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Morse index and stability of elliptic Lagrangian solutions in the planar three-body problem

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Abstract

We illustrate a new way to study the stability problem in celestial mechanics. In this paper, using the variational nature of elliptic Lagrangian solutions in the planar three-body problem, we study the relation between Morse index and its stability via Maslov-type index theory of periodic solutions of Hamiltonian system. For elliptic Lagrangian solutions we get an estimate of the algebraic multiplicity of unit eigenvalues of its monodromy matrix in terms of the Morse index, which is the key to understand the stability problem. As a special case, we provide a criterion to spectral stability of relative equilibrium.

Keywords: Planar three-body problem; Lagrangian solution; Linear stability; Morse index; Maslov-type index

1. Introduction

We consider the planar three-body problem. Let $q_1, q_2, q_3 \in \mathbf{R}^2$ be the position vectors of three particles with masses $m_1, m_2, m_3 > 0$ respectively. We denote by $\|\cdot\|$ the standard norm of

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vectors in Euclidian space. Suppose the particles interact each other under the gravity, then the Newton system of equations is

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i = 1, 2, 3, \tag{1}$$

where $U(q) = U(q_1, q_2, q_3) = \sum_{1 \leqslant i < j \leqslant 3} \frac{m_i m_j}{\|q_i - q_j\|}$ is the potential or force function. As far as periodic solution is concerned, it is the Euler–Lagrange equation of the action functional

$$\mathcal{A}(q) = \int_0^T \left[\sum_{i=1}^{3} \frac{m_i \|\dot{q}_i(t)\|^2}{2} + U\bigl(q(t)\bigr) \right] dt$$

defined on loop space $W^{1,2}(\mathbf{R}/T\mathbf{Z}, \hat{\mathcal{X}})$ for a fixed positive real number T as period, where

$$\hat{\mathcal{X}} := \left\{ q = (q_1, q_2, q_3) \in \bigl(\mathbf{R}^2\bigr)^3 \;\Big|\; \sum_{i=1}^{3} m_i q_i = 0,\ q_i \neq q_j,\ \forall i \neq j \right\}$$

is the configuration space of the planar three-body problem. In other words any periodic solution to (1) is a critical point of the action functional.

It is well known that (1) can be converted into a Hamiltonian system by Legendrian transformation. We denote by $p_1, p_2, p_3 \in \mathbf{R}^2$ the momentum vectors of the particles respectively. The Hamiltonian system corresponding to (1) is

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad i = 1, 2, 3, \tag{2}$$

with Hamiltonian function

$$H(p, q) = H(p_1, p_2, p_3; q_1, q_2, q_3) = \sum_{i=1}^{3} \frac{\|p_i\|^2}{2m_i} - U(q_1, q_2, q_3). \tag{3}$$

In 1772, in his well-known paper for the prize of Paris Royal Scientific Academy [14], Lagrange considered some special solutions, now named after him, of the three-body problem, namely the three bodies form an equilateral triangle at any instant of the motion and at the same time each body travels along a specific Keplerian orbit about the center of masses of the system. These solutions were found by Lagrange purely from mathematical interests, and only later it was realized that such a configuration can be used to analyze the Sun–Jupiter–Trojan asteroids system.

If the Keplerian motion is a circle with some appropriate frequency, then all the three bodies move around the center of masses with the same frequency. It would be an equilibrium in the coordinate system rotating around the center of masses in the same frequency. So it is called *relative equilibrium* or *Lagrangian circular orbit*.

When the Keplerian orbit is elliptic, following Meyer and Schmidt [26], we call this elliptic Lagrangian solution *elliptic relative equilibrium*.

Both of these two kinds of orbits are known as the homographic solutions. The equilateral triangle is an example of central configurations of three-body problem. In celestial mechanics, central configuration plays an important role because we can construct the homographic solutions of general n-body problem explicitly from central configurations and Keplerian orbits. Up to now this is the only known way to get exact solutions of the general n-body problem which is already known to Euler and Lagrange. For the state of arts on this topic, see [12].

In this paper we are mainly interested in the linear stability problem of Lagrangian solutions. We want to clarify its variational nature and understand it from the point of view of index theory of periodic solutions of Hamiltonian system.

For three-body problem with masses $m_1, m_2, m_3 > 0$, we define

$$\beta = \frac{27(m_1m_2 + m_1m_3 + m_2m_3)}{(m_1 + m_2 + m_3)^2}. \tag{4}$$

In 1843, Gascheau [10] proved that Lagrangian circular orbit in three-body problem is linearly stable if and only if $\beta < 1$. Later in 1875, Routh [30] also proved this result independently.

Because of curiosity to the relation between resonance and stability, Danby [5] considered linear stability of elliptic relative equilibrium in restricted three-body problem. Now the stability depends on the eccentricity e and mass ratio μ. He used first variational equations and numerical methods to get the bifurcation diagram of stability in the (e, μ)-plane. Later Schmidt [31] gave a purely analytical proof.

Danby [6] also started to study the linear stability of the elliptic relative equilibrium in the general three-body problem. He was very sketchy and reduced the problem to that of restricted case. In this general case the stability also depends on two parameters, namely the eccentricity e and β.

Later Roberts [28] made further progress by reducing all the symmetries and their first integrals. Then he applied perturbation techniques to small $e > 0$ rigorously and used numerical methods for large $e > 0$. He got the bifurcation diagram partially in the (e, β)-plane for the stability.

Recently Meyer and Schmidt [26] reconsidered the stability for small $e > 0$ case via different method. They depended heavily on the central configuration nature of the elliptic relative equilibrium. Their methods are very useful to us, and we will give the details later.

Martínez, Samà and Simó [25] studied the stability problem when $e > 0$ is small by using normal form theory and $e \lesssim 1$ by using blow-up technique in general homogeneous potential. They also gave much more complete bifurcation diagram numerically.

On the other hand, Maslov-type index theory [4,8,15,19,20,33] has been well developed to study the existence, multiplicity and stability of periodic solutions for general Hamiltonian system. It is a powerful tool to investigate periodic solutions of variational nature [24]. In the next section we will review basic facts about Maslov-type index theory we need.

The main idea to the stability problem of periodic solutions by Maslov-type index theory is based on the following fact: different ω-index [19] could give estimate of the ellipticity. The Bott-type iteration formula is essential to this purpose. Dell'Antonio, D'Onofrio and Ekeland [7] studied stability of the periodic solutions of the convex Hamiltonian system, and they proved there exists at least one elliptic closed characteristic on any symmetric closed hypersurface. Later Long built up the precise iteration formula for general Hamiltonian systems, and he proved on convex hypersurface in $\mathbf{R}^4$, both of them are elliptic if there are only 2 closed characteristics [18]. Great progress was made by Long and Zhu [24], and they proved that if the number of the closed characteristics on convex hypersurface in $\mathbf{R}^{2n}$ is finite, then there are at least $[\frac{n}{2}]+1$ closed characteristics and at least one of them is elliptic. It is natural to apply these ideas to the concrete classical Hamiltonian system–n-body problem. As a first step in this program, we use Maslov-type index theory to the stability of the elliptic Lagrangian solutions.

In the calculus of variation, Morse index is natural information adhere to the critical point. Fortunately for Lagrangian system, a celebrated result of Long [19] tells us that for periodic solution, this flexible Maslov-type index of corresponding first order Hamiltonian system is equal to its Morse index seen as critical point. In this paper, we give the relation between Morse index and stability, and compute the index.

For the stability analysis of solutions to the n-body problem, it is always important to clarify and factor out the effects from first integrals of the problem. Following Meyer and Schmidt [26], the fundamental solution of the elliptic relative equilibrium is decomposed into two parts symplectically, one of which is the same as that of the Keplerian solution and the other is the essential part to the stability. We first analyze the Poincaré map of the Keplerian solution, and we prove that the Poincaré map of Keplerian solution is decomposed into two 2×2 matrices $I_2=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. A theorem of Gordon which asserts that the Keplerian solution is a minimizer in the loop space under some topological constraint is crucial to study the property of the Morse index.

To the essential part, a theorem of Venturelli says that Lagrangian solution is the minimizer among the loops in its homology class. From this theorem and the analysis of the Keplerian solution, we could give criterion of the stability by Morse index. Let ϕ_k be the Morse index of the k-th iteration of the Lagrangian solution in the variational problem, and according to Venturelli $\phi_1=0$. Let $e(M)$ be the total algebraic multiplicity of all eigenvalues of M on the unite circle in the complex plan. For $M\in\mathrm{Sp}(2n)$, it is spectrally stable if $e(M)=2n$, and linearly stable if M is spectrally stable and semi-simple. We denote by $\gamma_2(t)$ the essential part of the fundamental solution. The Lagrangian solution is linearly stable (spectrally stable) if $\gamma_2(T)$ is linearly stable (spectrally stable). Our main theorems are:

Theorem 1.1. *For the monodromy matrix M corresponding to the elliptic Lagrangian solution x, $2\leqslant\phi_2\leqslant 4$ and,*

$$e(M)/2\geqslant\phi_2. \tag{5}$$

Moreover

(I) *If $\phi_2=4$, then the Lagrangian solution is spectrally stable;*
(II) *If $\phi_2=3$, then the Lagrangian solution is linearly unstable;*
(III) *If $\phi_2=2$, then the Lagrangian solution is spectrally stable if there exists some integer $k\geqslant 3$, such that $\phi_k>2(k-1)$;*
(IV) *If $\phi_k=2(k-1)$, for all $k\in\mathbf{N}$, then the Lagrangian solution is linearly unstable.*

Moreover, if the essential part of monodromy matrix at $2T$ is non-degenerate, we can get its normal forms at T.

Theorem 1.2. *Under the same setting as above theorem, if $\gamma_2(2T)$ is non-degenerate, then*

(I) *If $\phi_2=4$, then $\gamma_2(T)$ is linearly stable. Moreover, $\exists P\in\mathrm{Sp}(4)$, such that $\gamma_2(T)=P^{-1}(R(2\pi-\theta_1)\diamond R(2\pi-\theta_2))P$, with $\theta_1,\theta_2\in(0,\pi)$;*

(II) *If* $\phi_2 = 3$, *then* $\exists P \in \mathrm{Sp}(4)$, *such that* $\gamma_2(T) = P^{-1}(D(\lambda) \diamond R(2\pi - \theta))P$, *with* $\lambda < 0$, $\theta \in (0, \pi)$*;*

(III) *If* $\phi_2 = 2$, *and there exists some integer* $k \geqslant 3$, *such that* $\phi_k > 2(k-1)$, *then* $\gamma_2(T)$ *is linearly stable. Moreover,* $\exists P \in \mathrm{Sp}(4)$, *such that* $\gamma_2(T) = P^{-1}(R(2\pi - \theta_1) \diamond R(\theta_2))P$, *with* $0 < \theta_1 < \theta_2 < \pi$*;*

(IV) *If* $\phi_k = 2(k-1)$, *for all* $k \in \mathbf{N}$, *then* $\gamma_2(T)$ *is hyperbolic or spectrally stable and linearly unstable.*

Please refer to (13), (15) for the definition of $D(\lambda)$ and $R(\theta)$, to (6) for $\diamond$.

There are some numerical computations on the stability of Lagrangian solutions which depend on mass ratio β and eccentricity e [25,26,28], and a beautiful figure is given in [25]. We will explain this figure by the Morse index, and this is another confirmation that the index theory is a better tool to the stability problems.

This method can also be used to study recently discovered periodic orbits (see [1] for a survey and closely related [9]) in celestial mechanics by minimizing methods on various loop spaces. In another paper [13] we work on the celebrated figure-eight periodic solutions due to Chenciner and Montgomery [3] in the planar three-body problem with equal masses.

The paper is organized as follows. In Section 2, we recall Maslov-type index theory for symplectic paths in symplectic groups. In Section 3, we use the coordinate decomposition for elliptic Lagrangian solutions of Meyer and Schmidt [26] to give the decomposition of the symplectic paths of its fundamental solution matrices and factor out those from first integrals. Section 4 is the main part of the paper, and we give the criteria of stability via index. In Section 5, we consider in detail the Lagrangian circular orbits. At last, in Section 6, we give an explanation of the figure derived in [25] via Morse index.

2. Review of the Maslov-type index for symplectic matrix paths

In this section, we firstly recall briefly the Maslov-type index theory for symplectic matrix paths. All the details can be found in [19]. Our main goal in this paper is the relation of Morse index and the stability of elliptic Lagrangian solutions via this index.

Let $(\mathbf{R}^{2n}, \Omega)$ be the standard symplectic vector space with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, then $\Omega = \sum_{i=1}^n dx_i \wedge dy_i$. Let $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, where I_n is the identity matrix on $\mathbf{R}^n$.

As usual, the symplectic group $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n) = \left\{ M \in \mathrm{GL}(2n, \mathbf{R}) \;\middle|\; M^T J M = J \right\},$$

whose topology is induced from that of $\mathbf{R}^{4n^2}$. For $\tau > 0$ we are interested in paths in $\mathrm{Sp}(2n)$:

$$\mathcal{P}_\tau(2n) = \left\{ \gamma \in C\big([0, \tau], \mathrm{Sp}(2n)\big) \;\middle|\; \gamma(0) = I_{2n} \right\},$$

which is equipped with the topology induced from that of $\mathrm{Sp}(2n)$. For any $\omega \in \mathbf{U} := \{z \in \mathbf{C} \mid \|z\| = 1\}$ and $M \in \mathrm{Sp}(2n)$, the following real function was introduced in [17]:

$$D_\omega(M) = (-1)^{n-1} \overline{\omega}^n \det(M - \omega I_{2n}).$$

Thus for any $\omega \in \mathbf{U}$ the following codimension 1 hypersurface in $\mathrm{Sp}(2n)$ is defined [17]:

$$\mathrm{Sp}(2n)^0_\omega = \left\{ M \in \mathrm{Sp}(2n) \;\middle|\; D_\omega(M) = 0 \right\}.$$

For any $M \in \mathrm{Sp}(2n)^0_\omega$, we define a co-orientation of $\mathrm{Sp}(2n)^0_\omega$ at M by the positive direction $\frac{d}{dt}Me^{tJ}|_{t=0}$ of the path Me^{tJ} with $0 \leqslant t \leqslant \varepsilon$, ε small enough positive number. Let

$$\begin{aligned}
\mathrm{Sp}(2n)^*_\omega &= \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)^0_\omega,\\
\mathcal{P}^*_{\tau,\omega}(2n) &= \big\{\gamma \in \mathcal{P}_\tau(2n) \,\big|\, \gamma(\tau) \in \mathrm{Sp}(2n)^*_\omega\big\},\\
\mathcal{P}^0_{\tau,\omega}(2n) &= \mathcal{P}_\tau(2n) \setminus \mathcal{P}^*_{\tau,\omega}(2n).
\end{aligned}$$

For any two continuous paths ξ and $\eta : [0,\tau] \to \mathrm{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, we define their concatenation as:

$$\eta * \xi(t) = \begin{cases} \xi(2t), & \text{if } 0 \leqslant t \leqslant \tau/2,\\ \eta(2t-\tau), & \text{if } \tau/2 \leqslant t \leqslant \tau. \end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \left(\begin{smallmatrix} A_k & B_k \\ C_k & D_k \end{smallmatrix}\right)$ with $k = 1, 2$, the $\diamond$-product of M_1 and M_2 is defined [19] by the following $2(m_1+m_2) \times 2(m_1+m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}. \tag{6}$$

For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and 1, let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ for all $t \in [0,\tau]$.

We define a special continuous symplectic path $\xi_n \subset \mathrm{Sp}(2n)$ by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\diamond n} \quad \text{for } 0 \leqslant t \leqslant \tau. \tag{7}$$

Definition 2.1. (See [17,19].) For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, define

$$\nu_\omega(M) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I_{2n}). \tag{8}$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega\big(\gamma(\tau)\big). \tag{9}$$

If $\gamma \in \mathcal{P}^*_{\tau,\omega}(2n)$, define

$$i_\omega(\gamma) = \big[\mathrm{Sp}(2n)^0_\omega \colon \gamma * \xi_n\big], \tag{10}$$

where the right-hand side of (10) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}^0_{\tau,\omega}(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf\big\{i_\omega(\beta) \,\big|\, \beta \in U \cap \mathcal{P}^*_{\tau,\omega}(2n)\big\}. \tag{11}$$

Then

$$\big(i_\omega(\gamma), \nu_\omega(\gamma)\big) \in \mathbf{Z} \times \{0, 1, \dots, 2n\},$$

is called the index function of γ at ω.

Note that when $\omega = 1$, this index theory was introduced by Conley and Zehnder in [4] for the non-degenerate case with $n \geqslant 2$, Long and Zehnder in [21] for the non-degenerate case with $n = 1$, and Long in [16] and Viterbo in [34] independently for the degenerate case. The case for general $\omega \in \mathbf{U}$ was defined by Long in [17] in order to study the index iteration theory (cf. [19] for more details and references).

As in [17], let $\Omega^0(M)$ be the path-connected component containing $M = \gamma(\tau)$ of the set

$$\Omega(M) = \big\{N \in \mathrm{Sp}(2n) \,\big|\, \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and } \nu_\lambda(N) = \nu_\lambda(M)\ \forall \lambda \in \sigma(M) \cap \mathbf{U}\big\}, \tag{12}$$

where $\sigma(\cdot)$ denotes the spectrum of a matrix, that is the set of its total eigenvalues. Here $\Omega^0(M)$ is called the *homotopy component* of M in $\mathrm{Sp}(2n)$. For a continuous family of paths $\gamma_s(t)$ with $(s,t) \in [0,1] \times [0,T]$, $\gamma_s(T) \in \Omega^0(\gamma_0(T))$, then $i_w(\gamma_s)$ is independent of s.

In [17–19], the following symplectic matrices were introduced as *basic normal forms*:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda = \pm 2, \tag{13}$$

$$N_1(\lambda, a) = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \pm 1,\ a = \pm 1, 0, \tag{14}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in (0,\pi) \cup (\pi, 2\pi), \tag{15}$$

$$N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix} \quad \text{with } b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \tag{16}$$

where $\omega = e^{\sqrt{-1}\theta}$, $\theta \in (0,\pi) \cup (\pi, 2\pi)$, $b_i \in \mathbf{R}$ and $b_2 \neq b_3$.

As proved in [17], any $M \in \mathrm{Sp}(2n)$ can be connected to N in $\Omega^0(M)$, where

$$N = M_1 \diamond \cdots \diamond M_j \tag{17}$$

with M_i, $i = 1, \dots, j$ in basic normal form. For two paths, it is obvious that [19]

$$i_\omega(\gamma_1 \diamond \gamma_2) = i_\omega(\gamma_1) + i_\omega(\gamma_2), \quad \forall \omega \in \mathbf{U}. \tag{18}$$

Remark 2.2. The normal form of symplectic matrix is the Jordan block under the symplectic transform, and we remind the reader that for 2×2 matrix, the normal form is the same as the basic normal form in (13)–(15) [19].

Definition 2.3. (See [17,19].) For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, the *splitting numbers* $S_M^\pm(\omega)$ of M at ω are defined by

$$S_M^\pm(\omega) = \lim_{\epsilon \to 0^+} i_{\omega \exp(\pm\sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma), \tag{19}$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

For splitting numbers, we have:

Lemma 2.4. *(See [17,19].) Splitting numbers $S_M^{\pm}(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$ appeared in* (19)*. For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, splitting numbers $S_N^{\pm}(\omega)$ are constant for all $N \in \Omega^0(M)$.*

Lemma 2.5. *(See [17], [19, pp. 198–199].) For $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, $0 < \theta < \pi$, there hold*

$$S_M^{\pm}(\omega) = 0, \quad \textit{if } \omega \notin \sigma(M), \tag{20}$$

$$S_M^{\pm}(\omega) = S_M^{\mp}(\bar{\omega}), \tag{21}$$

$$0 \leqslant S_M^{\pm}(\omega) \leqslant \dim\ker(M - \omega I), \tag{22}$$

$$S_M^{+}(\omega) + S_M^{-}(\omega) \leqslant \dim\ker(M - \omega I)^{2n}, \quad \textit{if } \omega \in \sigma(M), \tag{23}$$

$$\big(S_{N_1(1,a)}^{+}(1), S_{N_1(1,a)}^{-}(1)\big) = \begin{cases} (1,1), & \textit{if } a = 0, 1, \\ (0,0), & \textit{if } a = -1, \end{cases} \tag{24}$$

$$\big(S_{N_1(-1,a)}^{+}(-1), S_{N_1(-1,a)}^{-}(-1)\big) = \begin{cases} (1,1), & \textit{if } a = 0, -1, \\ (0,0), & \textit{if } a = 1, \end{cases} \tag{25}$$

$$\big(S_{R(\theta)}^{+}\big(e^{\sqrt{-1}\theta}\big), S_{R(\theta)}^{-}\big(e^{\sqrt{-1}\theta}\big)\big) = (0,1), \tag{26}$$

$$\big(S_{R(2\pi-\theta)}^{+}\big(e^{\sqrt{-1}\theta}\big), S_{R(2\pi-\theta)}^{-}\big(e^{\sqrt{-1}\theta}\big)\big) = (1,0), \tag{27}$$

$$\big(S_{N_2(\omega,b)}^{+}(\omega), S_{N_2(\omega,b)}^{-}(\omega)\big) = \begin{cases} (1,1), & \textit{if } (b_2 - b_3)\sin\theta < 0, \\ (0,0), & \textit{if } (b_2 - b_3)\sin\theta > 0, \end{cases} \tag{28}$$

For any $M_i \in \mathrm{Sp}(2n_i)$ with $i = 0$ and 1, there holds

$$S_{M_0 \diamond M_1}^{\pm}(\omega) = S_{M_0}^{\pm}(\omega) + S_{M_1}^{\pm}(\omega), \quad \forall \omega \in \mathbf{U}. \tag{29}$$

For any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbf{N}$, we define its m-th iteration $\gamma^m : [0, m\tau] \to \mathrm{Sp}(2n)$ by

$$\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \text{for } j\tau \leqslant t \leqslant (j+1)\tau,\ j = 0, 1, \ldots, m-1. \tag{30}$$

The next Bott-type iteration formula is a basic tool to geometric multiplicity of periodic orbits.

Lemma 2.6. *(See [19, Theorem 9.2.1, p. 199].) For any $z \in \mathbf{U}$,*

$$i_z\big(\gamma^m\big) = \sum_{\omega^m = z} i_\omega(\gamma). \tag{31}$$

For $M \in \mathrm{Sp}(2n)$, we denote by $e(M)$ the total algebraic multiplicity of all eigenvalues of M on $\mathbf{U}$.

Definition 2.7. For $M \in \mathrm{Sp}(2n)$, we say M is linearly stable if $\|M^k\|$ is bounded for all $k \in \mathbf{N}$, and M is spectrally stable if $e(M) = 2n$.

Note that $M \in \mathrm{Sp}(2n)$ is linearly stable implies that M is spectrally stable and semi-simple, and this shows that M can be split into two-dimensional rotations.

Choose any path γ of symplectic matrices from I_{2n} to M, the deference of ω-index for ω in $\mathbf{U}$ could provide a lower bound for $e(M)$. A criteria which will be used later for the elliptic Lagrangian solutions is as follows:

Lemma 2.8. *Suppose* $\gamma \in \mathcal{P}_\tau(2n)$ *with* $\gamma(\tau) = M = P^{-1}(N_1(1,-1)^{\diamond j} \diamond M_1)P$ *for some* $P \in \mathrm{Sp}(2n)$*, then*

$$\frac{e(M)}{2} \geqslant j + \left|i_1(\gamma) - i_\omega(\gamma)\right| + \left|i_\omega(\gamma) - i_{-1}(\gamma)\right|, \quad \forall \omega \in \mathbf{U}. \tag{32}$$

Proof. Without loss of generality, we suppose that $\omega \in \mathbf{U}$ and $\mathrm{Im}(\omega) \geqslant 0$. By definition of splitting numbers,

$$i_1(\gamma) - i_\omega(\gamma) = -\left(S_M^+(1) + \sum_{\omega_0}\left(S_M^+(\omega_0) - S_M^-(\omega_0)\right) - S_M^-(\omega)\right), \tag{33}$$

where the sum is taken over all the eigenvalues ω_0 of M on $\mathbf{U}$ in the arc from 1 to ω along the upper semi circle. Similarly,

$$i_\omega(\gamma) - i_{-1}(\gamma) = -\left(S_M^+(\omega) + \sum_{\omega_0}\left(S_M^+(\omega_0) - S_M^-(\omega_0)\right) - S_M^-(-1)\right), \tag{34}$$

where the sum is taken over all the eigenvalues ω_0 of M on $\mathbf{U}$ in the arc from w to -1 along the upper semi circle.

Note that for any ω on $\mathbf{U}$,

$$S_M^\pm(\omega) = S_{M_1}^\pm(\omega). \tag{35}$$

So

$$\left|i_1(\gamma) - i_\omega(\gamma)\right| + \left|i_\omega(\gamma) - i_{-1}(\gamma)\right| \leqslant e(M_1)/2. \tag{36}$$

Since $M = P^{-1}(N_1(1,-1)^{\diamond j} \diamond M_1)P$, by the definition of $e(M)$ we have

$$e(M)/2 = j + e(M_1)/2. \tag{37}$$

This ends the proof. □

3. First integrals and decompositions of symplectic paths

Now we turn to the elliptic Lagrangian solutions of the planar three-body problem. As we stated before, any planar central configuration of the n-body problem gives rise to a solution where each body moves in a specific Keplerian orbit and at the same time the configurations formed by the bodies keep its similarity shape with respect to the center of masses. Meyer and

Schmidt [26] give a beautiful coordinate system in which the linear variational equation corresponding to this solution decouples into three subsystems. One of them refers to the motion of center of masses, another is Keplerian orbits and the last shows the nontrivial characteristic multipliers. The merit of this coordinate system is that the decomposition is symplectic, in other words, any two parts are mutual symplectic complements to each other. This fits quite well to the index theory of the last section.

Recall that we have fixed the center of masses once and for all at the beginning. It is well known that the solution to the linearized equation of the solution to any Hamiltonian system is a continuous path of symplectic matrices starting from identity matrix. Accordingly, in our case, the symplectic path $\gamma \in \mathcal{P}_T(8)$ of fundamental solution matrices of the Lagrangian solution decomposes into two symplectic paths $\gamma_1 \in \mathcal{P}_T(4)$ and $\gamma_2 \in \mathcal{P}_T(4)$, where γ_1 is the symplectic path of fundamental solution matrices of the Keplerian solution which corresponds to the first integrals of the energy and the angular momentum, and γ_2 is the essential part and our main concern which will be studied in details in the next section. In our notation of the last section,

$$\gamma = \gamma_1 \diamond \gamma_2. \tag{38}$$

Here, we suppose that T is the prime period of the Lagrangian solution. From [26], γ_1 is the fundamental solution of the Keplerian solution with prime period T.

Definition 3.1. The Lagrangian solution is spectrally (or linearly) stable if $\gamma_2(T)$ is spectrally (or linearly, respectively) stable.

We will show that the monodromy matrix of γ_1 can be decomposed into two 2×2 Jordan blocks of the form I_2 and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

At first, we compute the Jordan block of the energy first integral.

For this purpose we need some general fact about the periodic orbits of n-body problem. The key observation is that if $x(t)$ is a periodic solution of period T to (1), then $h^{-2/3}x(ht)$ is also a solution. We set $z_h(t) = (h^{1/3}\dot{x}(ht), h^{-2/3}x(ht))^T$, then $z_h(t)$ is a solution of the Hamiltonian system (2) and it satisfies

$$z_h(T_h) = z_h(0), \tag{39}$$

where $T_h = T/h$ is the period of the $z_h(t)$ and

$$H(z_h) = h^{2/3}H(z_1). \tag{40}$$

The next two lemmas are motivated by Ekeland and Long [8,19] in their studies of the closed characteristics on convex energy hypersurface.

Lemma 3.2. *The monodromy matrix M of fundamental solution path $\gamma(t)$ of a T-periodic solution to* (2) *with $\gamma(T) = M$ satisfies*

$$M\dot{z}(0)=\dot{z}(0), \tag{41}$$

$$-T\dot{z}(0)+M\frac{d}{dh}z_h(0)|_{h=1}=\frac{d}{dh}z_h(0)|_{h=1}. \tag{42}$$

Proof. From the definition of fundamental solution of $z_h(t)$, we have

$$M_h\left(\frac{d}{dh}z_h\right)(0)=\frac{d}{dh}z_h(T_h) \quad \text{with } M_h=\gamma_h(T_h). \tag{43}$$

Differentiating (39) with respect to h yields

$$\dot{z}_h(T_h)\frac{dT_h}{dh}+\frac{d}{dh}z_h(T_h)=\frac{d}{dh}z_h(0). \tag{44}$$

Plugging (43) to (44), and letting $h=1$ yield (42). This ends the proof. □

Lemma 3.3. *For any periodic solution $z(t)$ of the n-body problem* (2) *with monodromy matrix M, there exist $P\in \mathrm{Sp}(2n)$ and $M_1\in \mathrm{Sp}(2n-2)$, such that*

$$M=P^{-1}\big(N_1(1,1)\diamond M_1\big)P. \tag{45}$$

Proof. Let $\xi_1=T\dot{z}(0)$, $\xi_2=\frac{d}{dh}z_h(0)|_{h=1}$, direct computation shows that

$$\omega(\xi_1,\xi_2)=\left\langle J\cdot TJH'\big(z(0)\big),\frac{d}{dh}z_h(0)|_{h=1}\right\rangle=-T\frac{d}{dh}H(z_h)>0.$$

Note that here we have used the fact that for any periodic solution of n-body problem, it has negative energy.

So the space spanned by ξ_1,ξ_2 is the invariant symplectic subspace of M, and ξ_1,ξ_2 is the symplectic basis of this subspace. By Lemma 3.2, following Lemma 15.3.4 of [19, p. 328], M restricted to this subspace is $N_1(1,1)$. Since M is symplectic, we have the result. □

For the solution to the Kepler problem, by Lemma 3.3 and angular momentum as first integral, we know that the 2×2 matrix M_1 in the last lemma must have eigenvalues 1. Then M_1 must be symplectically similar to a matrix of the form $N_1(1,b)$. Note that in the negative energy hypersurface, all the solutions are elliptic orbits with period T, so the time T Hamiltonian map restricted to the fixed negative energy hypersurface is the identity map, then we have

$$\dim\ker(M-I)=3.$$

So the monodromy matrix has the form

$$M=P^{-1}\big(N_1(1,1)\diamond I_2\big)P, \tag{46}$$

for some $P\in \mathrm{Sp}(4)$.

The next two lemmas are very useful to study the Maslov-type index of the Keplerian solution. The first says that the periodic elliptic Keplerian orbits are local minimizers of the action functional. And the second relates the Morse index to Maslov-type index which is a general fact for Lagrangian system. More precisely,

Lemma 3.4. *(See Gordon [11].) Let T be some fixed positive real number. In the planar Kepler problem, the minimizer of the action functional on the subspace of $W^{1,2}(\mathbf{R}/T\mathbf{Z},\mathbf{R}^2)$-loops with winding number ± 1 with respect to the origin is realized by elliptic Keplerian orbits with prime period T.*

For $T>0$, suppose $x(t)$ is a critical point of the functional

$$F(x)=\int_0^T L(t,x,\dot{x}),\quad \forall x\in W^{1,2}\bigl(\mathbf{R}/T\mathbf{Z},\mathbf{R}^n\bigr),$$

where $L\in \mathbf{C}^2((\mathbf{R}/T\mathbf{Z})\times \mathbf{R}^{2n},\mathbf{R})$ and satisfies the Legendrian convexity condition $L_{p,p}(t,x,p)>0$. It is well known that $x(t)$ is a solution of the corresponding Euler–Lagrangian equation:

$$\frac{d}{dt}L_p(t,x,\dot{x})-L_x(t,x,\dot{x})=0; \tag{47}$$

$$x(0)=x(T),\qquad \dot{x}(0)=\dot{x}(T). \tag{48}$$

For such an extremal loop, define

$$\begin{aligned} P(t)&=L_{p,p}\bigl(t,x(t),\dot{x}(t)\bigr),\\ Q(t)&=L_{x,p}\bigl(t,x(t),\dot{x}(t)\bigr),\\ R(t)&=L_{x,x}\bigl(t,x(t),\dot{x}(t)\bigr). \end{aligned}$$

Note that

$$F''(x)=-\frac{d}{dt}\left(P\frac{d}{dt}+Q\right)+Q^T\frac{d}{dt}+R. \tag{49}$$

For $\omega\in\mathbf{U}$, set

$$D(\omega,T)=\bigl\{y\in W^{1,2}\bigl([0,T],\mathbf{C}^n\bigr)\bigm| y(0)=\omega y(T)\bigr\}.$$

We define the ω-Morse index $\phi_\omega(x)$ of x to be the dimension of the negative definite subspace of

$$\bigl\langle F''(x)y_1,y_2\bigr\rangle,\quad y_1,y_2\in D(\omega,T).$$

On the other hand, $s(t)=(\partial L/\partial\dot{x}(t),x(t))^T$ is the solution of the corresponding Hamiltonian system, and its fundamental solution is such that

$$\dot{\gamma}(t) = JB(t)\gamma(t); \tag{50}$$

$$\gamma(0) = I_{2n}, \tag{51}$$

with

$$B(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}(t)Q(t) \\ -Q(t)^T P^{-1}(t) & Q(t)^T P^{-1}(t)Q(t) - R(t) \end{pmatrix}. \tag{52}$$

Lemma 3.5. *(See Long [17], [19, p. 172].) For the ω-Morse index $\phi_\omega(x)$ of its solution $x(t)$ and Maslov-type index $i_\omega(\gamma)$ of solution $s(t) = (\partial L/\partial \dot{x}(t), x(t))^T$, we have*

$$\phi_\omega(x) = i_\omega(\gamma), \quad \forall \omega \in \mathbf{U}. \tag{53}$$

Proposition 3.6. *For the Keplerian orbit, its fundamental solution γ_1 satisfies*

$$i_1\left(\gamma_1^k\right) = 2(k-1), \quad \forall k \in \mathbf{N}. \tag{54}$$

Proof. Since Keplerian orbit is a local minimizer by Lemma 3.4, its Morse index is zero. By Lemma 3.5, the corresponding Maslov-type index is zero, that is

$$i_1(\gamma_1) = 0. \tag{55}$$

From (46), (24)

$$i_\omega(\gamma_1) = 2, \quad \forall \omega \in \mathbf{U}\backslash\{1\}. \tag{56}$$

The statement follows from the Bott-type iteration formula (31). □

Remark 3.7. The last proposition can be proved even if we don't know the normal form corresponding to the angular momentum. In fact, for some $P \in \mathrm{Sp}(4)$, M can be written as

$$M = P^{-1}\big(N_1(1,1) \diamond N_1(1,b)\big)P, \tag{57}$$

with $b = -1, 0, 1$. If $b = -1$, then $i_1(\gamma_1)$ must be odd [19, Theorem 4, pp. 179–180]. This is a contradiction to (55), so $b = 0$ or 1. From (24), in both cases, the splitting numbers are the same, so we have (56).

4. Index and stability of elliptic Lagrangian solutions

In this section, we will discuss the stability of the Lagrangian solution of the planar three-body problem.

Following Montgomery [27], the first homology group $H_1(\hat{\mathcal{X}})$ of the configuration space $\hat{\mathcal{X}}$ for the planar three-body problem is isomorphic to $\mathbf{Z}^3$. Three components of each element of $H_1(\hat{\mathcal{X}})$ are the winding numbers of each side of the triangle defined by the bodies undergoing along the loop. The next lemma is very useful, which is a generalization of Gordon's theorem in the last section.

Lemma 4.1. *(See Venturelli [32], see also [35].) Fix an element* $(k_1, k_2, k_3) \in H_1(\hat{\mathcal{X}}) \cong \mathbf{Z}^3$ *in the first homology group of the configuration space of the planar three-body problem. If* $(k_1, k_2, k_3) = (1, 1, 1)$ *or* $(-1, -1, -1)$*, the minimizers of the action functional among the loops of fixed period* $T \in \mathbf{R}_+$ *in this homology class are exactly the elliptic Lagrangian solutions with prime period* T *which form a critical manifold.*

For other variational characterizations of Lagrangian orbits under various constraint loop spaces, see for instance the papers [2,22,23].

Note also that for any elliptic Lagrangian orbits in the last lemma, each body travels along a Keplerian orbit with *the* same prime period T which is exactly a minimizer of action in the loop space with winding number 1 as characterized by Gordon's theorem.

Let $x(t)$ be such an elliptic Lagrangian solution, and $\gamma(t)$ the symplectic path of fundamental solution matrices to its linear variational equation.

We denote by ϕ_k the Morse index of the action at $x(t)$ on the loop space with period kT. By Lemma 4.1, we know that $x(t)$ is a local minimizer, so we have

$$\phi_1 = 0. \tag{58}$$

By Lemma 3.5, this means that

$$i_1(\gamma) = 0. \tag{59}$$

Following Meyer and Schmidt [26],

$$\dot{\gamma_2}(t) = JB(t)\gamma_2(t), \tag{60}$$

$$\gamma_2(0) = I_4, \tag{61}$$

with

$$B(t) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & \frac{2e\cos(t)-1-\sqrt{9-\beta}}{2(1+e\cos(t))} & 0 \\ 1 & 0 & 0 & \frac{2e\cos(t)-1+\sqrt{9-\beta}}{2(1+e\cos(t))} \end{pmatrix}, \tag{62}$$

where e is the eccentricity, and t is the truly anomaly.

Let

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} \frac{3+\sqrt{9-\beta}}{2(1+e\cos(t))} & 0 \\ 0 & \frac{3-\sqrt{9-\beta}}{2(1+e\cos(t))} \end{pmatrix},$$

and set

$$L(t, x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2 + (Qx, \dot{x}) + \frac{1}{2}(Rx, x), \quad x \in W^{1,2}\big(\mathbf{R}/T\mathbf{Z}, \mathbf{R}^2\big).$$

Obviously the origin in configuration space is the solution of the corresponding Euler–Lagrangian equation. By Legendrian transform, the corresponding Hamiltonian function is

$$H(t,z)=\frac{1}{2}\big(B(t)z,z\big),\quad z\in\mathbf{R}^4,$$

and the origin in phase space is the corresponding solution to this Hamiltonian system.

From Lemma 3.5,

$$i_\omega(\gamma_2)=\phi_\omega.$$

This implies that

$$i_\omega(\gamma_2)\geqslant 0,\quad \forall\omega\in\mathbf{U}. \tag{63}$$

So $i_1(\gamma_2^k)\geqslant 0$ for all $k\in\mathbf{N}$ by Bott-type iteration formula. Furthermore,

$$\phi_k=i_1\big(\gamma^k\big)=i_1\big(\gamma_1^k\big)+i_1\big(\gamma_2^k\big)\geqslant 2(k-1).$$

Based on this, we can prove

Lemma 4.2. *For any elliptic Lagrangian orbit $x(t)$ with fundamental solution $\gamma(t)=\gamma_1(t)\diamond\gamma_2(t)$, $\phi_k=2(k-1)$ for all $k\in\mathbf{N}$ is equivalent to $i_\omega(\gamma_2)=0$ for all $\omega\in\mathbf{U}$.*

Proof. From the Bott-type iteration formula (Lemma 2.6) applied to $\gamma_2(t)$ and Lemma 3.5, it is easy to see that $i_\omega(\gamma_2)=0$ for all $\omega\in\mathbf{U}$ implies $\phi_k=2(k-1)$ for all $k\in\mathbf{N}$.

On the other hand, if $\phi_k=2(k-1)$ for all $k\in\mathbf{N}$, then $i_1(\gamma_2^k)=0$ for all $k\in\mathbf{N}$, so $i_\omega(\gamma_2)=0$ for any ω in the set of the union of the k-th roots of 1 for all k, which is a dense subset of $\mathbf{U}$. By (22), $i_\omega(\gamma)$ is a sub-continuous integer-valued function, then the proof is complete. □

Lemma 4.3. *For any elliptic Lagrangian orbit $x(t)$,*

$$2\leqslant\phi_2\leqslant 4. \tag{64}$$

Proof. By Bott-type iteration formula (Lemma 2.6), we have

$$\phi_2=i_1\big(\gamma^2\big)=i_{-1}(\gamma)+i_1(\gamma).$$

Since $\gamma=\gamma_1\diamond\gamma_2$, we have

$$i_1(\gamma)=i_1(\gamma_1)+i_1(\gamma_2),\quad\text{and}\quad i_{-1}(\gamma)=i_{-1}(\gamma_1)+i_{-1}(\gamma_2).$$

By (59) and (55), we have

$$i_1(\gamma)=0,\quad\text{and}\quad i_1(\gamma_1)=0,$$

so

$$i_1(\gamma_2)=0. \tag{65}$$

Moreover, for Keplerian orbit γ_1, by (56), we have $i_{-1}(\gamma_1) = 2$. So

$$\phi_2 = i_{-1}(\gamma_2) + 2. \tag{66}$$

Since γ_2 is a path in Sp(4),

$$0 \leqslant i_{-1}(\gamma_2) \leqslant i_1(\gamma_2) + 2 = 2.$$

This completes the proof. □

This is the first part of our main Theorem 1.1. Since ϕ_2 is Morse index, by this lemma we know that the possible values of ϕ_2 can only be 2, 3 and 4. Now we can prove the left parts of the main Theorem 1.1.

Proof of Theorem 1.1. (5) is from Lemma 2.8. That is

$$e(M)/2 \geqslant \left| i_1(\gamma) - i_{-1}(\gamma) \right| = \phi_2.$$

Note that here we have used the fact $i_1(\gamma) = 0$ again.

(I) is directly from (5) and the definition of $e(M)$.

To prove (II), note that $i_{-1}(\gamma_2) = \phi_2 - 2 = 1$, $i_{-1}(\gamma_2) - i_1(\gamma_2) = 1$. This means that $\gamma_2(T)$ is unstable. In fact, if $\gamma_2(T)$ is stable, then there exists $P \in \mathrm{Sp}(4)$ such that $\gamma_2(T) = P^{-1}(R(\theta_1) \diamond R(\theta_2))P$. In this case, $i_{-1}(\gamma_2) - i_1(\gamma_2)$ must be even by Lemma 2.5. This is a contradiction.

For (III), $i_1(\gamma_2) = i_{-1}(\gamma_2) = 0$ by $\phi_2 = 2$. If for some $k > 2$, $\phi_k > 2(k-1)$, i.e. $i_1(\gamma_2^k) > 0$. From the iteration formula, there must be some $\omega \in \mathbf{U}$, such that $i_\omega(\gamma_2) > 0$. Since $i_\omega(\gamma_1) = 2$ for any $\omega \in \mathbf{U} \setminus \{1\}$, from (32), we have

$$e(M)/2 \geqslant 2 + 2i_\omega(\gamma_2) \geqslant 4$$

as desired.

For (IV), this happens only when $\gamma_2(T)$ has no eigenvalue on $\mathbf{U}$ which is the hyperbolic case; or when the eigenvalue ω of $\gamma_2(T)$ has splitting number $(S^+_{\gamma_2(T)}(\omega), S^-_{\gamma_2(T)}(\omega)) = (0, 0)$ which is the case of linearly unstable by checking the list in Lemma 2.5 although it is maybe spectrally stable. □

Now, in the presence of non-degenerate condition, we prove our second main theorem.

Proof of Theorem 1.2. Note that $\gamma_2(2T)$ is non-degenerate implies that $\dim \mathrm{Ker}(\gamma_2(T) \pm I_4) = 0$.

To (I), suppose $\omega_i = e^{\sqrt{-1}\theta_i} \in \sigma(\gamma_2(T) \cap \mathbf{U})$, $\theta_i \in (0, \pi)$. Since $i_{-1}(\gamma_2) = \phi_2 - 2 = 2$,

$$\begin{aligned} i_{-1}(\gamma_2) &= i_1(\gamma_2) + \sum_{\omega_i} \left(S^+_{\gamma_2(T)}(\omega_i) - S^-_{\gamma_2(T)}(\omega_i)\right) \\ &= \sum_{\omega_i} \left(S^+_{\gamma_2(T)}(\omega_i) - S^-_{\gamma_2(T)}(\omega_i)\right). \end{aligned}$$

Now we have

$$2 = \sum_{\omega_i} \left(S^+_{\gamma_2(T)}(\omega_i) - S^-_{\gamma_2(T)}(\omega_i)\right) \leqslant \sum_{\omega_i} \left(S^+_{\gamma_2(T)}(\omega_i) + S^-_{\gamma_2(T)}(\omega_i)\right) \leqslant 2.$$

These two formulas mean that $S^-_{\gamma_2(T)}(\omega_i) = 0$ for all i, hence there are exactly two ω_1 and ω_2 such that $S^+_{\gamma_2(T)}(\omega_i) = 1$ for $i = 1, 2$. By Lemma 2.5, we have the form as stated.

To prove (II), note that $\phi_2 = 3$ implies $i_{-1}(\gamma_2) = 1$. And by similar deduction as (I), this further implies that there exists only one eigenvalue $\omega_0 = \exp(i\theta_0), \theta_0 \in (0, \pi)$ with splitting number $(1, 0)$. By checking the basic normal forms, this shows that ω_0 must be semi-simple, so $\gamma_2(T) = P^{-1}(D(\lambda) \diamond R(2\pi - \theta))P$. On the other hand, since $i_1(\gamma_2) = 0$ which is even, there must hold $\det(\gamma_2(T) - I_4) > 0$ by the definition of our index, which in turn implies $\lambda < 0$.

For (III), by the Bott-type iteration formula (31), there exists $w_0 = \exp(i\theta_0), \theta_0 \in (0, \pi)$, such that $i_{w_0}(\gamma_2) > 0$. If ω_0 is not an eigenvalue of $\gamma_2(T)$, then the proof is the same as above; if ω_0 is an eigenvalue, then from the sub-continuity of $i_\omega(\gamma_2)$, there exists ω_1 near ω_0 which is not an eigenvalue of $\gamma_2(T)$ such that $i_{\omega_1}(\gamma_2) > 0$. This proves the statement.

To prove (IV), note that this happens only when there is no eigenvalue of $\gamma_2(T)$ on $\mathbf{U}$ or the eigenvalue has splitting number $(0, 0)$, these are the cases of hyperbolic or spectrally stable and not linearly stable. □

5. Stability of the Lagrangian circular orbits

In this section, we focus on Lagrangian circular orbits from the viewpoint of the Morse index. This is a special case of the last section, but we can make it more precise. Note that in [29], as an example of interesting questions to link variational techniques and classical stability calculations, Roberts posed the following problem: is it possible to use variational methods to derive the well-known stability inequality $\beta < 1$ for the Lagrange equilateral triangle solution? This section can be read as an answer to this problem. We get to this problem independently from our work of understanding the stability of figure-eight orbit from the point of view of Maslov-type index.

The criteria due to Gascheau now is well known. Namely Lagrangian circular orbit in planar three-body problem is linearly stable if and only if $\beta < 1$. It is spectrally stable at $\beta = 1$ [28], which will be quite clear by our analysis with the Maslov-type index.

Recall that

$$\beta = \frac{27(m_1m_2 + m_1m_3 + m_2m_3)}{(m_1 + m_2 + m_3)^2}.$$

First note that the Hamiltonian function (3) satisfies

$$H(-p, -q) = H(p, q) \tag{67}$$

and the Lagrangian circular orbit $x(t)$ satisfies

$$x(t + T/2) = -x(t). \tag{68}$$

As in the last section we denote by $\gamma(t)$ the symplectic path of fundamental solution matrices of $x(t)$. Direct computation shows that

$$\gamma(T) = \gamma(T/2)^2. \tag{69}$$

So the stability of $\gamma(T)$ is the same as that of $\gamma(T/2)$.

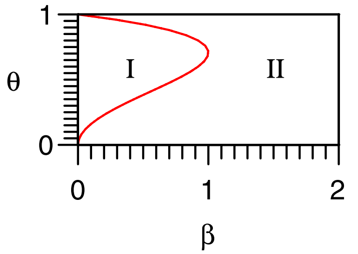

Fig. 5.1. Bifurcation diagram of Lagrangian circular obits in the plane (β,θ). This figure is based on the computations in [28]. The common boundary of regions I and II is defined by $(1-2\theta^2)^2 = 1-\beta$. In the region I or II the ω-index is $i_{e^{\sqrt{-1}\theta\pi}}(\gamma_2^{\frac{1}{2}}) = 1$ or 0 respectively.

Let $\gamma^{\frac{1}{2}} = \gamma(t), t \in [0, T/2]$, then $\gamma = (\gamma^{\frac{1}{2}})^2 = \gamma^{\frac{1}{2}} * \gamma^{\frac{1}{2}}$. By the Bott-type iteration formula (2.6),

$$i_1(\gamma) = i_1\bigl((\gamma^{\frac{1}{2}})^2\bigr) = i_1\bigl(\gamma^{\frac{1}{2}}\bigr) + i_{-1}\bigl(\gamma^{\frac{1}{2}}\bigr). \tag{70}$$

Since $i_1(\gamma) = \phi_1 = 0$ as in the last section, and $i_1(\gamma^{\frac{1}{2}}) \geqslant 0$, $i_{-1}(\gamma^{\frac{1}{2}}) \geqslant 0$ by its relation to Morse index, we have

$$i_1\bigl(\gamma^{\frac{1}{2}}\bigr) = i_{-1}\bigl(\gamma^{\frac{1}{2}}\bigr) = 0. \tag{71}$$

Note that Lagrangian circle solution is a local minimizer, so it must be a local minimizer on the Z_2 symmetry loop space. In fact, it is also a global minimizer for the Z_2 symmetry loop space [22].

Theorem 5.1. *The Lagrangian circular orbit is spectrally stable if there exists an $\omega \in \mathbf{U}$ such that $i_\omega(\gamma_2^{\frac{1}{2}}) \neq 0$. Moreover if $\gamma_2(T)$ is non-degenerate, then it is linearly stable.*

Proof. This is directly from (32), (71) and Theorem 1.1. □

So the problem of the stability of $x(t)$ is reduced to the computations of the Maslov-type index of $\gamma_2^{\frac{1}{2}}(t)$ with appropriate choice of $\omega \in \mathbf{U}$.

Based on the works of Roberts, Meyer and Schmidt [26,28], let $\kappa_1 = -\frac{1}{2}(1-\sqrt{1-\beta})$ and $\kappa_2 = -\frac{1}{2}(1+\sqrt{1-\beta})$, the eigenvalues of $\gamma_2(T/2)$ are $e^{\pm\sqrt{-1}\theta_1\pi}$ and $e^{\pm\sqrt{-1}\theta_2\pi}$, where $\theta_1 = \sqrt{-\kappa_1}$ and $\theta_2 = \sqrt{-\kappa_2}$. Then

$$0 \leqslant \theta_1 \leqslant \theta_2 \leqslant 1, \tag{72}$$

and $\theta_1 = \theta_2$ occurs when $\beta = 1$, in this case $\theta_1 = \theta_2 = \frac{\sqrt{2}}{2}$.

So for Fig. 5.1, $i_{e^{\sqrt{-1}\theta\pi}}(\gamma_2^{\frac{1}{2}}) = 1$ on region I, and $i_{e^{\sqrt{-1}\theta\pi}}(\gamma_2^{\frac{1}{2}}) = 0$ on region II. Recall that the index is minimizer under perturbations, so $i_{e^{\sqrt{-1}\theta\pi}}(\gamma_2^{\frac{1}{2}}) = 0$ on the common boundary curve of regions I and II. This shows that the orbit is spectrally stable if and only if $0 \leqslant \beta \leqslant 1$, moreover, it is linearly stable if $0 < \beta < 1$.

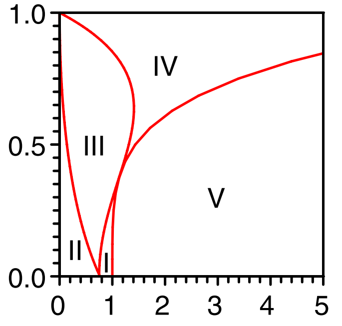

Fig. 6.1. Bifurcation diagram of Lagrangian elliptic orbits in the plane (β, e) with β defined by (4) and e the eccentricity. We draw the figure sketchily following Simò et al. [25] and please refer to their paper for the original one. The regions I, II are linearly stable, region III is hyperbolic–elliptic, region IV is hyperbolic with all the eigenvalues real, and region V is hyperbolic with complex conjugate eigenvalues.

From the Bott-type formula (Lemma 2.6)

$$i_{-1}(\gamma_2) = i_{e^{\sqrt{-1}\pi/2}}\left(\gamma_2^{\frac{1}{2}}\right) + i_{e^{-\sqrt{-1}\pi/2}}\left(\gamma_2^{\frac{1}{2}}\right), \tag{73}$$

so we have

$$\phi_2 = 2i_{e^{\sqrt{-1}\pi/2}}\left(\gamma_2^{\frac{1}{2}}\right) + 2. \tag{74}$$

Direct computation tells us that

$$\phi_2 = 4, \quad \text{if } 0 \leqslant \beta < 3/4, \tag{75}$$

$$\phi_2 = 2, \quad \text{if } \beta \geqslant 3/4. \tag{76}$$

6. Explanation of known numerical results

In this section, we will explain the numerical results from the viewpoint of Morse index. From Theorem 4.1 of [28] or [25], we know that for $\beta = 3/4$, two of the characteristic multipliers move off the unit circle as e increases away from 0. So for e small enough, there are four regions in the $\beta - e$ plan, and these four regions correspond to the four cases of Theorem 1.1.

Let $\lambda_1, \bar{\lambda}_1, \lambda_2$ and $\bar{\lambda}_2$ be the eigenvalues of $\gamma_2(T)$. By the numerical results of [25], $\lambda_1, \lambda_2 \in \mathbf{U}$ on regions I and II; $\lambda_1 \in \mathbf{R}$ and $\lambda_2 \in \mathbf{U}$ on region III; $\lambda_1, \lambda_2 \in \mathbf{R}$ on region IV; $\lambda_1 = \lambda_2^{-1} \in \mathbf{C} \setminus (\mathbf{R} \cup \mathbf{U})$ on region V. The boundaries correspond to $|\lambda_1| = 1$ and $\lambda_1 = \lambda_2^{-1} \in (\mathbf{R} \cup \mathbf{U})$. In the whole meaningful region $0 \leqslant e < 1$ and $0 < \beta \leqslant 9$, $\lambda_1, \lambda_2 \neq 1$.

Now we can analyze in detail the index on each region of Fig. 6.1. Let $P_0 = (3/4, 0)$ be the intersection point of regions I, II and III; P_1 be the intersection of the boundaries of regions I, III, IV and V; $P_2 = (1, 0)$. Let B_1 be the left boundary of region III not including P_0, B_2 be the left boundary of region I not including P_0 and P_1, B_3 be the right boundary of I not including P_1 and P_2, and B_4 be the left boundary of IV without P_1. Recall that $i_1(\gamma_2) = 0$, $\phi_2 = i_{-1}(\gamma_2) + 2$, $\gamma_2(2T) = \gamma_2(T)^2$.

(1) The region for $\gamma_2(2T)$ to be degenerate is the whole boundary of region III in Fig. 6.1.

This follows from the facts that $\ker(\gamma_2(2T) - I_4) = \ker(\gamma_2(T) - I_4) \oplus \ker(\gamma_2(T) + I_4)$ and $\lambda_1 = -1$ on the boundary of region III.

(2) $\phi_2 = 4$ on region II. This is from (75) and the homotopy invariance of the Maslov-type index.

(3) $\phi_2 = 3$ on region III. On this region $\lambda_1 \in \mathbf{R}$ and $\lambda_2 \in \mathbf{U}$. Suppose $\lambda_2 = e^{i\theta}$, so $i_{-1}(\gamma_2) = i_1(\gamma_2) + S^+_{R(\theta)}(\lambda_2) - S^-_{R(\theta)}(\lambda_2)$ must be odd which implies that ϕ_2 must be odd.

(4) $\phi_2 = 2$ on regions I, V and IV. This is from (76) and the homotopy invariance of the Maslov-type index.

(5) $\phi_2 = 2$ at P_0 by (76). From [28], $\dim \operatorname{Ker}(\gamma_2(T) + I_{2n}) = 2$. B_1 and B_2 are bifurcation locus from P_0.

(6) $\phi_2 = 3$ on B_1. Note that $i_{-1}(\gamma_2) = 2$ on region II. By definition, the Maslov-type index $i_{-1}(\gamma_2)$ is the minimizer of the (-1)-index for any small perturbation of γ_2. Since $\dim \operatorname{Ker}(\gamma_2(T) + I_{2n}) = 1$ on B_1, $i_{-1}(\gamma_2) = 2$ or 1 on B_1. We know also that $i_{-1}(\gamma_2) = 1$ on region III, so $i_{-1}(\gamma_2) = 1$ on B_1.

(7) $\phi_2 = 2$ on B_2, B_3, B_4 and P_0, P_1, P_2 because of perturbation definition of the Maslov-type index.

(8) $\phi_k = 2(k-1)$ for any $k \in \mathbf{N}$ on B_3, B_4, P_1 and P_2. In fact, $\phi_k = 2(k-1) + i_1(\gamma_2^k)$. Since $\gamma_2(T)$ is hyperbolic on the regions IV and V, $i_1(\gamma_2^k) = 0$ for any $k \in \mathbf{N}$ which is also true for the boundary by the property that Maslov-type index is defined to be the minimizer of any perturbation.

Based on the above explanations, we know that $\gamma_2(T)$ is spectrally stable and not linearly stable on B_1, B_2, B_3 and P_1, P_2. The norm forms on the regions I, II and III had been given in Theorem 1.2. Moreover, we can get the normal forms or basic normal forms on the boundaries.

(9) The normal form of $\gamma_2(T)$ on B_1 is $N_1(-1, 1) \diamond R(2\pi - \theta)$, for some $\theta \in (0, \pi)$.

(10) The normal form of $\gamma_2(T)$ on B_2 is $N_1(-1, -1) \diamond R(2\pi - \theta)$, for some $\theta \in (0, \pi)$.

(11) The normal form on B_4 is $N_1(-1, 1) \diamond D(\lambda)$ with $\lambda < 0$.

(12) The *basic* normal form on P_1 is $N_1(-1, 1) \diamond D(\lambda)$ with $\lambda < 0$. In fact, the algebraic multiplicity of -1 is 4.

(13) The *basic* normal form on P_2, B_3 is $N_2(e^{\sqrt{-1}\theta}, b)$, with $(b_2 - b_3)\sin\theta > 0$.

The analysis is quite similar to the proof of Theorem 1.2, and we give a simple explanation of them here.

For (9), since B_1 is the boundary of regions II and III, it has the norm form $N_1(-1, b) \diamond R(2\pi - \theta)$ with $b = \pm 1$, $0 < \theta < \pi$. Since $i_{-1}(\gamma_2) = 1$ is odd, by checking the splitting number of $N_1(-1, b)$ and $R(2\pi - \theta)$, we know that $b = 1$ in this case.

Similarly (10) is true by the fact that $i_{-1}(\gamma_2) = 0$ is even at B_2.

For (11), $\gamma_2(T)$ on region III has norm form $D(\lambda) \diamond R(2\pi - \theta)$ with $\lambda < 0$, and $\gamma_2(T)$ on region IV has two pairs of real eigenvalues. Since regions III and IV have common boundary B_4, the eigenvalues of $\gamma_2(T)$ on region IV must be negative, and the norm form on B_4 must have form $N_1(-1, b) \diamond D(\lambda)$ with $\lambda < 0$. Similar deduction as that of (9) shows that $b = 1$ by the fact $i_{-1}(\gamma_2) = 0$ is even at B_4.

For (12), first note that $\dim \ker(\gamma_2(T) + I_4) = 1$, and P_1 is the common boundary of linearly stable region I and region IV which have totally real eigenvalues, the algebraic multiplicity of -1 is 4. Clearly it is a boundary of B_4 which gives its basic norm form by (11).

For (13), $\phi_k = 2(k-1)$ for any $k \in \mathbf{N}$ on B_3 and P_2 which implies that $i_\omega(\gamma_2) = 0$ for any $\omega \in \mathbf{U}$; on the other hand, B_3 and P_2 are the boundaries of linearly stable region, so $\sigma(\gamma_2(T)) \cap$

$\mathbf{U} \neq \emptyset$. Then the splitting number of the eigenvalue is $(0, 0)$, and the only possible basic normal form is $N_2(e^{\sqrt{-1}\theta}, b)$, with $(b_2 - b_3)\sin\theta > 0$.

As far as Theorem 1.2 is concerned, we get a clear description of the known figure.

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