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Journal of Algebra 310 (2007) 428–433

JOURNAL OF Algebra

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The generating hypothesis for the stable module category of a p-group

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Received 29 September 2006

Available online 4 January 2007

Communicated by Luchezar L. Avramov

Abstract

Freyd's generating hypothesis, interpreted in the stable module category of a finite p-group G, is the statement that a map between finite-dimensional kG-modules factors through a projective if the induced map on Tate cohomology is trivial. We show that Freyd's generating hypothesis holds for a non-trivial finite p-group G if and only if G is either C_2 or C_3. We also give various conditions which are equivalent to the generating hypothesis.

Keywords: Generating hypothesis; Stable module category; Ghost map

1. Introduction

The generating hypothesis (GH) is a famous conjecture in homotopy theory due to Peter Freyd [6]. It states that a map between finite spectra that induces the zero map on stable homotopy groups is null-homotopic. If true, the GH would reduce the study of finite spectra X to the study of their homotopy groups $\pi_*(X)$ as modules over $\pi_*(S^0)$. Therefore it stands as one of

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.12.013

the most important conjectures in stable homotopy theory. This problem is notoriously hard; despite serious efforts of homotopy theorists over the last 40 years, the conjecture remains open, see [4,5]. Keir Lockridge [9] showed that the analogue of the GH holds in the derived category of a commutative ring R if and only if R is a von Neumann regular ring (a ring over which every R-module is flat). More recently, Hovey, Lockridge and Puninski have generalised this result to arbitrary rings [7]. Lockridge's result [9] applies to any tensor triangulated category where the graded ring of self maps of the unit object is graded commutative and is concentrated in even degrees. Note that this condition is not satisfied by the stable homotopy category of spectra. So in order to better understand the GH for spectra, we formulate and solve the analogue of Freyd's GH in the stable module category of a finite p-group. Here the ring of self maps of the unit object (the trivial representation k) is non-zero in both even and odd degrees.

To set the stage, let G be a non-trivial finite p-group and let k be a field of characteristic p. Consider the stable module category $\mathrm{StMod}(kG)$ of G. It is the category obtained from the category of kG-modules by killing the projectives. The objects of $\mathrm{StMod}(kG)$ are the left kG-modules, and the space of morphisms between kG-modules M and N, denoted $\underline{\mathrm{Hom}}_{kG}(M, N)$, is the k-vector space of kG-module homomorphisms modulo those maps that factor through a projective module. $\mathrm{StMod}(kG)$ has the structure of a tensor triangulated category with the trivial representation k as the unit object and Ω as the loop (desuspension) functor. The category $\mathrm{stmod}(kG)$ is defined similarly using the finite-dimensional kG-modules. A key fact [1] is that the Tate cohomology groups can be described as groups of morphisms in $\mathrm{StMod}(kG)$: $\widehat{H}^i(G, M) \cong \underline{\mathrm{Hom}}(\Omega^i k, M)$. In this framework, the GH for kG is the statement that a map $\phi : M \to N$ between finite-dimensional kG-modules is trivial in $\mathrm{stmod}(kG)$ if the induced map in Tate cohomology $\underline{\mathrm{Hom}}(\Omega^i k, M) \to \underline{\mathrm{Hom}}(\Omega^i k, N)$ is trivial for each i. Maps between kG-modules that are trivial in Tate cohomology will be called *ghosts*. It is shown in [2] that there are no non-trivial ghosts in $\mathrm{StMod}(kG)$ if and only if G is cyclic of order 2 or 3. The methods in [2] do not yield ghosts in $\mathrm{stmod}(kG)$. In this paper, we use induction to build ghosts in $\mathrm{stmod}(kG)$. Our main theorem says:

Theorem 1.1. *Let G be a non-trivial finite p-group and let k be a field of characteristic p. There are no non-trivial maps in* $\mathrm{stmod}(kG)$ *that are trivial in Tate cohomology if and only if G is either C_2 or C_3. In other words, the generating hypothesis holds for kG if and only if G is either C_2 or C_3.*

Note that the theorem implies that the GH for p-groups does not depend on the ground field k, as long as its characteristic divides the order of G.

We now explain the strategy of the proof of our main theorem. We begin by showing that whenever the GH fails for kH, for H a subgroup of G, then it also fails for kG. We then construct non-trivial ghosts over cyclic groups of order bigger than 3 and over $C_p \oplus C_p$. It can be shown easily that the only finite p-groups that do not have one of these groups as a subgroup are the cyclic groups C_2 and C_3. And for C_2 and C_3 we show that the GH holds.

For a general finite group G, the GH is the statement that there are no non-trivial ghosts in the thick subcategory generated by k. When G is not a finite p-group, our argument does not necessarily produce ghosts in $\mathrm{thick}(k)$ and the GH is an open problem.

In the last section we give conditions on a finite p-group equivalent to the GH. One of them says that the GH holds for kG if and only if the category $\mathrm{stmod}(kG)$ is equivalent to the full subcategory of finite coproducts of suspensions of k. We also show that if the GH holds for a finite p-group, then the Tate cohomology functor $\widehat{H}^*(G,-)$ from stmod(kG) to the category of graded modules over the ring $\widehat{H}^*(G,k)$ is full.

Throughout we assume that the characteristic of k divides the order of the finite group G. For example, when we write kC_3, it is implicitly assumed that the characteristic of k is 3. We denote the desuspension of M in StMod(kG) by $\Omega(M)$, or by $\Omega_G(M)$ when we need to specify the group in question. All modules are assumed to be left modules.

2. Proof of the main theorem

Suppose H is a subgroup of G. A natural question is to ask how the truth or falsity of the GH for H is related to that for G. We begin by addressing this question.

Proposition 2.1. *Let H be a subgroup of a finite group G. If ϕ is a ghost in* stmod(kH)*, then $\phi\uparrow^G$ is ghost in* stmod(kG)*. Moreover, if ϕ is non-trivial in* stmod(kH)*, then so is $\phi\uparrow^G$ in* stmod(kG).

Proof. It is well known that the restriction $\underline{\mathrm{Res}}_H^G$ and induction $\underline{\mathrm{Ind}}_H^G$ functors form an adjoint pair of exact functors; see [8, Corollary 5.4] for instance. Therefore, for any kH-module L, we have a natural isomorphism

$$\underline{\mathrm{Hom}}_{kH}\big((\Omega_G^i k)\downarrow_H, L\big) \cong \underline{\mathrm{Hom}}_{kG}\big(\Omega_G^i k, L\uparrow^G\big).$$

But since $(\Omega_G^i k)\downarrow_H \cong \Omega_H^i k$ in stmod(kH), the above isomorphism can be written as

$$\underline{\mathrm{Hom}}_{kH}\big(\Omega_H^i k, L\big) \cong \underline{\mathrm{Hom}}_{kG}\big(\Omega_G^i k, L\uparrow^G\big).$$

The proposition now follows from the naturality of this isomorphism. The second statement follows from the observation that ϕ is a retract of $\phi\uparrow^G\downarrow_H$. □

Proposition 2.1 implies that if G is a finite p-group, then the GH fails for kG whenever it fails for a subgroup of G.

We now state two lemmas which will be needed in proving our main theorem.

Lemma 2.2. *Let G be a finite p-group and let x be a central element in G. Then for any kG-module M, the map $x-1: M \to M$ is a ghost.*

Proof. Since x is a central element, multiplication by $x-1$ defines a kG-linear map. We have to show that for all n and all maps $f: \Omega^n k \to M$, the composition $\Omega^n k \xrightarrow{f} M \xrightarrow{x-1} M$ factors through a projective. To this end, consider the commutative diagram

$$\begin{array}{ccc} \Omega^n k & \xrightarrow{f} & M \\ {\scriptstyle x-1}\big\downarrow & & \big\downarrow{\scriptstyle x-1} \\ \Omega^n k & \xrightarrow[f]{} & M. \end{array}$$

Note that $x-1$ acts trivially on k, so functoriality of Ω shows that the left vertical map is stably trivial. By commutativity of the square, the desired composition factors through a projective. □

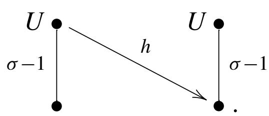
Lemma 2.3. *Let G be a finite p-group and let H be a non-trivial proper normal subgroup of G. If x is a central element in $G - H$, then multiplication by $x - 1$ on $k_H\uparrow^G$ is a non-trivial ghost, where k_H is the trivial kH-module. In particular, the GH fails for $k(C_p \oplus C_p)$.*

Proof. Since $k_H\uparrow^G\downarrow_H$ is a trivial kH-module, non-triviality of $x-1$ is easily seen by restricting to H. The fact that $x - 1$ is a ghost follows from Lemma 2.2. The last statement follows because $k_H\uparrow^G$ is finite-dimensional. □

Proof of Theorem 1.1. If $G \cong C_2$ and char $k = 2$, then $kC_2 \cong k[x]/(x^2)$, so by the structure theorem for modules over a PID it is clear that every kG-module is stably isomorphic to a sum of copies of k. Similarly, if $G \cong C_3$ and char $k = 3$, then one sees that every kG-module is stably isomorphic to a sum of copies of k and $\Omega(k)$. It follows that there are no non-trivial ghosts between finite-dimensional kG-modules if G is either C_2 or C_3.

Now suppose that G is not isomorphic to C_2 or C_3. It suffices to show that in these cases the GH fails for some subgroup of G. It is an easy exercise to show that if G is not isomorphic to C_2 or C_3, then G either has a cyclic subgroup of order at least four, or a subgroup isomorphic to $C_p \oplus C_p$ for some prime p. In Lemma 2.3 we have seen that the GH fails for $k(C_p \oplus C_p)$. We will be done if we can show that the GH fails for cyclic groups of order at least 4.

So let G be a cyclic group of order at least 4. Let σ be a generator for G and let M be a cyclic module of length two generated by U, so we have $(\sigma - 1)^2 U = 0$. Consider the map $h : M \to M$ which multiplies by $\sigma - 1$:



It is not hard to see that h is non-trivial, i.e., that it does not factor through the projective cover of M; this is where we use the hypothesis $|G| \geqslant 4$. The fact that h is a ghost follows from Lemma 2.2. □

3. Conditions equivalent to the generating hypothesis

Theorem 3.1. *The following are equivalent statements for a non-trivial finite p-group G.*

(1) *G is isomorphic to C_2 or C_3.*
(2) *There are no non-trivial ghosts in* stmod(kG). *That is, the GH holds for kG.*
(3) *There are no non-trivial ghosts in* StMod(kG).
(4) stmod(kG) *is equivalent to the full subcategory of the collection of finite coproducts of suspensions of k.*
(5) StMod(kG) *is equivalent to the full subcategory of arbitrary coproducts of suspensions of k.*

Proof. We have already seen that the statements (2) and (4) are equivalent to (1). The implications (5) ⇒ (3) ⇒ (2) are obvious. So we will be done if we can show that (1) ⇒ (5). This follows immediately from the following more general fact, due to Crawley and Jónsson [3], which was also proved independently by Warfield [10]. It states that if G has finite representation type (i.e., the Sylow p-subgroups are cyclic), then every kG-module is a direct sum of finite-dimensional kG-modules. □

We now state a dual version of the previous theorem. A map $d : M \to N$ between kG-modules is called a *dual ghost* if the induced map

$$\underline{\mathrm{Hom}}_{kG}\big(M, \Omega^i k\big) \leftarrow \underline{\mathrm{Hom}}_{kG}\big(N, \Omega^i k\big)$$

is zero for all i.

Theorem 3.2. *The following are equivalent statements for a non-trivial finite p-group G.*

(1) *G is isomorphic to C_2 or C_3.*
(2′) *There are no non-trivial dual ghosts in* $\mathrm{stmod}(kG)$.
(3′) *There are no non-trivial dual ghosts in* $\mathrm{StMod}(kG)$.
(4′) $\mathrm{stmod}(kG)$ *is equivalent to the full subcategory of the collection of finite products of suspensions of k.*
(5′) $\mathrm{StMod}(kG)$ *is equivalent to the full subcategory of retracts of arbitrary products of suspensions of k.*

Proof. Every finite-dimensional kG-module M is naturally isomorphic to its double dual M^{**}. Therefore, the exact functor $M \mapsto M^*$ gives a tensor triangulated equivalence between $\mathrm{stmod}(kG)$ and its opposite category. This shows that (2′) $\Leftrightarrow$ (2). In any additive category finite coproducts and finite products are the same, therefore (4′) $\Leftrightarrow$ (4). Thus, statements (1), (2′), and (4′) are equivalent. We will be done if we can show that (5′) $\Rightarrow$ (3′) $\Rightarrow$ (1) $\Rightarrow$ (5′).

(5′) $\Rightarrow$ (3′): Fix an arbitrary kG-module M. We have to show that there are no non-trivial dual ghosts out of M. Consider the full subcategory of all modules X such that there is no non-trivial dual ghost from M to X. This subcategory clearly contains arbitrary products of suspensions of k and is closed under taking retractions. So by assumption the subcategory has to be $\mathrm{StMod}(kG)$.

(3′) $\Rightarrow$ (1): (3′) clearly implies (2′). But we have already observed that (2′) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (5′): We know that (1) $\Rightarrow$ (5). It remains to show that (5) $\Rightarrow$ (5′). Let M be any kG-module. By (5), M is a coproduct $\bigoplus \Omega^s k$ of suspensions of k. We will complete the proof by showing that the canonical map

$$\bigoplus \Omega^s k \to \prod \Omega^s k$$

is a split monomorphism in $\mathrm{StMod}(kG)$. By (5), the fibre F of this map is a coproduct $\bigoplus \Omega^t k$ of suspensions of k. Since the objects $\Omega^t k$ are compact, one can show that the map $F \to \bigoplus \Omega^s k$ is zero and therefore the desired splitting exists. □

We end with a final observation. In the stable homotopy category of spectra, the GH says that the stable homotopy functor $\pi_*(-)$ from the category of finite spectra to the category of graded modules over the homotopy ring $\pi_*(S^0)$ of the sphere spectrum is faithful. Freyd showed [6] that if the GH is true, then $\pi_*(-)$ is also full. So it is natural to ask if the same is true in other algebraic settings in which the GH is being studied. Very recently, Hovey, Lockridge and Puninski [7] have given an example of ring R for which the homology functor $H_*(-)$ from the category of perfect complexes of R-modules to the category of graded R-modules is faithful, but not full. It turns out that from this point of view, the stable module category of a group behaves more like the stable homotopy category of spectra than the derived category of a ring. More precisely, we have the following result.

Theorem 3.3. *Let G be a finite p-group and let k be a field of characteristic p. If the GH holds for G, then the functor $\widehat{H}^*(G,-)$ from* $\mathrm{stmod}(kG)$ *to the category of graded modules over the graded ring $\widehat{H}^*(G,k)$ is full.*

Proof. We know by Theorem 1.1 that G has to be either C_2 or C_3. Therefore every finite-dimensional kG-module M is stably isomorphic to a finite sum of suspensions of k. In particular, $\widehat{H}^*(G,M)$ is a free $\widehat{H}^*(G,k)$-module of finite rank. It follows that the induced map

$$\underline{\mathrm{Hom}}_{kG}(M,X) \to \mathrm{Hom}_{\widehat{H}^*(G,k)}\big(\widehat{H}^*(G,M), \widehat{H}^*(G,X)\big)$$

is an isomorphism for all kG-modules X. Since M was an arbitrary finite-dimensional kG-module, we have shown that the functor $\widehat{H}^*(G,-)$ is full, as desired. □

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