On the Resolvent of Linear Nonautonomous Partial Functional Differential Equations*

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We give sufficient conditions for the existence of the resolvent operator for nonautonomous linear partial differential equations with delay, where the highest order derivatives are undelayed. Furthermore we analyse the connection between the resolvent and the solution operator of the homogeneous equation.

0. INTRODUCTION

In this paper we consider partial functional differential equations which can be written in the form

\[ y'(t) = A(t) y(t) + L(t) y(t) + h(t), \quad t \geq 0, \quad y_0 = \varphi, \tag{1} \]

where \( A(t) \) generates a strongly continuous evolutionary system in a real or complex Banach space \( B \) and \( L(t) : P \rightarrow B \) is a linear operator in a space \( P \) of functions \( \varphi : \mathbb{R}^- \rightarrow B \) which has some qualifications as a state space for infinite delay equations (cf. Sect. 1). Here we use the notations \( \mathbb{R}^- := (-\infty, 0], \mathbb{R}^+ := [0, \infty) \), and

\[ y(t) : = y(t + \lambda), \quad \lambda \in \mathbb{R}^-. \]

For the moment let us assume that the initial value problem (1) is well posed in a certain sense explained later in Section 1. Let \( \mathcal{L}(B) \) denote the Banach algebra of all linear bounded mappings of \( B \) into itself. We call a strongly continuous family

\[ X(t, s) \in \mathcal{L}(B), \quad 0 \leq s \leq t, \]

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a "resolvent operator" of (1), if the solution of (1) for the initial function \( \varphi = 0 \) is given by

\[
y(t) = \int_0^t X(t, \lambda) h(\lambda) \, d\lambda
\]

for every forcing function \( h \). The strongly continuous family \( T(t, s) \in \mathcal{L}(P), \quad 0 \leq s \leq t, \)

which is defined by

\[
T(t, s) \varphi := y^{s, \varphi}_t,
\]

\( y^{s, \varphi} \) being the solution of (1) with \( h = 0 \) and initial condition \( y_s = \varphi \), is called the "solution map" of (1).

In this paper it is our aim to make contributions to the following problems:

(a) What conditions on \( A(t) \) and \( L(t) \) imply existence of the resolvent operator?

(b) How is the solution map related to the resolvent operator?

If the delay is finite and if \( B := \mathbb{R}^n \) and \( P := C([-r, 0], B) \), an answer to these questions can be found in Hale's book on functional differential equations [3]. If \( B = \mathbb{R}^n \) but the delay is infinite, sufficient conditions for the existence and a representation of the resolvent operator have been given by Naito [7, 8] using an axiomatically defined phase space. If \( \dim B = \infty \), \( A(t) \) and \( L(t) \) are independent of \( t \), the delay is finite, and \( P := \mathbb{R} \times L^p((-r, 0), B) \) is the underlying phase space, the situation (among other interesting problems) has been clarified in various papers due to Schappacher, Kunisch, and Grimmer [1, 2, 5, 6]. Moreover, if \( L(t) \) has the special form

\[
L(t) \varphi := \int_{-r}^{0} C(\sigma) \varphi(\sigma) \, d\sigma,
\]

and if \( \dim B = \infty \), Grimmer and Schappacher [1, 2] have shown that the existence of the resolvent operator is equivalent to the existence of certain weak solutions of (1). Unfortunately, it is not clear whether their method can be generalized for more general delay operators \( L(t) \). In the present paper we use another method which still works for discrete delays which depend on the space and time variables. The basic idea consists in a generalization of Duhamel's principle, where the discontinuous initial function \( \varphi(0) := I, \varphi(\sigma) := 0, \sigma < 0, \) is approximated by smooth functions.
1. Preliminaries

In this section we explain some definitions which will be used throughout the paper, formulate the exact hypotheses, and draw some elementary conclusions from them.

**Definition 1.** A parametrized family of functions $H^e: \mathbb{R} \to [0, 1]$, $0 < e < 1$, is called a "smoothing of the Heaviside function" if

1. $H^e$ is $C^\infty$ on $\mathbb{R}$.
2. $H^e(t) = 0$ for $t \leq -e$ and $H^e(t) = 1$ for $t \geq 0$.
3. $H^e$ is nondecreasing.
4. $H^{e'}(t) \leq H^e(t)$ for all $t \in \mathbb{R}$ and $0 < e' \leq e$.

Since for partial functional differential equations there are many different kinds of past dependence which are of interest with respect to applications, e.g., delays depending on the space variables, we refrain from a specification of the delay operator $L(t)$ and give instead an axiomatic description of the underlying phase space.

**Definition 2.** A seminormed linear space $P$ of functions $\varphi: \mathbb{R}^- \to B$ over the same scalar field as $B$ is called an "admissible phase space" if

1. The equivalence classes $P/\mathcal{P}$, $\mathcal{P} := \{ \varphi \in P, |\varphi|_P = 0 \}$, where $| \cdot |_P$ denotes the seminorm of $P$, form a Banach space.
2. There exists a number $\bar{c} > 0$ such that $|\varphi(0)|_B \leq \bar{c} |\varphi|_P$ for all $\varphi \in P$, where $| \cdot |_B$ denotes the norm of $B$.
3. If $\varphi: \mathbb{R}^- \to B$ is continuous and has compact support, then $\varphi$ belongs to $P$.
4. There exists a continuous function $c: \mathbb{R}^+ \to (0, \infty)$ such that $|\varphi|_P \leq c(\tau) \max \{|\varphi(\sigma)|_B: -\tau \leq \sigma \leq 0\}$ for all continuous $\varphi: \mathbb{R}^- \to B$ with support in $[-\tau, 0]$ and all $\tau > 0$ (Without loss of generality we assume that $c$ is nondecreasing).
5. For every $\varphi \in P$ and $\lambda \geq 0$ let

\[
(S(\lambda) \varphi)(\sigma) := \begin{cases} 
\varphi(0), & -\lambda \leq \sigma \leq 0, \\
\varphi(\lambda - \sigma), & \sigma \leq -\lambda,
\end{cases}
\]

denote the "static continuation" of $\varphi$ by the amount $\lambda$. Let $P_0$ denote the closed subspace of $P$ consisting of all $\varphi \in P$ with $\varphi(0) = 0$ (cf. condition (2)). Then the operators $S(\lambda)$ map $P_0$ into itself and form a strongly continuous semigroup of bounded linear operators in $P_0$. 
For a discussion of these phase space axioms we refer to paper of Kappel and Schappacher [4]. It follows from these axioms (cf. Lemma 1) that the static continuation operators form a strongly continuous semigroup in $P$ also. Since we will include the case of time dependent delays vanishing for certain values of $t$ in our definition of the delay operator $L(t)$, it is not reasonable to assume that $P$ is a space consisting of functions on $\mathbb{R}^+ \setminus \{0\}$ instead of $\mathbb{R}^+$. Standard examples of admissible phase spaces are the following:

**Example 1.** $P$ consists of all continuous $\varphi: \mathbb{R}^+ \to B$ with the seminorm

$$|\varphi|_P := \max_{-r \leq \sigma \leq 0} |\varphi(\sigma)|_B,$$

where $r \geq 0$ is given.

**Example 2.** $P$ consists of all $\varphi: \mathbb{R}^+ \to B$ such that $e^{\alpha \sigma} |\varphi(\sigma)|_B$ is uniformly continuous and bounded for $\sigma \in \mathbb{R}^+$. $\omega \in \mathbb{R}$ is given. The seminorm of $P$ is

$$|\varphi|_P := \sup_{\sigma \leq 0} e^{\alpha \sigma} |\varphi(\sigma)|_B.$$

**Example 3.** $P$ consists of all continuous $\varphi: \mathbb{R}^+ \to B$ such that

$$\lim_{\sigma \to -\infty} e^{\alpha \sigma} |\varphi(\sigma)|_B = 0.$$

$P$ is a closed subspace of the space $P$ in Example 2.

**Example 4.** Let $r \geq 0$, $\kappa \geq 1$, and $\omega \in \mathbb{R}$ be given. Let $P$ consist of all strongly Lebesgue measurable $\varphi: \mathbb{R}^+ \to B$ such that $e^{\kappa \alpha \sigma} |\varphi(\sigma)|_B$ is integrable and $\varphi$ is continuous on $[-r, 0]$. The seminorm of $P$ is

$$|\varphi|_P := \max_{-r \leq \sigma \leq 0} |\varphi(\sigma)|_B + \left( \int_{-\infty}^{0} e^{\kappa \alpha \sigma} |\varphi(\sigma)|_B^\kappa \, d\sigma \right)^{1/\kappa}.$$

**Definition 3.** A linear subspace $P_1$ of $P$ is called an "admissible subspace" if $P_1$ is dense in $P$ and if a dense linear subspace $B_1$ of $B$ exists such that all functions

$$\psi(\sigma) = \gamma(\sigma) \, b, \quad \sigma \leq 0,$$

belong to $P_1$, when $b$ belongs to $B_1$ and $\gamma: \mathbb{R}^+ \to \mathbb{R}$ is $C^\infty$ and has its support in a compact subset of $\mathbb{R}^+ \setminus \{0\}$.

Now we are able to formulate the hypotheses on $A(t)$, $L(t)$, and the forcing function $h(t)$. 
(H1) Hypothesis on $A(t)$

The operators $A(t): \mathcal{D}(t) \to B$, $t \geq 0$, where $\mathcal{D}(t) \subset B$ is dense, are linear and generate a strongly continuous evolutionary system $U(t, s)$, $0 \leq s \leq t < \infty$, of bounded linear operators in $B$. This means that for all numbers $s$ and $t$, $0 \leq s \leq t < \infty$:

1. $U(t, s) \in \mathcal{L}(B)$,
2. $U(s, s) = I$ (identity), $U(t, t') U(t', s) = U(t, s)$, $s \leq t' \leq t$,
3. $U(t, s) a \in B$ is jointly continuous with respect to $t$ and $s$ for every fixed $a \in B$, and
4. $\lim_{h \to 0} h^{-1}(U(t+h, t) - I) a = A(t) a$ for all $a \in \mathcal{D}(t)$ and $\mathcal{D}(t)$ consists of all $a \in B$ such that this limit exists in $B$.

Conditions on the operators $A(t)$ which imply (H1) and examples are given in the book of Tanabe [10], for instance. If $A(t) = A$ is independent of $t$, the hypothesis (H1) is equivalent to the Hille-Yosida-Phillips condition for a closed densely defined linear operator $A$ to generate a strongly continuous semigroup of bounded linear operators in $B$.

(H2) Hypothesis on $L(t)$

There exists an admissible phase space $P$ together with an admissible subspace $P_1$ such that:

1. $L(t): P_1 \to B$ is linear for all $t \geq 0$ and $L(t) \varphi \in B$ is continuous in $t$ for every fixed $\varphi \in P_1$.
2. For every $\tau > 0$ there exists a continuous function $m_{\tau}: (0, \tau] \to \mathbb{R}^+$ such that $\int_0^\tau m_{\tau}(\lambda) \, d\lambda < \infty$ and

$$|U(t, \lambda) L(\lambda) \varphi|_B \leq m_{\tau}(t - \lambda) |\varphi|_P$$

for all $\varphi \in P_1$ and $0 \leq \lambda < t \leq \tau$.

It is important that in the estimate in condition (2) of (H2) the seminorm $|\varphi|_P$ appears instead of a seminorm of the subspace $P_1$. Therefore the operators

$$W(t, \lambda) := U(t, \lambda) L(\lambda), \quad 0 \leq \lambda < t < \infty,$$

have extensions to $\mathcal{L}(P)$, such that

$$|W(t, \lambda)|_{\mathcal{L}(P)} \leq m_{\tau}(t - \lambda) \quad \text{for} \quad 0 \leq \lambda < t \leq \tau.$$ (3)

If we denote this extension by $W$ again, it follows that $W(t, \lambda)$ is strongly continuous in $t$ and $\lambda$ for $0 \leq \lambda < t < \infty$. The use of a dense subspace $P_1$ of $P$ may be useful for the application to partial functional differential
equations which have time-delays in the arguments of spatial derivatives. However the assumption that the majorant $m_r$ is integrable at 0 excludes the case of delays in the highest order spatial derivatives (cf. Eq. 69) in Sect. 5).

Another hypothesis on $A(t)$ and $L(t)$ that is required for the proof of the existence of the resolvent operator is the following.

(H3) Hypothesis on $A(t)$ and $L(t)$

For every $a \in B_1$ and all numbers $0 \leq s \leq t < \infty$ there exists $\delta = \delta(a, s, t) > 0$ such for every sequence of $C^\infty$-functions $\gamma^n (n \in \mathbb{N})$ on $\mathbb{R}^-$ with range in $[0, 1]$ and support in intervals $I_n \subset (-\delta, 0)$ satisfying $I_{n+1} \subset I_n$ and diameter $I_n \to 0$ for $n \to \infty$, it follows that

$$\lim_{n \to \infty} \int_{s-\delta}^t W(t, \lambda) S(\lambda - s)(\gamma^n(\cdot) a)\, d\lambda = 0$$

in $B$ \hspace{1cm} (4)

(cf. Definition 3 of an admissible subspace).

The condition (4) is trivially satisfied if the sequence $\gamma^n$ converges to 0 in the seminorm of $P$. This holds in the case of the phase space of the example 4 with $r := 0$. Other conditions which imply (H3) are given in Proposition 5 in Section 5.

The following counterexample shows that the hypothesis (H3) is not a consequence of the hypotheses (H1) and (H2): \hspace{1cm}

EXAMPLE 5. Let $B := C_b(\mathbb{R}^+, \mathbb{R})$, the continuous and bounded real functions on $\mathbb{R}^+$ with the sup-norm, and $P := C_{u.b}(\mathbb{R}^-, B)$, the uniformly continuous and bounded functions on $\mathbb{R}^-$ with range in $B$, endowed with the sup-norm. For a given $\phi \in P$ we define $\phi(t, \xi) := \phi(t)(\xi)$, $t \leq 0$, $\xi \geq 0$, and for $t \geq 0$ and $\xi \geq 0$

$$(L(t) \phi)(\xi) := \phi(-\tau(t, \xi), \xi), \quad \tau(t, \xi) := t + e^{-\xi}.$$  \hspace{1cm}

Then together with $A(t) := 0$ for all $t \geq 0$, the hypotheses (H1) and (H2) hold. Now let $\gamma^n \in C^\infty(\mathbb{R}^-, [0, 1])$, $n \in \mathbb{N}$, be given such that the supports of the functions $\gamma^n$ are contained in $(-1/n, 0)$ and $\gamma^n(-1/2n) = 1$ for all $n \in \mathbb{N}$. Let $a \in B$ be defined by $a(\xi) := 1$ for all $\xi \geq 0$. Then we obtain for all $\xi \geq 0$ and $n \in \mathbb{N},$

$$\left( \int_0^t L(\lambda) S(\lambda)(\gamma^n(\cdot) a)\, d\lambda \right)(\xi) = t\gamma^n(-e^{-\xi}),$$

showing that (4) and hence (H3) is not true in this case.

We assume that the forcing $h$ satisfies the following hypothesis.
(H4) **Hypothesis on the Forcing Function h**

\( h: \mathbb{R}^+ \rightarrow B \) belongs to \( \mathcal{L}^p_{\text{loc}}(\mathbb{R}^+, B) \) for a certain number \( p \in (1, \infty] \) which may depend on \( h \), i.e., \( h \) is strongly Lebesgue measurable and \( |h(t)|^p_B \) is locally integrable (locally essentially bounded if \( p = \infty \)).

The reason for the assumption \( p > 1 \) is that our method for the verification of the variation of constants formula does not work in the case \( p = 1 \).

2. **Well Posedness of the Mild Problem**

In this section we provide some elementary lemmata and show in what sense the initial value problem (1) is well posed.

**Lemma 1.** Let \( P \) be an admissible phase space. Then there exist numbers \( M > 1 \) and \( \omega \in \mathbb{R} \) (only depending on \( P \)) such that for every function \( y: \mathbb{R} \rightarrow B \) and for every \( s \in \mathbb{R} \) such that \( y \) is continuous on \([s, \infty)\) and \( y_s \in P \) holds the following statements are true:

1. \( y_t \in P \) for all \( t \geq s \) and \( y_t \in P \) is continuous in \( t \) for all \( t \geq s \) ("Hale's property," cf. [4]).

2. The estimate

\[
|y_t|_P \leq M \max\left\{ e^{\omega(t-s)} |y_s|_P, \max_{s \leq \lambda \leq t} e^{\omega(t-\lambda)} |y(\lambda)|_B \right\}
\]

holds for all \( t \geq s \).

**Proof.** Let (P1)–(P5) denote the various conditions in Definition 2 of an admissible phase space. Choose \( y: \mathbb{R} \rightarrow B \) and \( s \in \mathbb{R} \) according to the hypothesis. If

\[
w(t) := \begin{cases} 0, & t \leq s - 1, \\ (1 + t - s) y(s), & s - 1 \leq t \leq s, \\ y(t), & s \leq t, \end{cases}
\]

it follows from (P3) and (P4) that \( w_t \in P \) for all \( t \in \mathbb{R} \) and \( w_t \) is continuous in \( t \). The representation

\[
y_t = w_t + S(t-s)(y_s - w_s), \quad t \geq s,
\]

together with (P5) and the assumption \( y_s \in P \) prove the first statement. Now by (P5) we find a number \( N \geq 1 \) such that

\[
|S(t)|_{\mathcal{L}^p(R_0)} \leq N, \quad 0 \leq t \leq 1.
\]
Then we obtain from (P2), (P4), (P5), (6), and (7) for \( s < I Q s + 1 \), after some elementary calculations, the estimate

\[
|y|_p \leq \tilde{M} \max \{ |y_s|_p, \max_{s \leq \lambda \leq t} |y(\lambda)|_B \},
\]

(8)

where \( \tilde{M} = N(1 + c(1) \tilde{c}) + c(2) \). Repeating the argument for \( s + 1 \) instead of \( s \) we get from (8) by induction for all \( n \in \mathbb{N}_0 \) and \( t = s + n + 1 \) with \( 0 \leq \sigma < 1 \) the estimate (note \( \tilde{M} \geq 1 \))

\[
|y|_p \leq \tilde{M} \max_{j=0, \ldots, n-1} \{ \tilde{M}^n \max_{s+j \leq \lambda \leq s+j+1} |y(\lambda)|_B \},
\]

(9)

Choosing \( \omega := \ln \tilde{M} \) and \( M := \tilde{M}^2 \) the estimate (5) follows from (9).

Lemma 1 gives the reason for the following definition which generalizes the idea of the relaxation property of fading memory spaces:

**Definition 4.** The infimum \( \omega_P \in \mathbb{R} \cup \{-\infty\} \) of all number such that (5) holds for a certain \( \tilde{M} \geq 1 \) is called the “relaxation exponent” of the space \( P \). We say \( P \) has the relaxation property if \( \omega_P < 0 \).

It is easy to see that phase spaces for problems with finite delay (i.e., there exists \( r > 0 \) such that \( |\varphi|_P = 0 \) for all \( \varphi \in P \) with \( \varphi(\lambda) = 0 \) for \(-r < \lambda \leq 0\)) have \( \omega_P = -\infty \). The phase spaces in Examples 2–4 have \( \omega_P = -\omega \).

As a basic tool for our study we require

**Lemma 2.** Let \( g(t, s) \in B, 0 \leq s \leq t \), be given such that \( g \) is jointly continuous in \( t \) and \( s \) and \( g(s, s) = 0 \) for all \( s \geq 0 \). Then for every \( s \geq 0 \) there is a unique solution \( u(t, s) \in B \) of the equation

\[
u(t, s) = g(t, s) + \int_s^t W(t, \lambda) u(s, \lambda) d\lambda, \quad t \geq s,
\]

(10)

\[
u(t, s) = 0, \quad t \leq s,
\]

which has the following properties:

1. \( u \) is jointly continuous in \( t \) and \( s \).
2. For each \( \tau > 0 \) there exist \( K_\tau > 0 \) and \( N_\tau > 0 \) such that

\[
|u(t, s)|_B \leq K_\tau \max_{s \leq \lambda \leq t} |g(\lambda, s)|_B
\]

(11)
and

$$|u(\cdot, s)|_\mathcal{P} \leq N_\tau \max_{\lambda \leq t} |g(\lambda, s)|_\mathcal{B}$$  \hspace{1cm} (12)

for $0 \leq s \leq t \leq \tau$.

(3) The mapping $(t, s) \rightarrow u(\cdot, s), \in \mathcal{P}$ is continuous for $0 \leq s \leq t$.

Proof. For a given number $\tau > 0$ let $C_{0,\tau}$ denote the Banach space of all continuous functions $\eta: (-\infty, \tau] \rightarrow \mathcal{B}$ such that $\eta = 0$ on $\mathbb{R}^-$ with the norm

$$|\eta|_\tau := \max_{0 \leq \lambda \leq \tau} e^{-q}\, |\eta(\lambda)|_\mathcal{B},$$

where $q > 0$ is for our disposal. Then we conclude from (H1), (H2), (3), and Lemma 1 that the operators $R^s: C_{0,\tau} \rightarrow C_{0,\tau}$, $0 \leq s \leq \tau$, defined by

$$(R^s\eta)(t) := g(t, s) + \int_s^t W(t, \lambda) \eta_\lambda d\lambda, \quad s \leq t,$$

$$(R^s\eta)(t) := 0, \quad t \leq s,$$

are well defined and continuous. Moreover, the mapping

$$[0, \tau] \ni s \rightarrow R^s\eta \in C_{0,\tau}$$

is continuous for every fixed $\eta$. For all $\eta_i \in C_{0,\tau}$ ($i = 1, 2$) and $s \in [0, \tau]$ we obtain

$$|R^s\eta_2 - R^s\eta_1|_\tau \leq L^{s,q} |\eta_2 - \eta_1|_\tau$$  \hspace{1cm} (13)

with

$$L^{s,q} := c(\tau) \int_0^\tau e^{-q\mu} \mathfrak{m}_\tau(\mu) d\mu.$$

Hence we get $L^{s,q} < 1$, if $q > 0$ is sufficiently large, and (13) shows that the operators $R^s$, $0 \leq s \leq \tau$, are uniformly contracting. Thus (10) has a unique solution $u$ and since $R^s\eta$ is continuous in $s$ for fixed $\eta$, claim (1) follows (uniform contraction principle). Now if $u^*$ denotes the fixed point of $R^s$ in $C_{0,\tau}$, we have the estimate

$$|u^*|_\tau \leq (1 - L^{s,q})^{-1} |R^s0|_\tau, \quad 0 \leq s \leq \tau,$$

which proves (11) with $K_\tau := (1 - L^{s,q})^{-1}$. Inequality (12) comes from inequality (11) and Lemma 1(5), showing claim (2). Finally claim (3) follows from claims (1) and (2) applying Lemma 1 again. $\blacksquare$

We remark that (H3) was not used in the proof of Lemma 2.
In the following we will distinguish the following two kinds of solutions of (1):

**Definition 5.** Let $s \geq 0$ and $\varphi \in P$ be given. Then we call a function $y: \mathbb{R} \to B$ a strong solution of the equation

$$y'(t) = A(t) y(t) + L(t) y_t + h(t), \quad t \geq s, \quad (14)$$

with initial data $s$ and $\varphi$, if $y_s = \varphi$, $y(t) \in D(t)$ for all $t \geq s$, $y_t \in P$, for all $t \geq s$, and $y$ is continuous, right-differentiable for all $t \geq s$, and (14) holds. A continuous function $y: \mathbb{R} \to B$ is called a mild solution of (14) with initial data $s$ and $\varphi$, if $y_s = \varphi$ and $y$ solves the integral equation

$$y(t) = u(t, s) \varphi(0) + \int_s^t u(t, \lambda) (L(\lambda) y_\lambda + h(\lambda)) \, d\lambda, \quad s \leq t. \quad (15)$$

(Note that by Lemma 1 and the remark after hypothesis (H2) the right-hand side of (15) is well defined.)

**Lemma 3.** Let (H1), (H2), and (H4) hold. Then a strong solution of (14) is a mild solution.

**Proof.** Since (15) is always satisfied for $t = s$ assume $s < t$. Then for each $\lambda \in [s, t)$ and $\delta \in (0, t - \lambda)$ we have

$$\delta^{-1} (U(t, \lambda + \delta) y(\lambda + \delta) - U(t, \lambda) y(\lambda)) - U(t, \lambda + \delta) (y(\lambda + \delta) - y(\lambda)) \delta^{-1} - U(t, \lambda + \delta) (U(\lambda, \lambda) - I) y(\lambda) \delta^{-1}.$$

Passing to the limit for $\delta \to 0$ in (16) shows that $U(t, \lambda) y(\lambda)$ is right-differentiable in $\lambda$ and

$$\frac{d}{d\lambda} U(t, \lambda) y(\lambda) = U(t, \lambda) y'(\lambda) - U(t, \lambda) A(\lambda) y(\lambda) = U(t, \lambda) (L(\lambda) y_\lambda + h(\lambda)). \quad (17)$$

As the right-hand side of (17) is locally integrable, we can integrate (17) over $[s, t]$ to obtain

$$U(t, t) y(t) - U(t, s) y(s) = \int_s^t U(t, \lambda) (L(\lambda) y_\lambda + h(\lambda)) \, d\lambda, \quad (18)$$

which is (15). (Note that continuity together with the existence of a locally Bochner-integrable right-derivative are enough to make the transformation from (17) into (18) correct; cf. [10]).
In the following we shall use the notion solution always in the sense of a mild solution. A contribution to the regularity problem of mild solutions is given in [9] for the case that $A(t)$ is independent of $t$.

**Lemma 4.** Assume (H1), (H2), and (H4). Then for every $s \geq 0$, $\varphi \in P$, and $h$ satisfying (H4) there exists a unique solution $y^{s,\varphi,h}$. Moreover the solution map

$$T(t, s) \varphi := y^{s,\varphi,0}, \quad 0 \leq s \leq t,$$

of the homogeneous equation ($h = 0$) forms a strongly continuous evolutionary system in $P$ (this means that conditions (1)–(3) of (H1) with $T$ instead of $U$ and $P$ instead of $B$ hold). In particular, for each $\tau > 0$ there is $R_\tau > 0$ such that

$$|T(t, s)|_{\mathcal{F}(P)} \leq R_\tau, \quad 0 \leq s \leq t \leq \tau. \quad (19)$$

**Proof.** It follows from Lemma 1 that the static continuation operators $S(t)$, $t \geq 0$, form a strongly continuous semigroup of bounded linear operators in $P$. We define

$$u(t, s) := 0, \quad t \leq s,$$

$$u(t, s) := y^{s,\varphi,h}(t) - (S(t - s) \varphi)(0), \quad t \geq s, \quad (20)$$

and

$$g(t, s) := (U(t, s) - I) \varphi(0) + \int_s^t W(t, \lambda) S(\lambda - s) \varphi \, d\lambda$$

$$+ \int_s^t U(t, \lambda) h(\lambda) \, d\lambda, \quad t \geq s. \quad (21)$$

Then because of

$$u(\, , s) = y^{s,\varphi,h} - S(t - s) \varphi, \quad t \geq s, \quad (22)$$

Eqs. (10) and (15) are equivalent. Thus the claim follows from Lemma 2 by standard arguments employing (H1), (H2), (H4), the admissibility of $P$, and the representations (20), (21), and (22).

It should be remarked that all results can be reformulated in such a way that the phase space $P$ is replaced by the Banach space $\bar{P} := P/P$ of the equivalence classes with respect to the seminorm in $P$. This is true since by means of the property 2 in Definition 2 the value $\varphi(0)$ depends only on the class to which $\varphi$ belongs and since every continuous mapping from $P$ into
an arbitrary seminormed space \( Q \) has a unique representation as a mapping from \( \tilde{P} \) into the equivalence classes of \( Q \). The assumption that \( \tilde{P} \) is a Banach space was used for the conclusion that a family of bounded linear operators in \( P \) which depends strongly continuously on real parameters is uniformly bounded on compact parameter sets.

3. Existence of the Resolvent and the Variation of Parameters Formula

The aim of this section is to prove

**Theorem 1.** Assume (H1)-(H4) and let a smoothing of the Heaviside function be given according to Definition 1. Define

\[
X^\varepsilon(t, s) a := (T(t, s)(H^\varepsilon(\cdot)) a_0(0), \quad a \in B, \quad 0 \leq s \leq t, \quad 0 < \varepsilon \leq 1. \tag{23}
\]

\((H^\varepsilon(\cdot)) a_0\) denotes the function \( H^\varepsilon(\lambda) a, \lambda \leq 0, \) which belongs to \( P \). Then the following statements hold:

1. *The limit*

\[
X(t, s) a := \lim_{\varepsilon \to 0} X^\varepsilon(t, s) a \tag{24}
\]

exists for all \( a \in B \) and \( 0 \leq s \leq t \).

2. \( X(t, s) \in \mathcal{L}(B) \) and \( X(t, s) \) is jointly strongly continuous with respect to \( t \) and \( s \) for \( 0 \leq s \leq t \).

3. For all \( s > 0 \), \( \varphi \in P \), and \( h \) according to (H4) the solution \( y^{s, \varphi, h} \) has the representation

\[
y^{s, \varphi, h}(t) = (T(t, s) \varphi)(0) + \int_s^t X(t, \lambda) h(\lambda) \, d\lambda, \quad t \geq s \quad \text{(variation of parameters formula).} \tag{25}
\]

The family \( X(t, s) \in \mathcal{L}(B), 0 \leq s \leq t, \) is called the resolvent.

**Remark.** The approximation of the resolvent \( X \) by the operators \( X^\varepsilon \) generalizes the Duhamel principle: The resolvent \( X \) is the solution of the homogeneous problem with the step initial function \( \phi(\lambda) := 0 \) for \( \lambda < 0 \) and \( \phi(0) = I \) (cf. [3]).

**Proof.** The proof of Theorem 1 proceeds in several steps.

1. Existence of the limit (24) for \( a \in B_1 \). Let \( a \in B_1 \) be fixed. For all
numbers \(0 < \varepsilon' < \varepsilon < 1\), \(s > 0\), and \(t \in \mathbb{R}\), we define \((X^\varepsilon(t, s) a : = H^\varepsilon(t - s) a, t \leq s)\),

\[
Y^\varepsilon(t, s) a := X^\varepsilon(t, s) a - H^\varepsilon(t - s) a,
\]

\[
Y^{\varepsilon,\varepsilon'}(t, s) a := Y^\varepsilon(t, s) a - Y^\varepsilon(t, s) a,
\]

\[
H^{\varepsilon,\varepsilon'} := H^\varepsilon - H^\varepsilon'.
\]

Then by (23) and the definition of \(T\) we find that \(Y^{\varepsilon,\varepsilon'}\) solves the integral equation

\[
Y^{\varepsilon,\varepsilon'}(t, s) a = \int_s^t W(t, \lambda)(H^{\varepsilon,\varepsilon'}(\cdot, s) a)_{\lambda - s} d\lambda + \int_s^t W(t, \lambda)(Y^{\varepsilon,\varepsilon'}(\cdot, s) a)_{\lambda - s} d\lambda,
\]

\[
0 \leq s \leq t,
\]

\[
Y^{\varepsilon,\varepsilon'}(t, s) a := 0, \quad t \leq s.
\]

Application of Lemma 2 to Eq. (26) yields for each \(\tau > 0\) and \(0 \leq s \leq t \leq \tau\) the estimate (cf. (11))

\[
|Y^{\varepsilon,\varepsilon'}(t, s) a|_B \leq K_\tau \max_{s \leq t'} |\beta^{\varepsilon,\varepsilon'}(t', s)|_B,
\]

where

\[
\beta^{\varepsilon,\varepsilon'}(t', s) := \int_s^{t'} W(t', \lambda)(H^{\varepsilon,\varepsilon'}(\cdot, s) a)_{\lambda - s} d\lambda.
\]

Hypothesis (H3) implies

\[
\lim_{\varepsilon \to 0, 0 < \varepsilon' < \varepsilon} \beta^{\varepsilon,\varepsilon'}(t, s) = 0
\]

for fixed \(t\) and \(s\), \(0 \leq s \leq t\). Next we will show that the limit in (29) is locally uniform with respect to \(t\) and \(s\). In particular, by (27) this then implies

\[
\lim_{\varepsilon \to 0, 0 < \varepsilon' < \varepsilon} Y^{\varepsilon,\varepsilon'}(t, s) a = 0
\]

and therefore also the existence of the limit (24) for \(t \geq s\). Now let us assume that the limit in (29) is not locally uniform in \(t\) and \(s\). Then we can find numbers \(\tau > 0\) and \(\rho > 0\) together with sequences \(0 \leq s_n \leq t_n \leq \tau\), \(0 < \varepsilon'_n < \varepsilon_n < 1\), \(n \in \mathbb{N}\), such that \(s_n \to s\), \(t_n \to t\), and \(\varepsilon_n \to 0\) for \(n \to \infty\) and

\[
\rho \leq |\beta^{\varepsilon_n,\varepsilon'_n}(t_n, s_n)|_B, \quad n \in \mathbb{N}.
\]
Since $P$ is admissible there exists a number $\hat{c} > 0$ such that

$$|(H^{n,\sigma}(\cdot) a)_{n}|_{p} \leq \hat{c}, \quad n \in \mathbb{N}, \quad 0 \leq \sigma \leq \tau. \quad (31)$$

Then we obtain from $(H3)$, $(28)$, $(30)$, and $(31)$

$$\rho \leq \hat{c} \int_{s_{n}}^{t_{n}} m_{s}(\lambda) d\lambda, \quad n \in \mathbb{N}, \quad (32)$$

which shows that the case $t = s$ is impossible. Therefore we can select number $\hat{s}$ and $\tilde{t}$ and $n_{0} \in \mathbb{N}$ such that $0 \leq s < \hat{s} < \tilde{t} < t \leq \tau$, and, for all $n \geq n_{0}$, $\hat{s} - \delta/2 < s_{n} \leq \hat{s}$, $\tilde{t} \leq t_{n}$, $\sigma_{n} < \delta/2$ ($\delta = \delta(a, \hat{s}, \tilde{t})$ and

$$\int_{s_{n}}^{\hat{s}} m_{s}(\lambda) d\lambda \leq \rho/(4\hat{c}), \quad \int_{\tilde{t}}^{t_{n}} m_{s}(\lambda) d\lambda \leq \rho/(4\hat{c}). \quad (33)$$

Now we obtain from $(28)$, $(30)$, $(31)$, and $(33)$ for all $n \geq n_{0}$ the estimate

$$\rho/2 \leq \left| \int_{s}^{\tilde{t}} W(t_{n}, \lambda) (H^{n,\sigma}(\cdot) a) e_{n_{0}} d\lambda \right|_{B}$$

$$\leq \left| U(t_{n}, \tilde{t}) \cdot \int_{s}^{\tilde{t}} W(\lambda, \tilde{t}) (S(\lambda - \hat{s})^{n}(\cdot) a) d\lambda \right|_{B}, \quad (34)$$

where

$$\gamma^{n}(\sigma) := H^{n,\sigma}(\hat{s} - s_{n} + \sigma), \quad \sigma \leq 0. \quad (35)$$

Since the operators $U(t_{n}, \tilde{t})$ are uniformly bounded by $(H1)$ and the sequence $\gamma^{n}$ has the properties which are assumed in $(H3)$, inequality $(34)$ contradicts $(H3)$.

**Step 2.** Existence of the limit $(24)$ for $a \in B$ and claim $(2)$. For every $\tau > 0$ we conclude from $(19)$ with the aid of the properties of $H^{e}$ and the phase space $P$

$$|X^{e}(t, s) a|_{B} \leq R_{t} c(1) \hat{c} |a|_{B}, \quad 0 \leq \varepsilon \leq 1, \quad a \in B,$$

$$0 \leq s \leq t \leq \tau. \quad (36)$$

showing that the operators $X^{e}(t, s)$ belong to $L(B)$ and having norms which are uniformly bounded in $\varepsilon \in (0, 1]$ locally in $t$ and $s$. As $B_{1}$ was dense in $B$, we get the existence of the limit $(24)$ in $B$ and the claim $X(t, s) \in L(B)$ for $0 \leq s \leq t$. As by the definition $(23)$ and Lemma 4 the operators $X^{e}(t, s)$ are strongly continuous in $t$ and $s$ for fixed $\varepsilon$, the strong continuity of the operators $X(t, s)$ in $t$ and $s$ follows from the fact that the
limit of the operators \( X^\varepsilon \) for \( \varepsilon \to 0 \) is locally uniform with respect to \( t \) and \( s \). (For the present, this holds if \( a \in B_1 \), but since the operators \( X(t, s) \) are locally uniformly bounded in \( t \) and \( s \) by means of (36), the strong continuity of \( X(t, s) \) in \( B_1 \) implies the strong continuity in \( B \).)

**Step 3. Verification of the variation of parameters formula.** By the linearity of the equation it is sufficient to prove (25) for the case \( \varphi = 0 \). The definition (23) of \( X^\varepsilon \) says that \( X^\varepsilon \) solves the following integral equation:

\[
X^\varepsilon(t, s) a = U(t, s) a + \int_s^t W(t, \mu)(X^\varepsilon(\cdot, s) a)_\mu d\mu, \quad 0 \leq s \leq t, (37)
\]

\[
X^\varepsilon(t, s) a = H^\varepsilon(t - s) a, \quad t \leq s.
\]

Standard arguments employing the properties of \( X^\varepsilon \), hypothesis (H4), and conditions (3) and (4) of an admissible phase space imply that the mapping

\[
\mathbb{R}^1 \times \mathbb{R}^1 \ni (\lambda, \mu) \mapsto (X^\varepsilon(\cdot, \lambda) h(\lambda))_\mu \in \mathcal{P}
\]

is strongly measurable and locally Bochner integrable, where

\[
\int_s^\mu (X^\varepsilon(\cdot, \lambda) h(\lambda))_\mu d\lambda = \left( \int_s^\mu X^\varepsilon(\cdot, \lambda) h(\lambda) d\lambda \right)_\mu. (38)
\]

holds for \( 0 \leq s \leq \mu \). Thus we obtain from (37) and Fubini's theorem (put \( s = \lambda \) and \( a = h(\lambda) \) in (37)) that

\[
w^{i.e., h}(t) := \int_s^t X^\varepsilon(t, \lambda) h(\lambda) d\lambda, \quad 0 \leq s \leq t, (39)
\]

\[
w^{e.s., h}(t) := 0, \quad t \leq s,
\]

satisfies for \( 0 \leq s \leq t \),

\[
w^{e.s., h}(t) = \int_s^t U(t, \lambda) h(\lambda) d\lambda
\]

\[
+ \int_s^t \left( \int_s^\mu W(t, \mu)(X^\varepsilon(\cdot, \lambda) h(\lambda))_\mu d\mu \right) d\lambda
\]

\[
= \int_s^t U(t, \mu) h(\mu) d\mu
\]

\[
+ \int_s^t W(t, \mu) \left( \int_s^\mu X^\varepsilon(\cdot, \lambda) h(\lambda) d\lambda \right)_\mu d\mu. (40)
\]
Now let us consider the quantities

\[ \phi^{e,\mu} := \phi^{e,s,h}(x) := \left( \int_{s}^{t} X_{e}(\cdot, \lambda) h(\lambda) \, d\lambda \right) \in P \]

for \( s \leq \mu \leq t \). For all \( \nu \leq 0 \) we conclude from (23) and (39)

\[ \phi^{e,\mu}(\nu) = \begin{cases} - \int_{a_{0}}^{s} H^{e}(\mu + \nu - \lambda) h(\lambda) \, d\lambda, & \text{if } a_{0} \leq \alpha_{1}, \\ 0, & \text{otherwise}, \end{cases} \]  

(41)

where \( \alpha_{0} := \max\{ s, \mu + \nu \} \), \( \alpha_{1} := \min\{ \mu, \mu + \nu + \epsilon \} \). Choose \( p > 1 \) according to (H4) and let \( q := p/(p - 1) \). Then because of \( \alpha_{1} - \alpha_{0} \leq \epsilon \) we get from (41) the estimate

\[ |\phi^{e,\mu}(\nu)|_{B} \leq \epsilon^{1/q} \left( \int_{0}^{\tau} |h(\lambda)|_{B}^{p} \, d\lambda \right)^{1/p}, \quad 0 \leq \nu \leq s \leq \mu \leq t. \]  

(42)

Let \( \tau > 0 \) be given and suppose \( 0 \leq \sigma \leq \tau \leq \tau \). As the support of \( \phi^{e,\mu} \) is contained in \( [-\tau - \epsilon, 0] \) for \( s \leq \mu \leq \tau \), we conclude from (42) and condition (4) in Definition 2 for \( 0 < \epsilon < 1 \):

\[ |\phi^{e,\mu}|_{B} \leq c(\tau + 1) \epsilon^{1/q} \left( \int_{0}^{\tau} |h(\lambda)|_{B}^{p} \, d\lambda \right)^{1/p}, \quad 0 \leq s \leq \mu \leq \tau \leq \tau. \]  

(43)

Rewriting (40) we get

\[ w^{e,s,h}(t) = \int_{s}^{t} W(t, \mu) w^{e,s,h} \, d\mu + \int_{s}^{t} U(t, \mu) h(\mu) \, d\mu \]

\[ - \int_{s}^{t} W(t, \mu) \phi^{e,\mu} \, d\mu. \]  

(44)

By means of (39) and the results of Step 2 it follows that

\[ \lim_{\epsilon \to 0} w^{e,s,h}(t) = w^{s,h}(t) := \int_{s}^{t} X(t, \lambda) h(\lambda) \, d\lambda, \quad t \geq s, \]  

(45)

locally uniform with respect to \( t \). Thus defining

\[ w^{s,h}(t) := 0, \quad t \leq s, \]

we conclude (cf. Definition 2)

\[ \lim_{\epsilon \to 0} w^{e,s,h} = w^{s,h} \text{ in } P \text{ locally uniform in } \mu. \]  

(46)
Gathering (43), (44), (45), and (46) together we conclude that \( w^{s,h} \) solves the equation

\[
\begin{align*}
\frac{d^{\mu}}{dt^\mu} w^{s,h}(t) &= \int_s^t \frac{d^{\mu}}{d\lambda^{\mu}} W(t, \lambda) \phi^{s,h}(\lambda) \, d\lambda + \int_s^t U(t, \lambda) h(\lambda) \, d\lambda, & t \geq s, \\
\phi^{s,h}(t) &= 0, & t \leq s,
\end{align*}
\]

showing that \( \phi^{s,h} \) is a solution of (15) with initial data \( s \) and \( \varphi = 0 \). By uniqueness (Lemma 4) we must have \( \phi^{s,0,h} = w^{s,h} \) which proves claim (3).

We remark that the properties of the resolvent together with the variation of parameters formula (25) define the resolvent uniquely. Namely, if \( X \) and \( \tilde{X} \) would be two resolvent satisfying claims (2) and (3) of Theorem 1, it follows from the uniqueness and linearity of the initial value problem that

\[
\int_s^t (X(t, \lambda) - \tilde{X}(t, \lambda)) h(\lambda) \, d\lambda = 0
\]

for all numbers \( 0 \leq s \leq t \) and all functions \( h \) satisfying (H4). Since \( X \) and \( \tilde{X} \) are locally uniformly bounded and strongly continuous in \( t \) and \( s \), (47) implies \( X(t, s) = \tilde{X}(t, s) \) for all numbers \( 0 \leq s \leq t \).

Lemma 4, (19) together with (23), and the admissibility of the phase space \( P \) (Definition 2) imply that the norms of the resolvent operators \( X(t, s) \) can be estimated by the norms of the solution operators \( T(t, s) \):

**Corollary 1.** The norm of the resolvent satisfies the estimate

\[
|X(t, s)|_{\mathcal{L}(B)} \leq \tilde{c} c(0) |T(t, s)|_{\mathcal{L}(P)},
\]

for all numbers \( 0 \leq s \leq t < \infty \), where \( \tilde{c} \) and \( c(0) \) are chosen according to Definition 2.

**4. Representation of the Solution of the Homogeneous Equation by the Resolvent**

So far we have given conditions that the contribution of the forcing term \( h \) to the solution of the homogeneous initial value problem can be written in the form

\[
\int_s^t X(t, \lambda) h(\lambda) \, d\lambda
\]
with an operator kernel $X$ (the resolvent) which is uniquely determined by the operators $A(t)$ and $L(t)$. This makes the application of local perturbation arguments available to nonlinear partial functional differential equations, provided that the linearization satisfies hypotheses (H1)–(H3) and estimates on the norm of $X(t, \lambda)$ are known. Corollary 1 says that estimates on the norm of the solution map in $L(P)$ always provide estimates on the norm of the resolvent in $L(B)$. However, since the resolvent corresponds to the solutions of initial value problems with fairly special initial functions (cf. (23)), it may sometimes be easier to obtain estimates of the norm of the resolvent in $L(B)$ than norms of the solution map in the generally more complicated space $L(P)$ Therefore conditions will be of interest which permit us to estimate the norms of $T(t, s)$ in $L(P)$ by the norms of $X(t, s)$ in $L(B)$. Since the case of an unbounded delay-operator $L(t)$ provides difficulties, we will restrict ourself to the following situation:

(H2') **Stronger Version of Hypothesis (H2)**

The operators $L(t): P \rightarrow B$ are linear and bounded for all $t \geq 0$ and $L(t) \varphi \in B$ is continuous in $t$ for every fixed $\varphi \in P$.

It is easy to see that hypotheses (H1) and (H2') together imply (H2).

Now we can prove

**Theorem 2.** Assume (H1), (H2'), (H3), and let a smoothing $H^\varepsilon$ of the Heaviside function be given. For each $\varepsilon \in (0, 1]$ and $\varphi \in P$ define

$$
\psi^\varepsilon(\lambda) := \varphi(\lambda) - H^\varepsilon(\lambda) \varphi(0), \quad \lambda \leq 0.
$$

Then we have for all $\varphi \in P$ and numbers $0 \leq s \leq t < \infty$ the following representation:

$$(T(t, s) \varphi)(0) = X(t, s) \varphi(0) + \lim_{\varepsilon \rightarrow 0} \int^t_s X(t, \lambda) L(\lambda) S(\lambda - s) \psi^\varepsilon \, d\lambda, \quad (50)$$

where the limit on the right-hand side of (50) exists in $B$ and is independent of the special choice of the smoothing $H^\varepsilon$. Moreover, assume that $P$ has the relaxation property (cf. Definition 4) and that there exists a number $\tilde{L} > 0$ such that

$$
|L(t)|_{\mathcal{L}(B)} \leq \tilde{L} \quad \text{for all } t \geq 0.
$$

Then for every number $\omega \in (0, -\omega_P)$ there exists a number $K > 0$ which is independent of $t$ and $s$ such that

$$
|T(t, s)|_{\mathcal{L}(P)} \leq Ke^{-\omega(t-s)}(1 + t - s) \sup_{s \leq \lambda' \leq t} e^{\omega(\lambda' - \lambda)} |X(\lambda', \lambda)|_{\mathcal{L}(B)},
$$

$$
0 \leq s \leq t. \quad (52)
$$
Remark. If an estimate of the form $|X(t, s)|_{\mathcal{X}(\phi)} \leq K_0 e^{\tilde{\omega}(t-s)}$ is known for $0 \leq s \leq t < \infty$, where $\tilde{\omega} > -\omega$, then we obtain from (52) the estimate

$$|T(t, s)|_{\mathcal{X}(\phi)} \leq KK_0(1 + t - s) e^{\tilde{\omega}(t-s)}, \quad 0 \leq s \leq t.$$  (53)

Proof of Theorem 2. Let $s \geq 0$ and $\varphi \in \mathcal{P}$ be given. By the definition (23) and the linearity of the equation we have

$$(T(t, s) \varphi)(0) = X^\varepsilon(t, s) \varphi(0) + (T(t, s) \varphi^\varepsilon)(0),$$  (54)

and

$$y^\varepsilon(t) := (T(t, s) \varphi^\varepsilon)(0), \quad t \geq s,$$

$$y^\varepsilon(t) := \varphi^\varepsilon(t - s), \quad t < s,$$

solves the equation

$$y^\varepsilon(t) = \int_s^t W(t, \lambda) y^\varepsilon_\lambda \, d\lambda, \quad t \geq s.$$  (55)

Because of

$$y^\varepsilon_\lambda = S(\lambda - s) \varphi^\varepsilon + w^\varepsilon_\lambda,$$

$$w^\varepsilon(t) := y^\varepsilon(t), \quad s \leq t,$$

$$w^\varepsilon(t) := 0, \quad t < s,$$

we get from (55) for $t \geq s$,

$$w^\varepsilon(t) = \int_s^t W(t, \lambda) w^\varepsilon_\lambda \, d\lambda + \int_s^t U(t, \lambda) L(\lambda) S(\lambda - s) \varphi^\varepsilon \, d\lambda.$$  (56)

Now we can apply the variation of parameters formula to (56) to obtain

$$y^\varepsilon(t) = w^\varepsilon(t) = \int_s^t X(t, \lambda) L(\lambda) S(\lambda - s) \varphi^\varepsilon \, d\lambda, \quad t \geq s \geq 0.$$  (57)

Hence (54) and (57) together yield for $0 \leq s \leq t$:

$$(T(t, s) \varphi)(0) = X^\varepsilon(t, s) \varphi(0) + \int_s^t X(t, \lambda) L(\lambda) S(\lambda - s) \varphi^\varepsilon \, d\lambda.$$  (58)

Because of Theorem 1 (claim (1)), (58) shows the representation (50) and the first claim by transition to the limit for $\varepsilon \to 0$.

Now let us assume (51). Let $\omega \in (0, -\omega_P)$ be given and choose $M \geq 1$ such that (5) holds. We conclude (cf. Definitions 1 and 2)

$$|\varphi^\varepsilon|_\rho \leq (1 + c(\varepsilon) \tilde{\varepsilon}) |\varphi|_\rho$$  (59)
and

\[ |S(v)\varphi|^p \leq Me^{-c(v)}|\varphi|^p, \quad v \geq 0. \]  

(60)

Then (50) yields for \(0 \leq s \leq t < \infty\) with \(\tilde{K} = \tilde{c} + (1 + c(0)\tilde{c})M\tilde{L}\):

\[ |(T(t, s)\varphi)(0)| \leq \tilde{K} \max \left\{ |X(t, s)|_{\mathcal{L}(P)}, \int_s^t |X(t, \lambda)|_{\mathcal{L}(P)} e^{-\omega(t-s)} \, d\lambda \right\} |\varphi|^p. \]  

(61)

Lemma 1 implies for \(0 < s < t\),

\[ |T(t, s)\varphi|^p \leq M \max \left\{ e^{-\omega(t-s)} |\varphi|^p, \max_{s \leq t' < t} e^{-\omega(t-t')} |(T(t', s)\varphi)(0)| \right\}. \]  

(62)

Now the existence of the claimed number \(K\) and inequality (52) follow from (61) and (62).

Under the hypotheses of Theorem 2 we find by comparing the inequalities (48) and (52) that the exponential growth-rates of the resolvent operators \(X(t, s)\) and the solution maps \(T(t, s)\) are the same for \(t - s \to \infty\), provided that they are greater than \(\omega_P\). The following trivial example shows that this restriction related to the relaxation exponent of the space \(P\) is essential and it is not caused by the technique of the proof only.

**Example 6.** Let \(P\) denote the space in the Example 2 with \(\omega_P = -\kappa < 0\). Choose \(\kappa > \omega\) and consider the equation (1) with \(A(t) = -\kappa a\) and \(L(t) := 0\). In this case we have

\[ X(t, s) = e^{-\kappa(t-s)I}, \quad t \geq s, \]

and hence \(|X(t, s)|_{\mathcal{L}(P)} \leq e^{-\kappa(t-s)}\). However an easy calculation shows

\[ |T(t, s)|_{\mathcal{L}(P)} \geq e^{-\omega(t-s)}, \quad 0 \leq s \leq t. \]

5. SOME SPECIAL CASES AND REMARKS

By the following proposition we make a contribution to the problem under which conditions on \(A(t)\) and \(L(t)\) the hypothesis H3 is satisfied:

**Proposition 1.** Assume \(W(t, \lambda) = W_0(t, \lambda) + W_1(t, \lambda), \quad 0 \leq \lambda < t\), where \(W_0\) satisfies (H2) and \(W_1\) satisfies (H2) and (H3). In addition suppose that \(W_0(t, \lambda)\) is continuous in \(\lambda\) with respect to the operator norm in \(\mathcal{L}(P, B)\) for \(0 \leq \lambda < t\) and all \(t > 0\). Then \(W\) satisfies hypothesis (H3).
Proof. Let $a \in B_1$, numbers $0 \leq s \leq t$, and sequences of functions $\gamma^n$, $n \in \mathbb{N}$, be given according to hypothesis (H3). By (H2) we find a continuous function $m: (0, t] \rightarrow \mathbb{R}^+$ such that

$$|W_0(t, \lambda)|_{\mathcal{L}(P, B)} \leq m(t - \lambda), \quad 0 \leq \lambda < t, \quad \int_0^t m(v) \, dv < \infty. \quad (63)$$

Let $\rho > 0$ be given. There exists a partition of the interval $[s, t]$, namely $s = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < t$ such that

$$\int_0^{t - \lambda_m} m(v) \, dv \leq \rho \quad (64)$$

and

$$|W_0(t, \lambda) - W_0(t, \lambda_j)|_{\mathcal{L}(P, B)} \leq \rho, \quad j = 1, \ldots, m, \quad \lambda_{j-1} \leq \lambda \leq \lambda_j. \quad (65)$$

Let $\delta = \delta(a, t, s)$ such that the support-intervals $I_n$ are contained in $(-\delta, 0)$. Then we have (cf. Definitions 2, 4)

$$|S(v)(\gamma^n(\cdot) a)|_B \leq c(t - s + \delta) |a|_B, \quad 0 \leq v \leq t - s, \quad n \in \mathbb{N}. \quad (66)$$

Hence (63)–(66) imply for all $n \in \mathbb{N},$

$$\left| \int_s^t W_0(t, \lambda) S(\lambda - s)(\gamma^n(\cdot) a) \, d\lambda \right. \nonumber$$

$$\left. - \sum_{j=1}^m W_0(t, \lambda_j) \int_{\lambda_{j-1}}^{\lambda_j} S(\lambda - s)(\gamma^n(\cdot) a) \, d\lambda \right|_B$$

$$\leq \rho c(t - s + \delta)(1 + t - s) |a|_B. \quad (67)$$

Since $\rho > 0$ was arbitrary, (67) shows that (H3) is valid if

$$\lim_{n \to \infty} \int_{\lambda_{j-1}}^{\lambda_j} S(\lambda - s)(\gamma^n(\cdot) a) \, d\lambda = 0 \quad (68)$$

for every $j \in \{1, \ldots, m\}$. Let $\varepsilon_n$ denote the length of the interval $I_n$ containing the support of $\gamma^n$ (cf. (H3)). Then we have for all $v \leq 0$ (note $0 \leq \gamma^n(\sigma) \leq 1$),

$$\left| \left( \int_{\lambda_{j-1}}^{\lambda_j} S(\lambda - s)(\gamma^n(\cdot) a) \, d\lambda \right)(v) \right|_B \leq \left| \int_{\lambda_{j-1}}^{\lambda_j} \gamma^n(\lambda - s + v) \, d\lambda \, a \right|_B \leq |a|_B \varepsilon_n$$

and therefore

$$\left| \int_{\lambda_{j-1}}^{\lambda_j} S(\lambda - s)(\gamma^n(\cdot) a) \, d\lambda \right|_p \leq c(t - s + \delta) \varepsilon_n |a|_B$$

which proves (68).
Proposition 1 shows that the verification of (H3) is necessary for $W - W_0$ only, when $W_0$ has the property that $W_0(t, \lambda)$ is continuous in $\lambda$ with respect to the operator norm $0 \leq \lambda < t$. The continuity of $W(t, \lambda)$ in $\lambda$ with respect to the operator norm is violated if $U(t, \lambda)$ is discontinuous in $\lambda$ with respect to the operator norm in $\mathcal{L}(B)$, e.g., if $U(t, \lambda) = \exp(A(t - \lambda))$ with hyperbolic $A$, or if $L(\lambda)$ is discontinuous in $\lambda$ with respect to the operator norm in $\mathcal{L}(P, B)$ which may happen in the case of time-dependent delays.

To demonstrate the verification of hypotheses (H1)–(H3) we are going to consider some special cases. First, we take the equation

$$y'(t) = Ay(t) + C(t)y(t - \tau(t)) + \int_{-\infty}^{0} G(t, v)y(t + v)dv + h(t)$$

in a real Hilbert space $B$, where the operators $A$, $C(t)$, $G(t, v)$, and the delay $\tau(t)$ have the following properties:

1. $A: D(A) \to B$ is linear, selfadjoint, and generates an analytic semigroup $\exp(At)$, $t \geq 0$, such that $0$ belongs to the resolvent of $A$.

2. The operators $C(t): D(C(t)) \to B$ and $G(t, v): D(G(t, v)) \to B$ are closed, linear, and densely defined for all $t \geq 0$ and $v \leq 0$.

3. There exists $x \in [0, 1)$ such that the domain $D((-A)^x)$ of $(-A)^2$ is contained in the domains $D(C(t))$, $D(G(t, v))$, $D(C^*(t))$, and $D(G^*(t, v))$ for all $t \geq 0$ and $v \leq 0$, where $C^*(t)$ and $G^*(t, v)$ denote the adjoint operators of $C(t)$ and $G(t, v)$.

4. $C(t) a \in B$ and $G(t, v) a \in B$ are continuous in $t \in \mathbb{R}^+$ and $(t, v) \in \mathbb{R}^+ \times \mathbb{R}^-$ for every fixed $a \in D((-A)^x)$.

5. There exists $\omega > 0$ such that for every $\tau > 0$ a continuous function $\beta_\tau: \mathbb{R}^- \to \mathbb{R}^+$ can be chosen satisfying

$$\int_{-\infty}^{0} e^{-\omega v} \beta_\tau(v) dv < \infty$$

and

$$|G(\lambda, v)(-A)^x| \mathcal{L}(B) \leq \beta_\tau(v), \quad |G^*(\lambda, v)(-A)^{-x}| \mathcal{L}(B) \leq \beta_\tau(v)$$

(70)

for all $\lambda \in [0, \tau]$ and $v \leq 0$ (note that by (1)–(3) and the closed graph theorem the operators $G(\lambda, v)(-A)^{-x}$ and $G^*(\lambda, v)(-A)^{-x}$ are bounded.)

6. $\tau: \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. There exists $\delta > 0$ such that for every $r > 0$ and $\sigma \geq -\delta$ the set $\{t \in [0, r]: t - \tau(t) = \sigma\}$ has the Lebesgue measure zero.
For the verification of (H1)-(H3) we take $P$ as the phase space of Example 2, $B_1 := D((-A)^{\alpha})$ with the graph norm

$$|a|_{B_1} := |a|_B + |(-A)^{\alpha} a|_B,$$

and $P_1$ the space of all $\varphi: \mathbb{R}^- \to B_1$ such that $e^{\alpha \nu} \varphi(v)$ is uniformly continuous and bounded in $B_1$ for all $v \leq 0$. Take

$$|\varphi|_{P_1} := \sup_{\nu \leq 0} e^{\alpha \nu} |\varphi(v)|_{B_1}.$$

**Proposition 2.** If (1)-(6) hold, then hypotheses (H1)-(H3) are satisfied.

**Proof.** It is easy to see that $P_1$ is an admissible subspace of the admissible phase space $P$. With $U(t, s) := \exp(A(t-s))$ hypothesis (H1) is certainly satisfied. To verify (H2), (1), we note

$$L(t) \varphi = C(t) \varphi(-\tau(t)) + \int_{-\infty}^{0} G(t, v) \varphi(v) \, dv, \quad t \geq 0, \quad \varphi \in P_1.$$

Because of

$$L(t) \varphi = C(t)(-A)^{-\alpha}(-A)^{\alpha} \varphi(-\tau(t))$$

$$+ \int_{-\infty}^{0} (G(t, v)(-A)^{-\alpha})(-A)^{\alpha} \varphi(v) \, dv,$$

we see by means of the property (5) and the boundedness of $C(t)(-A)^{-\alpha}$, that $L(t) \varphi$ is well defined for all $t \geq 0$ and $\varphi \in P_1$. Since the operator $C(t)(-A)^{-\alpha}: \mathbb{B} \to \mathbb{B}$ depends strongly continuously on $t$ by 4, the operators $C(t)(-A)^{-\alpha}$ are uniformly bounded on compact $t$-intervals. Together with the choice of $P_1$ and assumption (6) this shows that $C(t) \varphi(-\tau(t))$ is continuous in $t$ for every $\varphi \in P_1$. A similar argument employing (5) and Lebesgue's theorem on dominated convergence shows that

$$\int_{-\infty}^{0} G(t, v) \varphi(v) \, dv \in \mathbb{B}$$

is continuous in $t$ for every fixed $\varphi \in P_1$. To show (H2), (2), we use for every $\varphi \in P_1$ and $0 \leq \lambda < t$, the representation

$$W(t, \lambda) \varphi = (-A)^{\alpha} \exp(A(t-\lambda))((-A)^{-\alpha} C(\lambda)) \varphi(-\tau(\lambda))$$

$$+ (-A)^{\alpha} \exp(A(t-\lambda)) \int_{-\infty}^{0} ((-A)^{-\alpha} G(\lambda, v) \varphi(v) \, dv. \quad (71)$$
We note that
\[ |(-A)^{\alpha} \exp(Av)|_{L'(B)} \leq \frac{c}{v^2} \quad \text{for all } v > 0, \quad (72) \]
where \( c > 0 \) is independent of \( v \). Moreover the operators \((-A)^{-\alpha} C(\lambda)\) and \((-A)^{-\alpha} G(\lambda, v)\) have the bounded adjoint operators \( C^*(\lambda)(-A)^{-\alpha} \) and \( G^*(\lambda, v)(-A)^{-\alpha} \). Therefore we have
\[ |(-A)^{-\alpha} G(\lambda, v)|_{L'(B)} = |G^*(\lambda, v)(-A)^{-\alpha}|_{L'(B)} \leq \beta_\xi(v) \quad (73) \]
for all \( v \leq 0, 0 \leq \lambda \leq \tau \). Similarly for each \( \tau > 0 \) there exists \( \rho_\tau > 0 \) such that
\[ |(-A)^{-\alpha} C(\lambda)|_{L'(B)} = |C^*(\lambda)(-A)^{-\alpha}|_{L'(B)} \leq \rho_\tau, \quad 0 \leq \lambda \leq \tau. \quad (74) \]
Then (71)–(74) yield for \( 0 \leq \lambda < t \leq \tau \) and \( \varphi \in P_1 \),
\[ |W(t, \lambda) \varphi|_{B} \leq \frac{c}{(t-\lambda)^2} \rho_\tau \exp(\alpha \max_{0 \leq \lambda \leq \tau} \xi(\lambda)) |\varphi|_{P} \]
\[ + \frac{c}{(t-\lambda)^2} \int_{-\infty}^{0} e^{-\alpha \xi(\lambda)} \beta_\xi(v) \, dv \, |\varphi|_{P}. \quad (75) \]
Inequality (75) shows (H2), (2). To prove (H3) let \( a \in B_1 \), numbers \( 0 \leq s < t \), and sequences of functions \( \gamma^n \) with support in intervals \( I_n \subset (-\delta, 0), n \in \mathbb{N} \), according to (H3) be given. Define \( \gamma^n(\lambda) := 0 \) for \( \lambda \geq 0 \). Then we obtain
\[ \delta_n := \int_{s}^{t} W(t, \lambda) S(\lambda - s)(\gamma^n(\lambda - a)) \, d\lambda = \delta^1_n + \delta^2_n, \]
\[ \delta^1_n := \int_{s}^{t} \exp(A(\lambda - s)) \gamma^n(\lambda - s - \tau(\lambda)) C(\lambda) \, a \, d\lambda, \quad (76) \]
\[ \delta^2_n := \int_{s}^{t} \exp(A(\lambda - s)) \left( \int_{-\infty}^{0} (G(\lambda, v) \, a) \, \gamma^n(\lambda - s + v) \, dv \right) \, d\lambda. \]
For all \( n \in \mathbb{N} \) and \( \lambda \in [s, t] \) we define the sets
\[ J^0_n(\lambda) := \{ v < 0; \lambda - s + v \in I_n \}, \quad J^1_n := \{ \mu \in (s, t); \mu, s, \tau(\mu) \in I_n \}. \]
Then the properties of the intervals \( I_n \) according to (H3) imply
\[ J^0_n + 1(\lambda) \subset J^0_n(\lambda), \quad J^1_n + 1 \subset J^1_n, \quad n \in \mathbb{N}, \]
\[ \text{diameter } J^0_n(\lambda) \to 0 \text{ uniformly in } \lambda. \quad (77) \]
Because of

\[ \bigcap_{n \in \mathbb{N}} J_n^I = \left\{ \mu \in (s, t): \mu - s - \tau(\mu) \in \bigcap_{n \in \mathbb{N}} I_n \right\} \]

and the intersection of the intervals \( I_n \) is either empty or consists of exactly one point, property (6) implies

\[ \bigcap_{n \in \mathbb{N}} J_n^I \text{ has Lebesgue measure zero.} \]

Hence (77), (78), and Lebesgue's theorem on dominated convergence imply \( \delta_i' \to 0 \) for \( n \to \infty \) \((i = 1, 2)\). Then (76) shows that (H3) is valid.

It is remarkable that property (6) of the delay-function \( \tau(t) \) is not only a technical assumption. If it is violated the following counterexample shows that a jointly strongly continuous resolvent \( X(t, s) \) (Theorem 1) fails to exist:

**Example 6.** Take \( B = \mathbb{R}^1, A = 0, C(t) = I, G(t, \lambda) = 0, \) and

\[ \tau(t) := \begin{cases} 0, & 0 \leq t \leq 1, \\ t - 1, & 1 < t, \end{cases} \]

i.e., we are considering the delay equation \( y'(t) = y(t - \tau(t)) + h(t) \). Then the limit (24) yields for all number \( 0 \leq s \leq t < \infty \),

\[ X(t, s) = \begin{cases} e^{t-s}, & 0 \leq s \leq t \leq 1, \\ e^{t-s}t, & 0 \leq s \leq 1 < t, \\ 1, & 1 < s \leq t, \end{cases} \]

showing that \( X(t, s) \) is discontinuous in \( s \) at \( s = 1 \) if \( t > 1 \). It is easy to see that (80) is the only possible choice of a piecewise continuous resolvent for which the variation of parameters formula is satisfied for every continuous forcing function \( h(t) \).

However, weakening hypothesis (H3), we can show that a resolvent still exists which has all properties of the resolvent \( X(t, s) \) according to Theorem 1 despite of the dependence on \( s \) which is merely strongly Lebesgue-measurable. The weakened version of hypothesis (H3) reads as follows:

\[(H3') \text{ Weakened Version of (H3)}\]

For every \( a \in B_1 \) and all numbers \( 0 \leq s \leq t < \infty \) there exists \( \delta = \delta(a, s, t) > 0 \) such that for every sequence of \( C^\infty \)-functions \( \gamma^n \ (n \in \mathbb{N}) \) on
with range in \([0, 1]\) and support in intervals \(I_n \subset (-\delta, 0)\) satisfying \(I_{n+1} \subset I_n\) and \(\bigcap_{n \in \mathbb{N}} I_n = \emptyset\), condition (4) is true.

Then the following theorem still holds:

**Theorem 3.** Assume \((H1), (H2), (H3'), and (H4). Then Theorem 1 remains true except statement (2) which has to be replaced by the following:

\((2')\) For all \(\tau > 0\) there exists \(K_\tau > 0\) such that \(X(t, s) \in \mathcal{L}(B)\) and \(|X(t, s)|_{\mathcal{L}(B)} \leq K_\tau\) for all numbers \(0 \leq s < t \leq \tau\). If \(a \in B\) is given, then \(X(t, s) a \in B\) is continuous in \(t\) for every \(s \geq 0\) and \(t \geq s\) and strongly Lebesgue-measurable in \(s \in [0, t]\). Moreover, \((t, s) a \in B\) is continuous in \(s\) from the left for \(0 \leq s < t\).

If \((H1), (H2'), and (H3') hold, Theorem 2 remains true again, provided that the estimate (52) is replaced by the following:

\[
|T(t, s)|_{\mathcal{L}(B)} \leq K \inf_{\kappa} e^{-\omega(t-s)}(1 + t - s) \sup_{s \leq \lambda < \lambda' \leq t} e^{\omega(\lambda' - \lambda)} \kappa(\lambda', \lambda),
\]

where the infimum runs over all functions \(\kappa \in \mathcal{L}_{loc}^\infty([t^-2, \mathbb{R}^+])\) such that \(|X(\lambda', \lambda)|_{\mathcal{L}(B)} \leq \kappa(\lambda', \lambda), 0 \leq \lambda \leq \lambda'.

**Proof.** The proof goes on by checking up in what parts of the proofs of Theorems 1 and 2 we can do with the weaker hypothesis \((H3')\). To show the existence of the limit (24) for fixed \(s, t\) by means of (27) and (29) we only need that the limit (29) is locally uniform in \(t\) for every \(s \geq 0\). In what follows we can therefore let \(s, \ldots, s_n, \ldots\) for all \(n \in \mathbb{N}\). Then the functions \(\gamma_n\) according to (35) are different from zero at most in the open intervals

\[
I_n := (s - \tilde{s} - \varepsilon_n, s - \tilde{s}).
\]

Since \(\varepsilon_n \to 0\) for \(n \to \infty\), the intersection of the intervals \(I_n\) is empty. Hence \((H3')\) implies by the same argument as in the proof of Theorem 1 that the limit (29) is uniform in \(t\) for \(t \geq s\) and every fixed \(s \geq 0\). As a consequence, the limit (24) exists and \(X(t, s) a\) is continuous in \(t \geq s\) for every fixed \(s \geq 0\) and \(a \in B\). Since every approximation \(X^\epsilon(t, s) a\) is jointly continuous in \(t\) and \(s\), it follows that \(X(t, s) a\) is at least strongly Lebesgue-measurable in \(B\) with respect to \(t\) and \(s\). Now let \(a \in B\) and \(t > 0\) be fixed and assume that \(X(t, s) a\) is not continuous from the left in \(s \in (0, t)\). Then we can find \(\delta \in (0, t), \rho > 0,\) and a sequence of numbers \(s_n \in (0, \delta)\) such that \(s_n \to \delta\) for \(n \to \infty\) and

\[
4\rho \leq |X(t, s) a - X(t, s_n) a|_B, \quad n \in \mathbb{N}.
\]
Let \( \varepsilon_i \in (0, 1], \, i \in \mathbb{N}, \) be such that \( \varepsilon_i \to 0 \) for \( i \to \infty. \) Because of (24) and
\[
\lim_{n \to \infty} |X^{\varepsilon_i}(t, s) a - X^{\varepsilon_i}(t, s_n) a|_B = 0, \quad i \in \mathbb{N} \text{ fixed},
\]
we find \( i_0 \in \mathbb{N} \) and for every \( i \geq i_0 \) a number \( n_i \in \mathbb{N} \) such that
\[
|X^{\varepsilon_i}(t, s) a - X^{\varepsilon_i}(t, s_n) a|_B \leq \rho, \quad i \geq i_0, \tag{83}
\]
and
\[
|X(t, s) a - X^{\varepsilon_i}(t, s) a|_B \leq \rho, \quad i \geq i_0.
\]
As
\[
\lim_{j \to \infty} |X^{\varepsilon_i}(t, s_n) a - X(t, s_n) a|_B = 0
\]
for every fixed \( i \in \mathbb{N}, \) there exists for each \( i \geq i_0 \) a number \( k(i) \in \mathbb{N} \) such that
\[
|X^{\varepsilon_{k(i)}}(t, s_n) a - X(t, s_n) a|_B \leq \rho, \quad i \geq i_0. \tag{85}
\]
Gathering (82)-(85) together we find
\[
\rho \leq |X^{\varepsilon_i}(t, s_n) a - X^{\varepsilon_{k(i)}}(t, s_n) a|_B, \quad i \geq i_0.
\]
Renaming \( \varepsilon_i := \varepsilon_{k(i)} \) and \( s_i := s_n, \) we conclude from (27) the existence of \( \bar{\rho} > 0 \) and \( \bar{t}_i \in (s_n, t], \, i \in \mathbb{N}_0, \) such that \( t_i \to \bar{t} \leq t \) \( (i \to \infty) \) and
\[
\bar{\rho} \leq |\beta^{\varepsilon_{k(i)}}(t, s_i)|_B, \quad i \geq i_0. \tag{86}
\]
Now we can proceed as in the proof of Theorem 1 beginning with formula (30). As \( s_i \to \bar{s} \) for \( i \to \infty, \) the support of the \( \gamma^t \) is now contained in the intervals
\[
I_i := (s_i - \bar{s} - \varepsilon_i, \bar{s} - \bar{s}), \quad i \geq i_0. \tag{87}
\]
Suppose there exists \( \bar{s} \in \mathbb{R} \) such that \( \bar{s} \in I_i \) for all \( i \geq i_0. \) Then by (87) we must have \( \bar{s} = \bar{s} - \bar{s} \) and therefore
\[
s_i - \bar{s} - \varepsilon_i < \bar{s} - \bar{s} < \bar{s} - \bar{s}, \quad i \geq i_0,
\]
which is a contradiction. Hence the intersection of the open intervals \( I_i \) is empty and by (H3') we get a contradiction to (86) as in the proof of Theorem 1. Thus statement (2') holds. To prove the variation of parameters formula we can proceed as in the proof of Theorem 1, provided we can check that \( X(t, \lambda) h(\lambda), \, s \leq \lambda \leq t, \) is Bochner–Lebesgue-integrable in
By (H4) we can find a sequence of continuous functions $h_n: [s, t] \to B$ such that

$$h(\lambda) = \lim_{n \to \infty} h_n(\lambda) \quad \text{almost everywhere in } [s, t].$$

Then by means of statement (1) and the uniform boundedness of the operators $X^{1/n}(t, \lambda), n \in \mathbb{N}$, with respect to $\lambda \in [s, t]$, we get

$$X(t, \lambda) h(\lambda) = \lim_{n \to \infty} X^{1/n}(t, \lambda) h_n(\lambda), \quad s \leq \lambda \leq t,$$

almost everywhere in $\lambda$, showing that $X(t, \cdot) h(\cdot)$ is strongly Lebesgue-measurable since the approximations $X^{1/n}(t, \cdot) h_n$ are continuous. As the approximating functions $h_n$ can be chosen such that

$$\sup_{n \in \mathbb{N}} \int_s^t |h_n(\lambda)|_B d\lambda < \infty,$$

the Bochner–Lebesgue-integrability of $X(t, \cdot) h(\cdot)$ in $[s, t]$ follows. The proof of Theorem 2 goes through except that it is not clear whether $X(t, \lambda) h(\lambda)$ is Lebesgue-measurable in $\lambda \in [s, t]$. Therefore we replace $X(t, \lambda) h(\lambda)$ by a strongly Lebesgue-measurable majorant $\kappa(t, \lambda)$ to obtain the estimate (61) which by transition to the infimum over $\kappa$ leads to the estimate (81).

**PROPOSITION 3.** Assume that hypotheses (1)–(5) for Eq. (69) hold and replace the hypothesis (6) by the following weaker hypothesis:

$$(6') \quad \tau: \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous.}$$

Then (H1), (H2), and (H3') hold.

**Proof.** The arguments in the proof of Proposition 2 go through except that the intersection of the intervals $I_n$ is now assumed to be empty. Then the intervals $J_n$ are open and form a decreasing sequence with empty intersection. Hence the conclusion is valid again.

Now let us consider four other examples of partial functional differential equations which are not covered by equations of the type (69).

**EXAMPLE 7.** Consider the diffusion–reproduction equation

$$\frac{\partial}{\partial t} u(t, \xi) = A u(t, \xi) - d(t, \xi) u(t, \xi) + b(t, \xi) u(t - \tau(t, \xi), \xi) + h(t, \xi),$$

if $t \geq 0$, $\xi \in \Omega$, and $u(t, \xi) = 0$, if $t \geq 0$, $\xi \in \partial \Omega$, (88)
where $\Omega$ is a bounded domain in $\mathbb{R}^n$ having the $C^1$-extension property and $d, b, \tau : \mathbb{R}^+ + \bar{\Omega} \to \mathbb{R}^+$ are continuous. We suppose that $h$ belongs to $L^2_{\text{loc}}(\mathbb{R}^+ \times \bar{\Omega}, \mathbb{R})$ and is continuous in $t$ in the $L^2$-mean. Choose $B := L^2(\bar{\Omega}, \mathbb{R})$. Then hypotheses (H1) and (H4) are certainly satisfied, where $A := A$ generates an analytic semigroup in $B$. If $G(t, \xi, \eta)$, $t > 0, \xi, \eta \in \Omega$, denotes the Green function of the Laplacian in $\Omega$ with respect to Dirichlet boundary conditions, we have

$$U(t, s) = \exp(A(t-s))a, \quad \exp(At)a(\xi) = \int_{\Omega} G(t, \xi, \eta) a(\eta) \, d\eta, \quad 0 < t, \quad a \in B.$$

Let $P$ be the space of Example 2 with arbitrary $\omega > 0$ and let $P_1 := P$. Then for every $\varphi \in P$ ($\varphi(v)(\xi) =: \varphi(v, \xi)$)

$$(L(t) \varphi)(\xi) := -d(t, \xi) \varphi(0, \xi) + b(t, \xi) \varphi(-\tau(t, \xi), \xi), \quad \xi \in \bar{\Omega},$$

defines an operator $L(t) \in \mathcal{L}(P, B)$ which depends strongly continuously on $t$. Thus (H2) holds and if $d, b,$ and $\tau$ are bounded in $\mathbb{R}^+ \times \bar{\Omega}$, (H2') holds, too. For a given $a \in B, 0 \leq s < t,$ and $\gamma^n$ accordingly to (H3) we obtain for all $\xi$ noting that the support of $\gamma^n$ is contained in $I_n = \mathbb{R} \setminus \{0\}$:

$$(\gamma^n(\lambda) := 0, \lambda \geq 0),$$

$$(\gamma^n(\lambda) := 0, \lambda \geq 0),$$

$$(\gamma^n(\lambda) := 0, \lambda \geq 0),$$

$$q^n(\lambda) := \left( \int_s^t W(t, \lambda) S(\lambda - s)(\gamma^n(\cdot) a) \, d\lambda \right)(\xi) = \int_s^t \int_{\Omega} G(t-\lambda, \xi, \eta) b(\lambda, \eta) a(\eta) \gamma^n(\lambda - s - \tau(\lambda, \eta)) \, d\eta \, d\lambda. \quad (89)$$

From (89) we see that the functions $q^n$ converge to zero in the $L^2$-norm, if the Lebesgue measure of the sets

$$J^n := \{(\lambda, \eta) \in [s, t] \times \bar{\Omega} : \lambda - s - \tau(\lambda, \eta) \in I_n \}$$

in $\mathbb{R}^+ + \bar{\Omega}$ goes to zero for $n \to \infty$. As by the continuity of $\tau$ and the boundedness of $[s, t] \times \bar{\Omega}$ this is always true if the intersection of the intervals $I_n$ is empty, (H3') holds. The stronger hypothesis (H3) holds if we assume the existence of $\delta > 0$ such that

$$\{ (\lambda, \eta) \in [0, r] \times \bar{\Omega} : \lambda - \tau(\lambda, \eta) = \sigma \} \quad \text{has Lebesgue measure zero} \quad (90)$$

for every $r > 0$ and $\sigma > -\delta$. If $h : \mathbb{R}^+ \times \bar{\Omega} \to \mathbb{R}$ is continuous, we can also take the space $B = C(\bar{\Omega})$. With aid of the integrability and smoothness properties of $G$ we obtain exactly the same conclusion since the functions $q^n$ in (89) converge to zero in $C(\bar{\Omega})$ under the same hypotheses on $\tau$, also.
EXAMPLE 8. Let us consider the migration-reproduction equation

$$\frac{\partial}{\partial t} u(t, \xi) = \int_{\Omega} K(\xi, \eta) u(t - \tau_0(t, \xi, \eta), \eta) \, d\eta - d(t, \xi) u(t, \xi) + b(t, \xi) u(t - \tau_1(t, \xi), \xi) + h(t, \xi), \quad t \geq 0, \quad \xi \in \bar{\Omega},$$

(91)

where $\bar{\Omega}$ is a bounded domain in $\mathbb{R}^n$, the functions $b, d, \tau_1: \mathbb{R}^+ \times \bar{\Omega} \to \mathbb{R}$, $K: \bar{\Omega}^2 \to \mathbb{R}$, $h: \mathbb{R}^+ \times \bar{\Omega} \to \mathbb{R}$, and $\tau_0: \mathbb{R}^+ \times \bar{\Omega}^2 \to \mathbb{R}^+$ are assumed to be continuous. We take $B := C(\bar{\Omega}, \mathbb{R})$, $p > 1$, $B_1 := B$, $P_1 := P$, $P$ being the phase space of Example 2 for a given number $\omega \in \mathbb{R}$. Let $A := 0$. Then (H1) and (H4) hold trivially ($U(t, s) = I$). The operators

$$(L(t) \varphi)(\xi) := \int_{\Omega} K(\xi, \eta) \varphi(-\tau_0(t, \xi, \eta), \eta) \, d\eta - d(t, \xi) \varphi(0, \xi) + b(t, \xi) \varphi(-\tau_1(t, \xi), \xi), \quad t \geq 0, \quad \xi \in \bar{\Omega},$$

belong to $\mathcal{L}(P, B)$ for all $t \geq 0$ and depend strongly continuously on $t$. Hence (H2) holds. If in addition $d, b, \tau_0$, and $\tau_1$ are bounded, (H2') holds, too (If $\omega \leq 0$, we can refrain from the boundedness of $\tau_0$ and $\tau_1$). To verify (H3) (H3') we have to look for conditions such that for given $Q \in B_1$ and sequences $\gamma^n$ according to (H3) (H3') ($\gamma^n(\lambda) := 0$, $\lambda \geq 0$)

$$q^n(\xi) := \left( \int_{s'}^t L(\lambda) S(\lambda - s)(\gamma^n(\cdot) a) \, d\lambda \right)(\xi)$$

$$= \int_{s'}^t \int_{\Omega} K(\xi, \eta) \gamma^n(\gamma - s - \tau_0(\lambda, \xi, \eta)) a(\eta) \, d\eta \, d\lambda$$

$$+ \int_{s'}^t b(\lambda, \xi) \gamma^n(\lambda - s - \tau_1(\lambda, \xi)) \, d\lambda \, a(\xi)$$

converges to zero in $B$ for $n \to \infty$. One finds that this is satisfied if for every sequence of support-intervals $I_n$ according to (H3) (H3') the Lebesgue measure of the sets

$$\{(\lambda, \eta) \in [s, t] \times \bar{\Omega}: \lambda - \tau_0(\lambda, \xi, \eta) - s \in I_n\},$$

$$\{\lambda \in [s, t]: \lambda - \tau_1(\lambda, \xi) - s \in I_n\}$$

converge to zero uniformly in $\xi \in \Omega$ for all $t > s \geq 0$. Examples for $\tau_0$ and $\tau_1$ which fulfil this condition in both cases are

$$\tau_0(\lambda, \xi, \eta) := c(\lambda)^{-1} |\xi - \eta|, \quad \|\cdot\| \text{ norm in } \mathbb{R}^n.$$
where $c(\lambda) > 0$ for $\lambda \geq 0$ is differentiable and bounded, and

$$\tau_1(\lambda, \xi) = \tau_1(\xi),$$

where $\tau_1(\xi) > 0$ is Fréchet differentiable in $\Omega$.

**Example 9.** Consider the equation

$$\frac{\partial}{\partial t} u(t, \xi) = -g(\xi) \frac{\partial}{\partial \xi} u(t, \xi) + d(t, \xi) u(t, \xi) + \int_0^1 K(t, \xi, \eta) u(t - \bar{\tau}(t, \xi, \eta), \eta) \, d\eta + h(t, \xi),$$

$$0 < \xi < 1, \quad t \geq 0,$$  \hspace{1cm} (92)

under the following hypotheses:

(i) $g: (0, 1] \rightarrow (0, \infty)$ is continuous, bounded, and fulfills

$$\int_0^1 g(\eta)^{-1} \, d\eta = \infty.$$

(ii) $d: \mathbb{R}^+ \times (0, 1) \rightarrow \mathbb{R}$ is continuous. There exists a continuous

$$\mu: (0, 1) \rightarrow \mathbb{R}^+$$

such that for every $\tau > 0$ and $t \in [0, \tau]$ the function

$$[0, \tau] \ni t \rightarrow d(t, \cdot) + \mu(\cdot) \in C_b((0, 1), \mathbb{R})$$

is continuous (with respect to the sup-norm in $C_b$).

(iii) The function $\mu$ in (ii) can be chosen such that

$$\int_0^{1/2} \mu(\eta) g(\eta)^{-1} \, d\eta < \infty$$

and

$$\int_1^{1/2} \mu(\eta) \, d\eta = \infty.$$

(iv) $\bar{\tau}: \mathbb{R}^+ \times [0, 1]^2 \rightarrow \mathbb{R}^+$ is continuous.

(v) $K: \mathbb{R}^+ \times [0, 1]^2 \rightarrow \mathbb{R}$ is Lebesgue measurable, $K(t, 1, \eta) = 0$ for all $t \geq 0$, $0 \leq \eta \leq 1$. For all $\tau > 0$ the mapping

$$[0, \tau] \times [0, 1] \ni (t, \xi) \rightarrow K(t, \xi, \cdot) \in L^1([0, 1], \mathbb{R})$$

is well defined and continuous.

(vi) $h(t, \cdot) \in C_b((0, 1), \mathbb{R})$ for all $t \geq 0$ and the mapping $\mathbb{R}^+ \ni t \rightarrow h(t, \cdot) \in C_b((0, 1), \mathbb{R})$ is continuous. $h(t, 1) = 0$ holds for all $t \geq 0$.

To verify hypotheses (H1)--(H4) define $B := C_{b, 0}((0, 1), \mathbb{R})$, the bounded, continuous real functions on $(0, 1]$ which vanish at $\xi = 1$. The norm of $B$ is the sup-norm. Choose $\omega > 0$ arbitrarily and let $P$ the phase space of Example 2, $P_1 := P$. Define $A(t) := A$, where

$$(Aa)(\xi) := -g(\xi) a'(\xi) - \mu(\xi) a(\xi), \quad 0 < \xi < 1, \quad a \in D(A).$$
Then by (i) and (iii) we see that $A$ generates a strongly continuous semigroup in $B$ which is given by

$$((\exp(At)a)(\xi)) = \exp \left( - \int_{\xi}^{\xi'} \mu(\eta) g(\eta)^{-1} \, d\eta \right) a(\xi'),$$

$$\xi' := G^{-1}(G(\xi) - t), \quad 0 < \xi < 1, \quad ((\exp(At)a)(\xi)) = 0, \quad \xi = 1,$$

where

$$G(\xi) := - \int_{\xi}^{1} g(\eta)^{-1} \, d\eta, \quad 0 < \xi \leq 1.$$

(We require the boundedness of $g$ to ensure that $\exp(At)a$ is continuous in $t$ with respect to the norm of $B$.) Hence (H1) holds. (H4) is trivially satisfied by means of condition (vi). For $t \geq 0$, $0 < \xi \leq 1$, and $\varphi \in P$ we have

$$(L(t) \varphi)(\xi) = (d(t, \xi) + \mu(\xi)) \varphi(0, \xi)$$

$$+ \int_{0}^{t} K(t, \xi, \eta) \varphi(- \tau(t, \xi, \eta), \eta) \, d\eta.$$ 

It follows from (ii) and (v) that $L(t)$ belongs to $L(P, B)$ and is strongly continuous in $t$. Thus (H2) holds. (H2') holds if in addition $d + \mu$ is bounded on $\mathbb{R}^+ \times (0, 1)$ and

$$\sup_{t \geq 0, 0 < \xi < 1} \int_{0}^{t} |K(t, \xi, \eta)| \exp(\omega \tau(t, \xi, \eta)) \, d\eta < \infty.$$ 

A similar argument as before shows that (II3) (II3') is satisfied if for all numbers $0 \leq s < t$ and support-intervals $I_n$ according to (H3) (H3') the Lebesgue measure of the sets

$$\{(\lambda, \eta) \in [s, t] \times (0, 1): \lambda - s - \tau(\lambda, G^{-1}(G(\xi) - t + \lambda), \eta) \in I_n\}$$

converges to zero for $n \to \infty$ uniformly in $\xi \in (0, 1)$.

**REFERENCES**


