The nullity and rank of linear combinations of idempotent matrices

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Abstract

Baksalary and Baksalary [J.K. Baksalary, O.M. Baksalary, Nonsingularity of linear combinations of idempotent matrices, Linear Algebra Appl. 388 (2004) 25–29] proved that the nonsingularity of \( P_1 + P_2 \), where \( P_1 \) and \( P_2 \) are idempotent matrices, is equivalent to the nonsingularity of any linear combinations \( c_1 P_1 + c_2 P_2 \), where \( c_1, c_2 \neq 0 \) and \( c_1 + c_2 \neq 0 \). In the present note this result is strengthened by showing that the nullity and rank of \( c_1 P_1 + c_2 P_2 \) are constant. Furthermore, a simple proof of the rank formula of Groß and Trenkler [J. Groß, G. Trenkler, Nonsingularity of the difference of two oblique projectors, SIAM J. Matrix Anal. Appl. 21 (1999) 390–395] is obtained.

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1. Introduction

If \( A \) is a matrix (linear transformation) on \( \mathbb{C}^n \), we write \( \mathcal{N}(A) \) and \( \mathcal{R}(A) \) for the nullspace and the range of \( A \). The rank of \( A \), \( \text{rk}(A) \), is the dimension of \( \mathcal{R}(A) \), and the nullity of \( A \), \( \text{nul}(A) \), is the dimension of \( \mathcal{N}(A) \). Let \( \mathcal{P} \) be the set of all \( n \times n \) complex idempotent matrices \( P \) \( (P^2 = P) \).

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In this note we consider nontrivial linear combinations of given $P_1, P_2 \in \mathcal{P}$, that is, matrices of the form

$$c_1 P_1 + c_2 P_2, \quad c_1, c_2 \neq 0, \quad c_1 + c_2 \neq 0.$$ 

Groß and Trenkler [2] proved that the nonsingularity of $P_1 - P_2$ implies the nonsingularity of $P_1 + P_2$. Their methods rely strongly on relations for the ranks of matrices developed by Marsaglia and Styan [5], while Koliha et al. [3] obtained new simple proofs, without reference to rank theory, and pointed out explicitly a condition, which combined with the nonsingularity of $P_1 + P_2$ implies the nonsingularity of $P_1 - P_2$. Baksalary and Baksalary [1] proved that the nonsingularity of $P_1 + P_2$ is equivalent to the nonsingularity of any linear combinations $c_1 P_1 + c_2 P_2$, where $c_1, c_2 \neq 0$ and $c_1 + c_2 \neq 0$. In the present note this result is strengthened by showing that the nullity and rank of $c_1 P_1 + c_2 P_2$ are constant. Furthermore, we obtain a simple proof of the rank formula of Groß and Trenkler [2].

2. Results

First a useful auxiliary result whose proof is left to the reader.

**Lemma 2.1.** If $P_1, P_2 \in \mathcal{P}$, we define $A$ as the restriction of $(I - P_1)P_2$ to $\mathcal{N}(P_1)$, that is,

$$A: \mathcal{N}(P_1) \to [(I - P_1)P_2] \mathcal{N}(P_1), \quad x \mapsto Ax = (I - P_1)P_2 x.$$ 

Then

$$\mathcal{N}(A) = \mathcal{N}[(I - P_1)P_2] \cap \mathcal{N}(P_1), \quad R(A) = R[(I - P_1)P_2(I - P_1)].$$

We start our observations with the following result.

**Theorem 2.2.** Let $P_1, P_2 \in \mathcal{P}$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $c_1 + c_2 \neq 0$. If $A$ is defined as in Lemma 2.1, then $\mathcal{N}(c_1 P_1 + c_2 P_2)$ is isomorphic to $\mathcal{N}(A)$.

**Proof.** In the proof we use Lemma 2.1.

Let $\mathcal{N} = \mathcal{N}(c_1 P_1 + c_2 P_2)$ and $c \neq 0$. First we show that

$$\mathcal{N} \cong (I - P_1) \mathcal{N} \quad \text{and} \quad \mathcal{N}(A) \cong (c I - P_2) \mathcal{N}.$$ 

(2.2)

Let $x \in \mathcal{N}$ and $(I - P_1)x = 0$. Then $x = P_1 x$, and $(c_1 + c_2)P_2 x = P_2 (c_1 P_1 + c_2 P_2) x = 0$. Therefore $P_2 x = 0$, and so $x = c_1^{-1} (c_1 P_1 + c_2 P_2) x = 0$. Hence $I - P_1$ restricted to acting from $\mathcal{N}$ to $(I - P_1) \mathcal{N}$ is an isomorphism.

If $x \in \mathcal{N}(A)$ and $(c I - P_2)x = 0$, then $P_1 x = 0$, $P_2 x = P_1 P_2 x$ and $P_2 x = c x$. Thus, $P_2 x = P_1 c x = 0$, that is $x = 0$. Hence, $c I - P_2$ restricted to acting from $\mathcal{N}(A)$ to $(c I - P_2) \mathcal{N}(A)$ is an isomorphism.

Next we prove that

$$(I - P_1) \mathcal{N} \subseteq \mathcal{N}(A) \quad \text{and} \quad (c I - P_2) \mathcal{N}(A) \subseteq \mathcal{N} \quad \text{for some} \ c \neq 0.$$ 

(2.3)

Suppose that $x \in \mathcal{N}$. Then $P_1 x = -(c_2/c_1) P_2 x$, and

$$A(I - P_1)x = (I - P_1)P_2(I - P_1)x = (I - P_1)(P_2 x - P_2 P_1 x)$$

$$= (I - P_1)(P_2 x + (c_2/c_1)P_2 x)$$

$$= \frac{c_1 + c_2}{c_1 c_2} (I - P_1)(c_1 P_1 x + c_2 P_2 x) = 0,$$

that is, $(I - P_1)x \in \mathcal{N}(A)$. This proves the first inclusion in (2.3).
Suppose that \( x \in \mathcal{N}(A) \) and set \( c = 1 + c_1/c_2 \). Then \( P_1 x = 0 \) and \( P_1 P_2 x = P_2 x \). Thus

\[
(c_1 P_1 + c_2 P_2)(cI - P_2)x = c_1 c P_1 x - c_1 P_1 P_2 x - c_2 P_2 x + c_2 c P_2 x \\
= -(c_1 + c_2) P_2 x + (c_1 + c_2) P_2 x = 0,
\]

that is, \((cI - P_2)x \in \mathcal{N}\). This proves the second inclusion in (2.3).

The proof is completed by combining (2.2) with (2.3). \( \Box \)

The following theorem subsumes a recent result of Baksalary and Baksalary [1, Theorem 1].

**Theorem 2.3.** Let \( P_1, P_2 \in \mathcal{P} \), let \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \), \( c_1 + c_2 \neq 0 \), and let \( A \) be defined as in Lemma 2.1. Then the nullity of \( c_1 P_1 + c_2 P_2 \) is constant, equal to \( \text{nul}(c_1 P_1 + c_2 P_2) = \text{nul}(P_1 + P_2) = \text{nul}(A) = \dim[\mathcal{N}((I - P_1) P_2) \cap \mathcal{N}(P_1)]. \)

In particular, \( c_1 P_1 + c_2 P_2 \) is nonsingular if and only if \( P_1 + P_2 \) is.

**Proof.** The result follows from Theorem 2.2 and from the fact that \( A \in \mathbb{C}^{n \times n} \) is nonsingular if and only if \( \text{nul}(A) = 0 \). \( \Box \)

Furthermore, as a corollary of Theorem 2.2 we obtain the following theorem on the constancy of the rank of \( c_1 P_1 + c_2 P_2 \), which in the case of \( c_1 = c_2 = 1 \) yields a simple proof of Groß-Trenkler’s [2] rank formula for the sum of oblique projections (see also [3]).

**Theorem 2.4.** Let \( P_1, P_2 \in \mathcal{P}, c_1, c_2 \in \mathbb{C} \setminus \{0\}, c_1 + c_2 \neq 0, \) and let \( A \) be defined as in Lemma 2.1. Then the rank of \( c_1 P_1 + c_2 P_2 \) is constant, equal to

\[
\text{rk}(c_1 P_1 + c_2 P_2) = \text{rk}(P_1 + P_2) = \text{rk}(P_1) + \text{rk}(A) \\
= \text{rk}(P_1) + \text{rk}((I - P_1) P_2 (I - P_1)] \\
= n - \dim[\mathcal{N}((I - P_1) P_2) \cap \mathcal{N}(P_1)]. \tag{2.4}
\]

**Proof.** Since \( \text{rk}(c_1 P_1 + c_2 P_2) = n - \text{nul}(c_1 P_1 + c_2 P_2) \), the rank of \( c_1 P_1 + c_2 P_2 \) is constant, equal to the rank of \( P_1 + P_2 \) by using Theorem 2.3. According to Lemma 2.1 and Theorem 2.2,

\[
\text{rk}(A) = \text{nul}(P_1) - \text{nul}(A) = n - \text{rk}(P_1) - \text{nul}(A),
\]

which implies \( \text{rk}(P_1 + P_2) = n - \text{nul}(P_1 + P_2) = n - \text{nul}(A) = \text{rk}(P_1) + \text{rk}(A). \) Then (2.4) follows from Lemma 2.1. \( \Box \)

**Open problem.** In [4] we studied Fredholm properties of the difference of orthogonal projections in a Hilbert space. Suppose that \( P_1, P_2 \) are orthogonal (or oblique) projections in a Hilbert space. Is it true that \( P_1 + P_2 \) is Fredholm if and only if any linear combinations \( c_1 P_1 + c_2 P_2 \) is Fredholm, where \( c_1, c_2 \in \mathbb{C} \setminus \{0\} \) and \( c_1 + c_2 \neq 0? \)

**References**