ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 330 (2007) 24–33
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Persistence and global stability in discrete models of Lotka–Volterra type

Yoshiaki Muroya [1]

Department of Mathematical Sciences, Waseda University, 3-4-1 Ohkubo Shinjuku-ku, Tokyo 169-8555, Japan

Received 24 December 2002

Available online 22 August 2006

Submitted by H.L. Smith

Abstract

In this paper, we establish new sufficient conditions for global asymptotic stability of the positive equilibrium in the following discrete models of Lotka–Volterra type:

$$\begin{cases} N_i(p+1) = N_i(p)\exp\left\{c_i - a_i N_i(p) - \sum_{j=1}^{n} a_{ij} N_j(p-k_{ij})\right\}, & p \geqslant 0,\ 1 \leqslant i \leqslant n, \\ N_i(p) = N_{ip} \geqslant 0, \quad p \leqslant 0, \quad \text{and} \quad N_{i0} > 0, & 1 \leqslant i \leqslant n, \end{cases}$$

where each N_{ip} for $p \leqslant 0$, each c_i, a_i and a_{ij} are finite and

$$\begin{cases} a_i > 0, \quad a_i + a_{ii} > 0, \quad 1 \leqslant i \leqslant n, \quad \text{and} \\ k_{ij} \geqslant 0, \quad 1 \leqslant i, j \leqslant n. \end{cases}$$

Applying the former results [Y. Muroya, Persistence and global stability for discrete models of nonautonomous Lotka–Volterra type, J. Math. Anal. Appl. 273 (2002) 492–511] on sufficient conditions for the persistence of nonautonomous discrete Lotka–Volterra systems, we first obtain conditions for the persistence of the above autonomous system, and extending a similar technique to use a nonnegative Lyapunov-like function offered by Y. Saito, T. Hara and W. Ma [Y. Saito, T. Hara, W. Ma, Necessary and sufficient conditions for permanence and global stability of a Lotka–Volterra system with two delays, J. Math. Anal. Appl. 236 (1999) 534–556] for $n = 2$ to the above system for $n \geqslant 2$, we establish new conditions for global asymptotic stability of the positive equilibrium. In some special cases that $k_{ij} = k_{jj}$, $1 \leqslant i, j \leqslant n$, and $\sum_{j=1}^{n} a_{ji} a_{jk} = 0$, $i \neq k$, these conditions become $a_i > \sqrt{\sum_{j=1}^{n} a_{ji}^2}$, $1 \leqslant i \leqslant n$, and improve the well-

E-mail address: ymuroya@waseda.jp.

[1] Research was supported by Waseda University Grant for Special Research Projects 2005A-077 and 2006B-146, and Scientific Research (c), No. 16540207 of Japan Society for the Promotion of Science.

doi:10.1016/j.jmaa.2006.07.070

known stability conditions $a_i > \sum_{j=1}^{n} |a_{ji}|$, $1 \leqslant i \leqslant n$, obtained by K. Gopalsamy [K. Gopalsamy, Global asymptotic stability in Volterra's population systems, J. Math. Biol. 19 (1984) 157–168].

Keywords: Persistence; Global asymptotic stability; Discrete model of autonomous Lotka–Volterra type

1. Introduction

Consider the persistence and global asymptotic stability of the following discrete models of Lotka–Volterra type:

$$\begin{cases} N_i(p+1) = N_i(p)\exp\left\{c_i - a_i N_i(p) - \sum_{j=1}^{n} a_{ij} N_j(p-k_{ij})\right\}, & p \geqslant 0, \\ N_i(p) = N_{ip} \geqslant 0, \quad p \leqslant 0, \quad \text{and} \quad N_{i0} > 0, \quad 1 \leqslant i \leqslant n, \end{cases} \tag{1.1}$$

where each N_{ip} for $p \leqslant 0$, each c_i, a_i and a_{ij} are finite and

$$\begin{cases} a_i > 0, \quad a_i + a_{ii} > 0, \quad 1 \leqslant i \leqslant n, \quad \text{and} \\ k_{ij} \geqslant 0, \quad 1 \leqslant i, j \leqslant n. \end{cases} \tag{1.2}$$

Recently, making the best use of the symmetry of the system and an extended La Salle's invariance principle, Saito, Hara and Ma [9] has shown necessary and sufficient conditions for permanence and global stability of a symmetrical Lotka–Volterra type predator–prey system with two delays. This improves the well-known sufficient condition on the global asymptotic stability of the positive equilibrium in the system obtained by Gopalsamy [4]. Saito [8] also established the necessary and sufficient condition for global stability of a Lotka–Volterra cooperative or competition system with delays for two species. On the other hand, Xu and Chen [10] has offer new techniques to obtain sufficient conditions of the persistence and global stability for a time-dependent pure-delay-type Lotka–Volterra predator–prey model for three species. On the other hand, Muroya [5,6] established conditions for the persistence and global stability of delay differential system and discrete system for n species, respectively, which are some extensions of the averaged condition offered by Ahmad and Lazer [1,2].

In this paper, applying Lemma 2.2 and Theorem 1.2 in Muroya [6] on sufficient conditions for the persistence of nonautonomous discrete Lotka–Volterra systems to the discrete system (1.1)–(1.2), we first obtain conditions for the persistence of the above autonomous system, and extending a similar technique to use a nonnegative Lyapunov-like function offered by Saito, Hara and Ma [9] for $n = 2$ to the above system for $n \geqslant 2$, we establish new conditions for global asymptotic stability of the positive equilibrium. This is a discrete version of Muroya [7]. In some special cases, these conditions improve the well-known stability result obtained by Gopalsamy [4].

Put

$$a_{ij}^{+} = \max(a_{ij}, 0), \qquad a_{ij}^{-} = \min(a_{ij}, 0), \tag{1.3}$$

and

$$\begin{cases} A_0 = \operatorname{diag}(a_1, a_2, \ldots, a_n), \quad B^{-} = \left[a_{ij}^{-}\right], \quad B^{+} = \left[a_{ij}^{+}\right] \quad \text{and} \\ D^{+} = \operatorname{diag}\left(a_{11}^{+}, a_{22}^{+}, \ldots, a_{nn}^{+}\right) \text{ are } n \times n \text{ matrices}, \quad \text{and} \\ \boldsymbol{c} = [c_i] \text{ is an } n\text{-dimensional vector}, \end{cases} \tag{1.4}$$

and assume that

$$\begin{cases} A_0 + D^+ + B^- \text{ is an } M\text{-matrix}, & (A_0 + D^+ + B^-)^{-1}\boldsymbol{c} > \mathbf{0} \quad \text{and} \\ \boldsymbol{c} > (B^+ - D^+)(A_0 + D^+ + B^-)^{-1}\boldsymbol{c}, \end{cases} \tag{1.5}$$

where a real $n \times n$ matrix $A = [a_{ij}]$ with $a_{ij} \leqslant 0$ for all $i \neq j$ is called an *M-matrix* if A is nonsingular and $A^{-1} \geqslant \mathbf{0}$ (see, for example, Berman and Plemmons [3]).

Applying Lemma 2.2 and Theorem 2.2 in Muroya [6] on the sufficient conditions of the persistence of nonautonomous discrete Lotka–Volterra systems to the system (1.1)–(1.2), we first obtain the following theorem.

Theorem 1.1. (See Muroya [6].) *For the system* (1.1)–(1.2)*, if the condition* (1.5) *is satisfied, then all solutions $N_i(p)$, $1 \leqslant i \leqslant n$, of the system are positive and the system is persistent, that is,*

$$0 < \liminf_{p \geqslant 0} N_i(p) \leqslant \limsup_{p \geqslant 0} N_i(p) < +\infty, \quad 1 \leqslant i \leqslant n. \tag{1.6}$$

In particular, all solutions $N_i(p)$, $1 \leqslant i \leqslant n$, of the system are bounded above, that is,

$$\limsup_{p \to \infty} N_i(p) \leqslant \bar{N}_i, \quad 1 \leqslant i \leqslant n, \tag{1.7}$$

where $\bar{N}_i$, $1 \leqslant i \leqslant n$, are defined by

$$\tilde{c}_i = c_i - \sum_{j=1}^{i-1} a_{ij}^- \bar{N}_j, \quad \tilde{N}_i = \tilde{c}_i / a_i, \quad \bar{N}_i = \begin{cases} \tilde{c}_i / a_i, & \tilde{c}_i \leqslant 1, \\ e^{\tilde{c}_i - 1} / a_i, & \tilde{c}_i > 1. \end{cases} \tag{1.8}$$

By Theorem 1.1 and extending a similar technique to use a nonnegative Lyapunov-like function offered by Saito, Hara and Ma [9] for $n = 2$ to the above system for $n \geqslant 2$, we get the following results.

Theorem 1.2. *For the system* (1.1)–(1.2)*, in addition to* (1.5) *and* (1.7)*, assume*

$$\tilde{c}_i < 1, \quad 1 \leqslant i \leqslant n, \tag{1.9}$$

and suppose that there exists a positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ and

$$a_i > \sqrt{\sum_{j=1}^{n} |a_{ji}| \left(\sum_{k=1}^{n} |a_{jk}| \right)}, \quad 1 \leqslant i \leqslant n. \tag{1.10}$$

Then, the positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ of (1.1) *is globally asymptotically stable for any $k_{ij} \geqslant 0$, $1 \leqslant i, j \leqslant n$.*

In particular, if

$$k_{ij} = k_{jj}, \quad 1 \leqslant i, j \leqslant n, \quad \text{and} \quad a_i > \sqrt{\sum_{k=1}^{n} \left| \sum_{j=1}^{n} a_{ji} a_{jk} \right|}, \quad 1 \leqslant i \leqslant n, \tag{1.11}$$

then the positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ of (1.1) *is globally asymptotically stable for any $k_{ii} \geqslant 0$, $1 \leqslant i \leqslant n$.*

Moreover, if

$$\sum_{j=1}^{n} a_{ji} a_{jk} = 0, \quad i \neq k, \tag{1.12}$$

then the last inequalities of (1.11) *becomes*

$$a_i > \sqrt{\sum_{j=1}^{n} a_{ji}^2}, \quad 1 \leqslant i \leqslant n. \tag{1.13}$$

Thus, in the cases of (1.11) and (1.12), the condition (1.13) is weaker than the following sufficient condition on the global asymptotic stability of the positive equilibrium of the system

$$a_i > \sum_{j=1}^{n} |a_{ji}|, \quad 1 \leqslant i \leqslant n, \tag{1.14}$$

which was obtained by Gopalsamy [4], and this extends some of results in Saito, Hara and Ma [9] for $n = 2$ to $n \geqslant 2$.

The organization of this paper is as follows. In Section 2, applying the results in Muroya [6], we offer conditions for the persistence of system (1.1)–(1.2), and using a nonnegative Lyapunov-like sequence, we establish conditions for the global asymptotic stability of positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ of the system (1.1)–(1.2).

2. Proof of theorems

In this section, we prove Theorems 1.1 and 1.2. Muroya [6] consider the following discrete system of nonautonomous Lotka–Volterra type:

$$\begin{cases} N_i(p+1) = N_i(p)\exp\left\{c_i(p) - a_i(p)N_i(p) - \displaystyle\sum_{j=1}^{n}\sum_{l=0}^{m} a_{ij}^l(p)N_j(p-k_l)\right\}, \\ \quad p = 0, 1, 2, \ldots, \\ N_i(p) = N_{ip} \geqslant 0, \quad p \leqslant 0, \quad \text{and} \quad N_{i0} > 0, \quad 1 \leqslant i \leqslant n, \end{cases} \tag{2.1}$$

where each $c_i(p), a_i(p)$ and $a_{ij}^l(p)$ are bounded for $p \geqslant 0$ and

$$\begin{cases} \inf\limits_{p \geqslant 0} a_i(p) > 0, \quad a_{ii}^0(p) \equiv 0, \quad 1 \leqslant i \leqslant n, \\ a_{ij}^l(p) \geqslant 0, \quad 1 \leqslant i \leqslant j \leqslant n, \ 0 \leqslant l \leqslant m, \\ k_0 = 0, \quad \text{integers } k_l \geqslant 0, \quad 1 \leqslant l \leqslant m. \end{cases} \tag{2.2}$$

For a given sequence $\{g(p)\}_{p=0}^{\infty}$, we set

$$\begin{aligned} g_M &= \sup\bigl\{g(p) \bigm| p = 0, 1, 2, \ldots\bigr\}, \\ g_L &= \inf\bigl\{g(p) \bigm| p = 0, 1, 2, \ldots\bigr\}, \end{aligned} \tag{2.3}$$

and for integers $0 \leqslant p_1 < p_2$, we set

$$A[g, p_1, p_2] = \frac{1}{p_2 - p_1} \sum_{p=p_1}^{p_2 - 1} g(p). \tag{2.4}$$

The *lower* and *upper averages* of $g(p)$, denoted by $m[g]$ and $M[g]$, respectively, are defined by

$$\begin{aligned} m[g] &= \lim_{q \to \infty} \inf\bigl\{A[g, p_1, p_2] \bigm| p_2 - p_1 \geqslant q\bigr\} \quad \text{and} \\ M[g] &= \lim_{q \to \infty} \sup\bigl\{A[g, p_1, p_2] \bigm| p_2 - p_1 \geqslant q\bigr\}. \end{aligned} \tag{2.5}$$

Put

$$\begin{aligned}
&a^l_{ijL} = a^{l-}_{ijL} + a^{l+}_{ijL}, \quad a^{l-}_{ijL} \leqslant 0 \leqslant a^{l+}_{ijL},\\
&a^l_{ijM} = a^{l-}_{ijM} + a^{l+}_{ijM}, \quad a^{l-}_{ijM} \leqslant 0 \leqslant a^{l+}_{ijM},\\
&b_{ijL} = \sum_{l=0}^{m} a^l_{ijL}, \qquad b^-_{ijL} = \sum_{l=0}^{m} a^{l-}_{ijL},\\
&b_{ijM} = \sum_{l=0}^{m} a^l_{ijM} \quad \text{and} \quad b^+_{ijM} = \sum_{l=0}^{m} a^{l+}_{ijM}, \quad 1 \leqslant i, j \leqslant n.
\end{aligned} \tag{2.6}$$

Let

$$\begin{aligned}
&A_L = \operatorname{diag}(a_{1L}, a_{2L}, \ldots, a_{nL}), \qquad B^-_L = \left[b^-_{ijL}\right], \qquad B^+_M = \left[b^+_{ijM}\right],\\
&D^+_L = \operatorname{diag}\left(b^+_{11L}, b^+_{22L}, \ldots, b^+_{nnL}\right) \quad \text{and}\\
&D^+_M = \operatorname{diag}\left(b^+_{11M}, b^+_{22M}, \ldots, b^+_{nnM}\right)\\
&\quad \text{are } n \times n \text{ matrices,} \quad \text{and}\\
&\underline{\boldsymbol{c}} = \left[m[c_i]\right] \quad \text{and} \quad \bar{\boldsymbol{c}} = \left[M[c_i]\right]\\
&\quad \text{are } n\text{-dimensional vectors.}
\end{aligned} \tag{2.7}$$

Assume that

$$\left(A_L + D^+_L + B^-_L\right)^{-1} \bar{\boldsymbol{c}} > \mathbf{0} \quad \text{and} \quad \underline{\boldsymbol{c}} > \left(B^+_M - D^+_M\right)\left(A_L + D^+_L + B^-_L\right)^{-1} \bar{\boldsymbol{c}}, \tag{2.8}$$

and put

$$\begin{aligned}
&\tilde{c}_{iM} = c_{iM} - \sum_{j=1}^{i-1} b^-_{ijL} \bar{N}_j, \quad \tilde{N}_i = \tilde{c}_{iM}/a_{iL},\\
&\bar{N}_i = \begin{cases} \tilde{c}_{iM}/a_{iL}, & \tilde{c}_{iM} \leqslant 1,\\ \exp(\tilde{c}_{iM} - 1)/a_{iL}, & \tilde{c}_{iM} > 1. \end{cases}
\end{aligned} \tag{2.9}$$

Muroya [6] obtained the following two results (see Muroya [6, Lemma 2.2 and Theorem 1.2]).

Lemma 2.1. *Assume that for Eq.* (2.7) *and* $c_M = (c_{1M}, c_{2M}, \ldots, c_{nM})^T$,

$$\left(A_L + B^-_L\right)^{-1} \boldsymbol{c}_M > \mathbf{0}. \tag{2.10}$$

Then, any solution of the system (2.1)–(2.2) *is bounded above, and it holds that*

$$\lim_{p\to\infty} N_i(p) \leqslant \bar{N}_i, \quad 1 \leqslant i \leqslant n, \tag{2.11}$$

where $\bar{N}_i$, $1 \leqslant i \leqslant n$, *are defined by* (2.9).

Note that (2.8) implies (2.10).

Lemma 2.2. *For the system* (2.1)–(2.2), *if the condition* (2.8) *is satisfied, then all solutions* $N_i(p)$, $1 \leqslant i \leqslant n$, *of the system are bounded above. Moreover, if there exists a nonempty subset* $Q \in \{1, 2, \ldots, n\}$ *such that*

$$c_{iL} - \sum_{j \notin Q} b^+_{ijM} \bar{N}_j > 0, \quad \text{for any } i \in Q, \tag{2.12}$$

then the system (2.1)–(2.2) *is persistent for solutions, that is,*

$$0 < \liminf_{p \geqslant 0} N_i(p) \leqslant \limsup_{p \geqslant 0} N_i(p) < +\infty, \quad 1 \leqslant i \leqslant n. \tag{2.13}$$

Note that for the system (1.1)–(1.2), (1.5) corresponds to (2.8) in system (2.1)–(2.2) and implies $\boldsymbol{c} > \mathbf{0}$ and for the set $Q = \{1, 2, \ldots, n\}$, it holds that

$$c_i - \sum_{j \notin Q} a_{ij}^{+} \bar{N}_j > 0, \quad \text{for any } i \in Q, \tag{2.14}$$

which implies (2.12).

Proof of Theorem 1.1. Put

$$l_{ij} = \begin{cases} (i-1) \times (i-1) + j, & i > j, \\ (j-1) \times (j-1) + 2j - i, & i \leqslant j, \end{cases}$$

and

$$\bar{a}_{ij}^{l} = \begin{cases} a_{ij}, & l = l_{ij}, \\ 0, & \text{otherwise}, \end{cases} \qquad k_l = \begin{cases} k_{ij}, & l = l_{ij}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, we have

$$\sum_{j=1}^{n} a_{ij} N_j(t - k_{ij}) = \sum_{j=1}^{n} \sum_{l=1}^{n^2} \bar{a}_{ij}^{l} N_j(t - k_l).$$

Thus, the system (1.1)–(1.2) is a special autonomous case of system (2.1)–(2.2). We can apply the results in Lemmas 2.1 and 2.2 to Eqs. (1.1)–(1.2) and obtain the conclusion of the theorem. This completes the proof. □

Proof of Theorem 1.2. Since by Theorem 1.1, the condition (1.9) implies that $\bar{N}_i = \tilde{N}_i < 1/a_i$, $1 \leqslant i \leqslant n$, we have that there is a positive integer p_0 such that for $p \geqslant p_0$, $N_i(p) < \bar{N}_i$, $1 \leqslant i \leqslant n$. Consider a nonnegative Lyapunov-like sequence $\{v(p)\}_{p=0}^{\infty}$ such that for $p \geqslant 0$,

$$\begin{aligned} v(p) = {} & \sum_{i=1}^{n} 2a_i \left\{ \frac{N_i(p)}{N_i^*} - 1 - \ln\bigl(N_i(p)/N_i^*\bigr) \right\} N_i^* \\ & + \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ji}| \left(\sum_{k=1}^{n} |a_{jk}| \right) \sum_{q=p-k_{ji}}^{p-1} \bigl(N_i(q) - N_i^*\bigr)^2. \end{aligned}$$

Then,

$$\begin{aligned} & v(p+1) - v(p) \\ & \quad = \sum_{i=1}^{n} 2a_i \left\{ \bigl(N_i(p+1) - N_i(p)\bigr) - N^* \ln \frac{N_i(p+1)}{N_i(p)} \right\} \\ & \qquad + \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ji}| \left(\sum_{k=1}^{n} |a_{jk}| \right) \bigl\{ \bigl(N_i(p) - N_i^*\bigr)^2 - \bigl(N_i(p - k_{ji}) - N_i^*\bigr)^2 \bigr\}. \end{aligned} \tag{2.15}$$

Since

$$\begin{aligned}
&N_i(p+1)-N_i(p)\\
&\quad=N_i(p)\left\{\exp\left(\ln\frac{N_i(p+1)}{N_i(p)}\right)-1\right\}\\
&\quad=N_i(p)\left\{\ln\frac{N_i(p+1)}{N_i(p)}+\frac{\exp\left(\theta\ln\frac{N_i(p+1)}{N_i(p)}\right)}{2!}\left(\ln\frac{N_i(p+1)}{N_i(p)}\right)^2\right\},\quad 0<\theta<1,
\end{aligned}$$

where for $p\geqslant p_0$ and $1\leqslant i\leqslant n$,

$$N_i(p)\exp\left(\theta\ln\frac{N_i(p+1)}{N_i(p)}\right)\leqslant\max\bigl(N_i(p),N_i(p+1)\bigr)<\frac{1}{a_i},$$

one can verify that

$$\begin{aligned}
&2a_i\left\{\bigl(N_i(p+1)-N_i(p)\bigr)-N^*\ln\frac{N_i(p+1)}{N_i(p)}\right\}\\
&\quad\leqslant 2a_i\bigl(N_i(p)-N_i^*\bigr)\ln\frac{N_i(p+1)}{N_i(p)}+\left(\ln\frac{N_i(p+1)}{N_i(p)}\right)^2,
\end{aligned}\tag{2.16}$$

and by (2.1), we have that

$$\ln\frac{N_i(p+1)}{N_i(p)}=-a_i\bigl(N_i(p)-N_i^*\bigr)-\sum_{j=1}^{n}a_{ij}\bigl(N_j(p-k_{ij})-N_j^*\bigr).$$

We have that $x-1-\ln x\geqslant 0$, for any $x>0$. By Theorem 1.1, each $N_i(p)$, $1\leqslant i\leqslant n$, are bounded above and below by positive constants for $p\geqslant 0$.

Therefore, it follows from (1.6) that for any $p\geqslant\bar{k}=\max\{k_{ij}\mid k_{ij}\geqslant 0,\ 1\leqslant i,j\leqslant n\}$, $0\leqslant v(p)<+\infty$.

Let

$$p_i=a_i\bigl(N_i(p)-N^*\bigr)\quad\text{and}\quad q_{ij}=a_{ij}\bigl(N_j(p-k_{ij})-N_j^*\bigr).$$

Then, $\ln\frac{N_i(p+1)}{N_i(p)}=-(p_i+\sum_{j=1}^{n}q_{ij})$, and

$$\begin{aligned}
\ln\frac{N_i(p+1)}{N_i(p)}&=2p_i\left(-p_i-\sum_{j=1}^{n}q_{ij}\right)\\
&=-\left(p_i+\sum_{j=1}^{n}q_{ij}\right)^2+\sum_{j=1}^{n}q_{ij}^2+2\sum_{j=2}^{n}\sum_{k=1}^{j-1}q_{ij}q_{ik}-p_i^2,
\end{aligned}$$

and for $r_{ji}=N_j(p-k_{ij})-N_j^*(p)$, we have that

$$\begin{aligned}
2\sum_{i=1}^{n}\sum_{j=2}^{n}\sum_{k=1}^{j-1}q_{ij}q_{ik}&=2\sum_{i=1}^{n}\sum_{j=2}^{n}\sum_{k=1}^{j-1}a_{ij}a_{ik}r_{ji}r_{ki}\leqslant\sum_{i=1}^{n}\sum_{j=2}^{n}\sum_{k=1}^{j-1}|a_{ij}a_{ik}|\bigl(r_{ji}^2+r_{ki}^2\bigr)\\
&=\sum_{j=2}^{n}\sum_{i=1}^{n}|a_{ij}|\left(\sum_{k=1}^{j-1}|a_{ik}|\right)r_{ji}^2+\sum_{k=1}^{n-1}\sum_{i=1}^{n}|a_{ik}|\left(\sum_{j=k+1}^{n}|a_{ij}|\right)r_{ki}^2
\end{aligned}$$

$$= \sum_{i=2}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{i-1}|a_{jk}|\right)r_{ji}^2 + \sum_{i=1}^{n-1}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=i+1}^{n}|a_{jk}|\right)r_{ki}^2$$
$$\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k\neq i}|a_{jk}|\right)r_{ij}^2.$$

Therefore,

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n}q_{ij}^2 + 2\sum_{j=2}^{n}\sum_{k=1}^{j-1}q_{ij}q_{ik}\right) \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)r_{ij}^2$$

and

$$\sum_{i=1}^{n}2a_i\left\{N_i(p+1) - N_i(p) - N_i^*\ln\frac{N_i(p+1)}{N_i(p)}\right\}$$
$$\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)r_{ij}^2 - \sum_{i=1}^{n}p_i^2$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)\left(N_i(p-k_{ji}) - N_i^*\right)^2 - \sum_{i=1}^{n}a_i^2\left(N_i(p) - N_i^*\right)^2.$$

Thus, by (2.15), we obtain

$$v(p+1) - v(p) \leqslant -\sum_{i=1}^{n}\left\{a_i^2 - \sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)\right\}\left(N_i(p) - N_i^*\right)^2$$
$$\leqslant -\delta\sum_{i=1}^{n}\left(N_i(p) - N_i^*\right)^2,$$

where by (1.10),

$$\delta = \min_{1\leqslant i\leqslant n}\left\{a_i^2 - \sum_{j=1}^{n}|a_{ji}|\left(\sum_{k=1}^{n}|a_{jk}|\right)\right\} > 0.$$

Then,

$$v(p+1) + \delta\sum_{q=0}^{p}\sum_{i=1}^{n}\left(N_i(q) - N_i^*\right)^2 \leqslant v(0), \quad \text{for any } p \geqslant 0,$$

and

$$\sum_{p=0}^{\infty}\sum_{i=1}^{n}\left(N_i(p) - N_i^*\right)^2 \leqslant \frac{v(0)}{\delta} < +\infty,$$

from which we conclude that $\sum_{i=1}^{n}(N_i(p) - N_i^*)^2 = 0$. This result implies that the positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ of (1.1) is globally asymptotically stable for any $k_{ij} \geqslant 0$, $1 \leqslant i, j \leqslant n$.

In particular, if (1.11) holds, then for $r_j = r_{jj} = N_j(p - k_{jj}) - N_j^*$, $1 \leqslant j \leqslant n$, we have that

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} q_{ij}^2 + 2\sum_{j=2}^{n}\sum_{k=1}^{j-1} q_{ij}q_{ik}\right) = \sum_{i=1}^{n}\left\{\sum_{j=1}^{n} a_{ij}^2 r_j^2 + 2\sum_{j=2}^{n}\sum_{k=1}^{j-1} a_{ij}r_j a_{ik}r_k\right\}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ij}r_j a_{ik}r_k = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ij}a_{ik}\right) r_j r_k$$

and

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\left|\left(\sum_{i=1}^{n} a_{ij}a_{ik}\right) r_j r_k\right| \leqslant \sum_{j=1}^{n}\sum_{k=1}^{n}\left|\sum_{i=1}^{n} a_{ij}a_{ik}\right| \frac{r_j^2 + r_k^2}{2}$$
$$= \sum_{j=1}^{n}\left(\sum_{k=1}^{n}\left|\sum_{i=1}^{n} a_{ij}a_{ik}\right|\right)\frac{r_j^2}{2} + \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\left|\sum_{i=1}^{n} a_{ij}a_{ik}\right|\right)\frac{r_k^2}{2}$$
$$= \sum_{j=1}^{n}\left(\sum_{k=1}^{n}\left|\sum_{i=1}^{n} a_{ij}a_{ik}\right|\right) r_j^2$$
$$= \sum_{i=1}^{n}\left(\sum_{k=1}^{n}\left|\sum_{j=1}^{n} a_{ji}a_{jk}\right|\right) r_i^2.$$

Thus, by (2.15), we obtain

$$v(p+1) - v(p) \leqslant -\sum_{i=1}^{n}\left\{a_i^2 - \sum_{k=1}^{n}\left|\sum_{j=1}^{n} a_{ji}a_{jk}\right|\right\}\left(N_i(p) - N_i^*\right)^2$$
$$\leqslant -\delta_1 \sum_{i=1}^{n}\left(N_i(p) - N_i^*\right)^2,$$

where by (1.11),

$$\delta_1 = \min_{1 \leqslant i \leqslant n}\left\{a_i^2 - \sum_{k=1}^{n}\left|\sum_{j=1}^{n} a_{ji}a_{jk}\right|\right\} > 0.$$

Then,

$$v(p+1) + \delta_1 \sum_{q=0}^{p}\sum_{i=1}^{n}\left(N_i(q) - N_i^*\right)^2 \leqslant v(0), \quad \text{for any } p \geqslant 0,$$

and

$$\sum_{p=0}^{\infty}\sum_{i=1}^{n}\left(N_i(p) - N_i^*\right)^2 \leqslant \frac{v(0)}{\delta_1} < +\infty,$$

from which we conclude that $\sum_{i=1}^{n}(N_i(p) - N_i^*)^2 = 0$. This result implies that the positive equilibrium $\boldsymbol{N}^* = (N_1^*, N_2^*, \ldots, N_n^*)$ of (1.1) is globally asymptotically stable for any $k_{ii} \geqslant 0$, $1 \leqslant i \leqslant n$.

Moreover, if (1.12) holds, then it is evident that the last inequalities of (1.11) becomes (1.13). □

Acknowledgment

The author thanks the anonymous referee for his or her valuable comments to make this paper in the present form.

References

[1] S. Ahmad, A.C. Lazer, Necessary and sufficient average growth in a Lotka–Volterra system, Nonlinear Anal. 34 (1998) 191–228.
[2] S. Ahmad, A.C. Lazer, Average conditions for global asymptotic stability in a nonautonomous Lotka–Volterra system, Nonlinear Anal. 40 (2000) 37–49.
[3] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.
[4] K. Gopalsamy, Global asymptotic stability in Volterra's population systems, J. Math. Biol. 19 (1984) 157–168.
[5] Y. Muroya, Persistence and global stability for nonautonomous Lotka–Volterra delay differential systems, Comm. Appl. Nonlinear Anal. 8 (2002) 31–45.
[6] Y. Muroya, Persistence and global stability for discrete models of nonautonomous Lotka–Volterra type, J. Math. Anal. Appl. 273 (2002) 492–511.
[7] Y. Muroya, Persistence and global stability in Lotka–Volterra delay differential systems, Appl. Math. Lett. 17 (2004) 795–800.
[8] Y. Saito, The necessary and sufficient condition for global stability of Lotka–Volterra cooperative or competition system with delays, J. Math. Anal. Appl. 268 (2002) 109–124.
[9] Y. Saito, T. Hara, W. Ma, Necessary and sufficient conditions for permanence and global stability of a Lotka–Volterra system with two delays, J. Math. Anal. Appl. 236 (1999) 534–556.
[10] R. Xu, L. Chen, Persistence and global stability for a delayed nonautonomous predator–prey system without dominating instantaneous negative feedback, J. Math. Anal. Appl. 262 (2001) 50–61.