(b))$ for $F(a)$ in (F, A).
Thus f^{-1} is onto. Hence f^{-1} is bijective. □

Theorem 5.12. *Let $f : (F, A) \to (G, B), g : (G, B) \to (H, C)$ be two bijective soft set functions. Then $g \circ f : (F, A) \to (H, C)$ is also a bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Proof. Let $F(a_1), F(a_2)$ be two distinct elements of (F, A). Then $F(a_1) \neq F(a_2)$

$$\Rightarrow f(F(a_1)) \neq f(F(a_2)), \quad \text{since } f \text{ is injective}$$
$$\Rightarrow g(f(F(a_1))) \neq g(f(F(a_2))), \quad \text{since } g \text{ is injective}$$
$$\Rightarrow (gf)(F(a_1)) \neq (g \circ f)(F(a_2)).$$

Hence $g \circ f$ is one–one.

Let $H(c)$ be an element of (H, C).Then there exists $G(b)$ in (G, B) such that $g(G(b)) = H(c)$ as g is onto. Again since f is onto there exists $F(a)$ in (F, A) such that $f(F(a)) = G(b)$.
Then $g(f(F(a))) = H(c)$ for every $H(c)$ in (H, C)

$$(g \circ f)(F(a)) = H(c)$$

$\therefore$ $g \circ f$ is onto. Hence $g \circ f$ is bijective.

Since $f, g, g \circ f$ are bijective they are invertible and for any relation R and S we have $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$, so we have in this case $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

6. Conclusion and future work

Soft set theory is a general method for solving problems of uncertainty. In the present paper the theoretical point of view of soft set is discussed. We extend the concepts of relations and functions in soft set theory context. We have also made an attempt to explain the equivalent version of some theories on relations and functions in the background of soft sets. All these concepts are basic supporting structures for research and development on soft set theory.

With the motivation of ideas presented in this paper one can think of multi soft sets and soft multisets, where a multiset is a set with repetitions. Research on the theoretical aspects of these generalized concepts seems to be more useful and need more attention. An attempt can be made in this direction by focusing on the theoretical foundation of these generalized concepts which are quite useful tools for soft computing. Further studies on the topology generated by the soft set relation may be done so that we may brood over the topological side of soft sets. Moreover a soft set relation can be extended in fuzzy soft sets and thus one can get more affirmative solution in decision making problems in real life situations.

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