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Exponential stability of simultaneously triangularizable switched systems with explicit calculation of a common Lyapunov function

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ABSTRACT

In this note, a common quadratic Lyapunov function is explicitly calculated for a linear hybrid system described by a family of simultaneously triangularizable matrices. The explicit construction of such a function allows not only obtaining an estimate of the convergence rate of the exponential stability of the switched system under arbitrary switching but also calculating an upper bound for the output during its transient response. Furthermore, the presented result is then extended to the case where the system is affected by parametric uncertainty, providing the corresponding results in terms of the nominal matrices and uncertainty bounds.

1. Introduction

The analysis and control of switched systems have attracted much interest during the last decade [1–4]. As pointed out in [1,5], there are three main problems concerned with switched systems, namely: (1) to find conditions for stability (especially asymptotic and exponential) for arbitrary switching signals (i.e. not being constrained to a dwell time [3]); (2) to identify the class of switching signals which provides stabilization; and (3) to construct stabilizing switching laws. The first problem has attracted considerable interest since a switched system stable under any switching law offers a great flexibility when designing a control scheme and it is the one considered in this work. In particular, we will consider the linear continuous-time switched system:

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t)\\ y(t) &= C^{\mathrm{T}}x(t)\end{aligned}\tag{1}$$

where $x \in \mathbb{R}^n$ and $A_{\sigma(t)} \in \mathcal{A}$ with

$$\mathcal{A} = \left\{A^{(i)}(p_i) \in \mathbb{R}^{n\times n} : p_i \in \mathcal{P}_i \subseteq \mathbb{R}^{n_i}, i \in \bar{N} = \{1, 2, \dots, N\}\right\} \equiv \left\{A_p; p \in \mathcal{P}\right\}$$

which defines a finite set of real matrices, each one being potentially parameterized by a vector p_i belonging to a compact subset $\mathcal{P}_i \subseteq \mathbb{R}^{n_i}$ for $i \in \bar{N}$ and $\mathcal{P} = \bar{N} \times \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_N$. The switching signal $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant function which selects the element of $\mathcal{A}$ parameterizing the linear system (1) at each time instant. The second notation introduced above for $\mathcal{A}$ will be used throughout the note for the sake of simplicity. Also, it will be assumed that $\mathcal{A}$ is a compact set with respect to the usual topology of $\mathbb{R}^{n\times n}$ (generated by any induced norm on $\mathbb{R}^{n\times n}$) and that

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all the matrices in $\mathcal{A}$ are strictly stable (i.e., all their eigenvalues with negative real parts) [6]. Earlier works (e.g., [7]) showed that commutativity relations within $\mathcal{A}$ played an important role in guaranteeing the stability of (1) under arbitrary switching. In this sense, a set of pairwise commuting matrices allows constructing a common Lyapunov function which guarantees the exponential stability of (1). Moreover, the works in [8,9] showed that the simultaneous triangularizability of $\mathcal{A}$ is sufficient for exponential stability under arbitrary switching. This fact can be easily understood by converting all the matrices to their common triangular form and solving the resulting system starting from the last state vector component up to the first one (see, e.g., [6]). Thus, a number of simultaneous triangularizability conditions have been investigated in the literature concerning switched systems including the well-known Lie-type algebraic ones [5,6,10]. Moreover, the reader can consult [11] for a comprehensive treatise on simultaneous triangularizability of matrices where a great number of criteria along with their proofs are collected. However, it is of interest to calculate a common Lyapunov function (eventually quadratic) for such a system (1). In [6], the problem of obtaining such a common Lyapunov function is treated. Nevertheless, the construction performed in [6] has an important drawback concerned with the complexity of the explicit calculation of the Lyapunov function which requires evaluating the principal minors of a (generally) $n \times n$ matrix. The technique developed in [6] is also used in [5] to prove the stability of mixed continuous–discrete-time systems and in [4] to discuss the convergence rate of exponentially stable switched systems. Additionally, a similar technique is used in [8] related to the joint spectral radius calculation for a family of simultaneously triangularizable matrices.

In this note, a mathematical construction alternative to that included in [5,6,8,4] for a common quadratic Lyapunov function for a set of simultaneously triangularizable switched system is presented. To this end, the set of matrices is represented in a special basis where simultaneous triangularizability with all the off-diagonal elements as small as desired is achieved. Apart from the intrinsic mathematical interest of the easy calculation of a common Lyapunov function, the approach allows us to estimate the worst case convergence rate for the exponential stability of the switched system. Moreover, the method presented goes one step further since it provides not only an estimate of the convergence rate but also a complete upper bounding to the norm of the state whence an estimate of the maximum amplitude of the transient behavior of the output can be extracted.

The note is organized as follows. Section 2 establishes the existence of a particular basis where the upper triangular matrices satisfy a certain property. Exponential stability results are introduced and proved in Section 3 on the basis of the facts presented in Section 2. Robustness issues are considered in Section 4 while a simulation example is commented on in Section 5. Finally, conclusions end the note.

Throughout the note, A^* denotes the conjugate transpose of A and $\lambda_i(*)$ the i-th eigenvalue of $(^*)$.

2. Simultaneous triangularization

This section establishes the concept of a set of simultaneously triangularizable matrices and introduces the special basis used for subsequent developments. Despite the definitions and results being quite standard, they are summarized in this section in order to make the note self-contained.

Definition 1. Let $\mathcal{A} = \left\{A_p;\, p \in \mathcal{P}\right\}$ be a set of matrices. Then, $\mathcal{A}$ is said to be simultaneously triangularizable if there exists an (in general) complex nonsingular transformation T such that $\tilde{A}_p = TA_pT^{-1}$ is upper triangular for all $p \in \mathcal{P}$,

$$\tilde{A}_p = \begin{pmatrix} \lambda_1(A_p) & \cdots & \tilde{a}_{ij}^{(p)} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n(A_p) \end{pmatrix} \tag{2}$$

where $\lambda_i(A_p)$; $i = 1, \ldots, n$, denote the eigenvalues of A_p. ■

Note that the common triangularization property does not ensure either that the absolute values of all the nonzero entries of the off-diagonal lie below a prescribed upper bound or that all the associated dynamic systems (1) via (2) possess a common Lyapunov function. The following linear algebra proposition, proved here for convenience, is the starting point of the subsequently presented results:

Proposition 1. *Let $\mathcal{A} = \left\{A_p;\, p \in \mathcal{P}\right\}$ be a compact set of simultaneously triangularizable matrices and $\varepsilon > 0$ arbitrary. Then, there exists a complex nonsingular transformation $T = T(\varepsilon)$ such that $\tilde{A}_p = TA_pT^{-1}$ is upper triangular with $\left|\tilde{a}_{ij}^{(p)}\right| < \varepsilon$ for $j > i$ and all $p \in \mathcal{P}$.*

Proof. If $\mathcal{A}$ is simultaneously diagonalizable, then the proposition is true with $\varepsilon = 0$. Thus, consider the nontrivial case when it is simultaneously triangularizable but not diagonalizable. Hence, there exists a nonsingular complex transformation matrix Q and a basis $V = \{v_1, \ldots, v_n\}$ such that $\bar{A}_p = QA_pQ^{-1}$ is upper triangular for all $p \in \mathcal{P}$. Now, define a new basis $W = \{w_1, \ldots, w_n\}$ by $w_i = v_i/r^i$ with $r \in \mathbb{R}$, $r \neq 0$, for the case $n \geq 2$. The matrix of each linear transformation $\tilde{A}_p$ with respect to this basis can be calculated for all $p \in \mathcal{P}$ as

$$\tilde{A}_p w_i = \tilde{A}_p \frac{v_i}{r^i} = \frac{1}{r^i}\tilde{A}_p v_i = \frac{1}{r^i}\left(\lambda_i v_i + \sum_{j=1}^{i-1} a_{ij}^{(p)} v_j\right) = \lambda_i w_i + \sum_{j=1}^{i-1} a_{ij}^{(p)} \frac{1}{r^{i-j}} w_j$$

for $i = 2, 3, \dots, n$. It can be seen that the above equation defines the elements of (2) with $\tilde{a}_{ij}^{(p)} = a_{ij}^{(p)} \frac{1}{r^{i-j}}$. It is clear that the choice $|r| > \max_{\substack{1 \le i,j \le n \\ p \in \mathcal{P}}} \frac{\left|a_{ij}^{(p)}\right|}{\varepsilon}$ would lead to off-diagonal entries being smaller than any prescribed ε for all $p \in \mathcal{P}$. ■

The property of ε being as small as desired will allow us to construct a common Lyapunov function and to obtain an estimate of the upper bounding to the state of the switched system in the following Section 3 in an easy way.

3. Common Lyapunov function calculation

In this section we concentrate on applying Proposition 1 to explicitly construct a common Lyapunov function in a mathematical setting alternative to that investigated in [5,6,4] which also allows us to obtain an explicit bound to the norm of the solutions to (1).

Theorem 1. *Let $\mathcal{A} = \left\{A_p; p \in \mathcal{P}\right\}$ be a compact set of simultaneously triangularizable strictly stable real $n \times n$ matrices and $\varepsilon \in \left(0, \frac{2\mu}{(n-1)}\right)$ with $\mu = \min_{\substack{p \in \mathcal{P} \\ 1 \le i \le n}} \left|Re\lambda_i\left(A_p\right)\right| > 0$. Then, $\mathcal{A}$ possesses a common Lyapunov function and there exists a real constant $k = k(\varepsilon) > 0$ such that any solution of (1) satisfies $\|x(t)\| \le k e^{-\left(\mu - \frac{\varepsilon}{2}(n-1)\right)t} = k' e^{-\left(\mu - \frac{\varepsilon}{2}(n-1)\right)t} \|x(0)\|$ under any switching rule.*

Proof. Firstly, it will be proved that there exists a function $V(t) = z^{\mathrm{T}}(t)Pz(t)$ such that it is a common Lyapunov function for the complete family:

$$\dot{z}(t) = \tilde{A}_p z(t); \quad p \in \mathcal{P} \tag{3}$$

with $\tilde{A}_p$ being defined by Proposition 1 through the nonsingular time-invariant state transformation $z(t) = T(\varepsilon)x(t)$. Without loss of generality, let $\tilde{A}_p$ be the matrix with the largest absolute eigenvalue for $p \in \mathcal{P}$. The time derivative of V is $\dot{V}(t) = z^{\mathrm{T}}(t)\left(\tilde{A}_p^* P + P\tilde{A}_p\right)z(t)$. As is well-known (see, e.g., [12]) the Lyapunov equation

$$\tilde{A}_p^* P + P\tilde{A}_p = -I \tag{4}$$

provides a tighter estimate of the convergence rate for the exponential stability of $\tilde{A}_p$ than any other choice for the negative definite matrix in the right-hand side of (4). Then, according to Proposition 1,

$$W_p = \tilde{A}_p - D_p = \begin{pmatrix} 0 & \tilde{a}_{12}^{(p)} & \cdots & \tilde{a}_{1n}^{(p)} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \tilde{a}_{n-1,n}^{(p)} \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

with $\left|\tilde{a}_{ij}^{(p)}\right| < \varepsilon$ where $D_p = \mathrm{diag}(\lambda_1(\tilde{A}_p), \dots, \lambda_n(\tilde{A}_p))$.

Therefore, the left-hand side of (4) becomes

$$\left(D_p^* + W_p^*\right)P + P\left(D_p + W_p\right) = \left(D_p^* P + PD_p\right) + \left(W_p^* P + PW_p\right) \tag{5}$$

Now, consider the solution to the auxiliary Lyapunov equation $D_p^* P + PD_p = -I$,

$$P = \int_0^\infty e^{D_p^* t} e^{D_p^t} dt = \int_0^\infty e^{(D_p^* + D_p)t} dt = \mathrm{diag}\left(\frac{1}{2\left|Re\lambda_1(D_p)\right|}, \dots, \frac{1}{2\left|Re\lambda_n(D_p)\right|}\right) > 0 \tag{6}$$

due to $\left[D_p, D_p^*\right] = 0$, since they are diagonal matrices and every matrix is strictly stable. The original Lyapunov equation (5) becomes

$$\tilde{A}_p^* P + P\tilde{A}_p \le -I + \lambda_{\max}(P)\left(W_p^* + W_p\right) = -I + \lambda_{\max}(P)\begin{pmatrix} 0 & \cdots & \tilde{a}_{ij}^{(p)} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{ij}^{(p)*} & \cdots & 0 \end{pmatrix} \le -I + \varepsilon\lambda_{\max}(P)\Delta \tag{7}$$

with $\Delta = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$. This matrix possesses two distinct eigenvalues, namely, -1 with multiplicity $(n-1)$ and $(n-1)$ with multiplicity unity. Hence, (7) is written as

$$\tilde{A}_p^* P + P\tilde{A}_p \leq -I + \lambda_{\max}(P)\,(n-1)\,\varepsilon I \tag{8}$$

so for $\varepsilon < \frac{1}{(n-1)\lambda_{\max}(P)}$ the right-hand side of (8) is still negative definite. Furthermore, if ε is arbitrarily small, which can be easily achieved according to Proposition 1, the left-hand side of (8) is arbitrarily near to the negative identity matrix. Furthermore, it will be shown that V is indeed a Lyapunov function for any $A_q \in \mathcal{A}$. For this, consider

$$\tilde{A}_q^* P + P\tilde{A}_q = \left(D_q^* P + PD_q\right) + \left(W_q^* P + PW_q\right) \tag{9}$$

since $\tilde{A}_q = D_q + W_q$. Thus,

$$D_q^* P + PD_q = PD_q^* + PD_q = \mathrm{diag}\left(\frac{Re\lambda_1(\tilde{A}_q)}{\left|Re\lambda_1(\tilde{A}_p)\right|}, \ldots, \frac{Re\lambda_n(\tilde{A}_q)}{\left|Re\lambda_n(\tilde{A}_p)\right|}\right) = -Q_q < 0 \tag{10}$$

for all $q \in \mathcal{P}$ since all the matrices in $\mathcal{A}$ are strictly stable. Then,

$$\tilde{A}_q^* P + P_p\tilde{A} = -Q_q + \left(W_q^* P + PW_q\right) \leq -Q_p + \lambda_{\max}(P)\left(W_q^* + W_q\right) \leq -Q_p + \mu\varepsilon\Delta. \tag{11}$$

As before, for the so defined ε, the right-hand side of (11) is still negative definite and V is a common Lyapunov function for $\mathcal{A}$. In conclusion, it has been proved that (3) possesses a common Lyapunov functions since the above result holds for any $q \in \mathcal{P}$. Finally,

$$-\dot{V} \geq (1 - \lambda_{\max}(P)\,(n-1)\,\varepsilon)\, z^{\mathrm{T}} z \geq \frac{1 - \lambda_{\max}(P)\,(n-1)\,\varepsilon}{\lambda_{\max}(P)} z^{\mathrm{T}} P z = \gamma V \tag{12}$$

with $\gamma = \frac{1-\lambda_{\max}(P)(n-1)\varepsilon}{\lambda_{\max}(P)} > 0$ for $0 < \varepsilon < \frac{1}{(n-1)\lambda_{\max}(P)}$. Thus,

$$-\dot{V} \geq \gamma V \Leftrightarrow \dot{V} + \gamma V \leq 0 \Rightarrow V(t) \leq V(0)\mathrm{e}^{-\gamma t}. \tag{13}$$

Note that $\lambda_{\max}(P) = \frac{1}{2\min_{1\leq i\leq n}\left|Re\lambda_i(\tilde{A}_p)\right|}$ and, therefore, $\frac{\gamma}{2} = \min_{1\leq i\leq n}\left|Re\lambda_i(\tilde{A}_p)\right| - \frac{\varepsilon}{2}(n-1)$ in (13). Hence, since $\tilde{A}_p$ is the matrix corresponding to the largest (negative) eigenvalue, $\mu = \min_{\substack{1 \leq i \leq n \\ p \in \mathcal{P}}} \left|Re\lambda_i(\tilde{A}_p)\right|$ which is well-defined due to the compactness of $\mathcal{A}$and $\frac{\gamma}{2} = \mu - \frac{\varepsilon}{2}(n-1)$. Finally, undoing the state transformation

$$x(t)^{\mathrm{T}} T^{\mathrm{T}} P T x(t) \leq V(0)\mathrm{e}^{-\gamma t} \Rightarrow \|x(t)\|_2 \leq \left(\frac{x(0) T^{\mathrm{T}} P T x(0)}{\lambda_{\min}(T^{\mathrm{T}} P T)}\right)^{1/2} \mathrm{e}^{-\frac{\gamma}{2}t}.$$

where $T = T(\varepsilon)$ and, finally, using the fact that all the norms in a finite dimensional space are equivalent, the desired result is obtained. ■

Remarks. 1. Note that the current explicit construction of the Lyapunov function for the original system (1) relies on the knowledge of the transformation matrix T. However, if the set of matrices is simultaneously triangularizable, then the convergence rate is calculated by routinely numerical dominant eigenvalue computation since the convergence rate depends on such an eigenvalue but not on T.

2. The method goes one step further since the constant $k(\varepsilon)$ upper bounding the norm of the state is explicitly given by

$$k(\varepsilon) = \left(\frac{x(0) T^{\mathrm{T}} P T x(0)}{\lambda_{\min}(T^{\mathrm{T}} P T)}\right)^{1/2} \tag{14}$$

which depends on the transformation matrix. Taking into account that $y(t) = C^{\mathrm{T}} x(t)$ implies $|y(t)| \leq \|C\|_2\,\|x(t)\|_2$, then the knowledge of the transformation matrix along with the proposed approach permits us to obtain an estimate of the maximum amplitude of the transient response of the output (which does not depend on the specific choice for the basis of the state variables). In fact, the maximum amplitude of the output is given by $|y(0)| \leq \|C\|_2\,\|x(0)\|_2 \leq \|C\|_2\,k(\varepsilon)$ due to the exponential convergence of the norm of the state to zero. Therefore, a tight estimate of the maximum amplitude can be obtained by minimizing $k(\varepsilon)$. Hence, this explicit construction of the Lyapunov function allows obtaining this extra information for system (1) which is not available from the mathematical approaches in [5,6,4].

3. Note that the upper bounding of the state norm given by Theorem 1 also holds if $\varepsilon > \frac{2\mu}{n-1}$. However, the upper bounding is divergent in that case since it corresponds to an exponential with positive exponent and it is not adequate for calculating a convergence rate for the system's trajectories.
4. The compactness assumption on $\mathcal{A}$ can be relaxed to a boundedness one. In this case, the maximum and minimum operators are substituted by the supremum and infimum ones, respectively, in the above equations. ■

Note that the approach presented does not require evaluating all the minors of the matrices $\tilde{A}_p$ for all $p \in \mathcal{P}$, in contrast to the method presented in [6,4], to obtain a solution to the Lyapunov equation (4). Since calculating the determinant of an $m \times m$ matrix involves the sum of $m!$ terms, the saving in computation with the proposed approach is apparent. Additionally, the so constructed Lyapunov function provides us easily with an estimate of the upper bound for the state norm, which is difficult to obtain following the guidelines in [6,4]. Furthermore, the skeleton proof of Theorem 1 suggests the following result on exponential stability of switched systems, whose conditions are easy to verify a priori.

Theorem 2. *Let $\mathcal{A} = \{A_p;\, p \in \mathcal{P}\}$ be a set of strictly stable real $n \times n$ matrices satisfying $A_p = A_q + \varepsilon \Lambda_{pq}$ for a sufficiently small $\varepsilon > 0$ with $\left\|\Lambda_{pq}\right\| < \infty$ for all $p, q \in \mathcal{P}$. Then, (1) is exponentially stable under any switching law.*

Proof. It is now proven that all the members of $\mathcal{A}$ possess, at least, a common Lyapunov function. For this, fix $p \in \mathcal{P}$ and solve the Lyapunov equation $A_p^{\mathrm{T}} P_p + P_p A_p = -I$. Then, $V_p = x^{\mathrm{T}} P_p x$ is a Lyapunov equation for any other $q \in \mathcal{P}$ since

$$A_q^{\mathrm{T}} P_p + P_p A_q = A_p^{\mathrm{T}} P_p + P_p A_p + \varepsilon(\Lambda_{pq}^{\mathrm{T}} P_p + P_p \Lambda_{pq}) = -I + \varepsilon(\Lambda_{pq}^{\mathrm{T}} P_p + P_p \Lambda_{pq}). \tag{15}$$

Thus, for a sufficiently small ε, the right-hand side of (16) is negative definite and V_p is a common Lyapunov function for all matrices in $\mathcal{A}$. Thus, the exponential stability of (1) under any switching rule is proved. ■

4. Robustness issues

Consider now the case when the system is affected by parametric uncertainty. In this case, the exponential stability of the system can still be guaranteed by merging the previously presented Theorems 1 and 2. For this, the following uncertainty sets are defined:

$$\mathcal{B}_p = \left\{ M \in \mathbb{R}^{n\times n} : \left| \max_{1\le i\le n} Re\lambda_i(A_p) - \max_{1\le i\le n} Re\lambda_i(M) \right| \le \varepsilon_p,\ \varepsilon_p \ge 0 \right\} \tag{16}$$

for every matrix $A_p \in \mathcal{A}$ and some fixed $\varepsilon_p \ge 0$. These sets are composed of all the matrices whose maximum eigenvalue is nearer than the prescribed bound ε_p to the maximum eigenvalue of A_p and corresponds to the presence of an uncertainty in each matrix $A_p \in \mathcal{A}$. Hence, the set $\mathcal{A}$ constitutes the set of nominal matrices and $\mathcal{B}_p$ denotes the uncertainty in the matrix $A_p \in \mathcal{A}$. Note, in particular, that $A_p \in \mathcal{B}_p$ and all the matrices in $\mathcal{B}_p$ are strictly stable. Then, the extended uncertainty set can be defined by $\mathcal{B} = \cup_{p\in\mathcal{P}} \mathcal{B}_p$ which is a covering of the original nominal set $\mathcal{A}$?. Hence, the system parameterizations (1) in the presence of uncertainty turn out to be $A_{\sigma(t)} \in \mathcal{B}$. Thus, with the above notation:

Theorem 3. *Let $\mathcal{A} = \{A_p;\, p \in \mathcal{P}\}$ be a compact set of simultaneously triangularizable strictly stable real $n \times n$ matrices, $\varepsilon \in \left(0, \frac{2\mu}{(n-1)}\right)$ with $\mu = \min_{\substack{p\in\mathcal{P}\\ 1\le i\le n}} \left|Re\lambda_i\left(A_p\right)\right| > 0$ and $\mathcal{B}$ defined above. Define $\varepsilon^* = \max_{p\in\mathcal{P}} \varepsilon_p$. Then, there exists a real constant $k(\varepsilon) > 0$ such that any solution to (1) with $A_{\sigma(t)} \in \mathcal{B}$ satisfies $\|x(t)\| \le k e^{-\left(\mu - \frac{\varepsilon}{2}(n-1) - \varepsilon^*\right)t} \|x(0)\|$ under any switching rule provided that $\varepsilon^* < \mu - \frac{\varepsilon}{2}(n-1)$.* ■

The proof can be sketched by following the same steps as are given in the proof of Theorem 1 and the result presented in Theorem 2 and, therefore, it is omitted here.

5. Simulation example

Consider the switched linear system (1) with simultaneously triangularizable parameterizations $A_p = Q\tilde{A}_p Q^{-1}$, $p = 1, 2, 3$ with $C^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and

$$\tilde{A}_1 = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{pmatrix}, \qquad \tilde{A}_2 = \begin{pmatrix} -1.5 & 6 & 4 \\ 0 & -1.5 & 1 \\ 0 & 0 & -5 \end{pmatrix},$$

$$\tilde{A}_3 = \begin{pmatrix} -2.5 & 1 & -2 \\ 0 & -2.4 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & -1 & 3 \\ -2 & 1 & 1 \\ -1 & -1 & 10 \end{pmatrix}.$$

In this case, the set of potential parameterizations of the system is composed of three different constant matrices indexed by the discrete index $p = 1, 2, 3$. The switching sequence is arbitrary. The evolution of the state of the switched system (1)

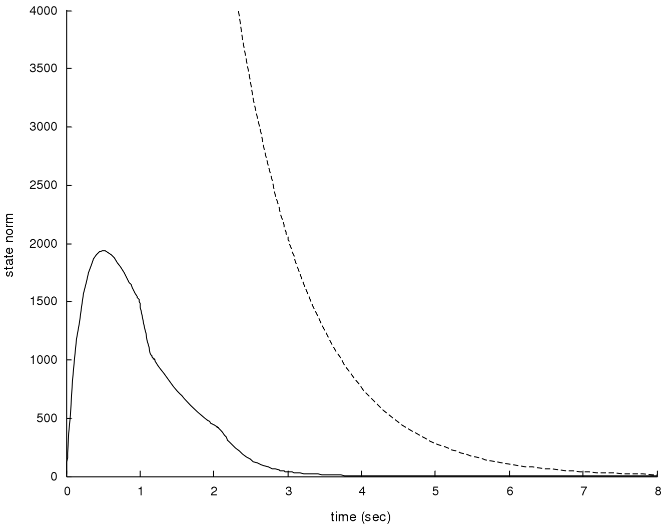

Fig. 1. Upper bounding of the state norm.

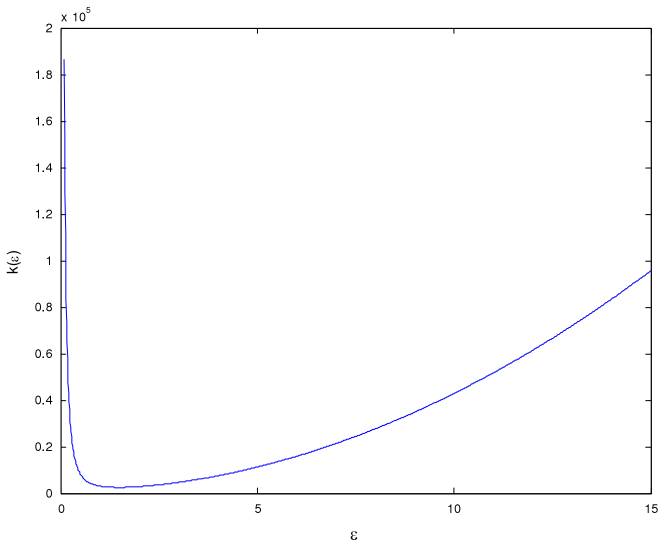

Fig. 2. Graphical representation of $k(\varepsilon)$.

defined by the above matrices through time is upper bounded by the function $\bar{x}(t) = 2 \times 10^4 e^{-0.9t}$ with $\varepsilon = 0.1$ as shown in Fig. 1 where, according to Theorem 1, the estimate of the worst case convergence rate is slightly smaller than the maximum eigenvalue of the set A_p, $p = 1, 2, 3$. Note that the selection of ε satisfies the requirements of Theorem 1 since $\varepsilon = 0.1 < \frac{2\mu}{(n-1)} = 1$ from the eigenvalues of A_p, $p = 1, 2, 3$, above. Furthermore, from Fig. 2 it can be appreciated how the selection of ε near zero implies a large estimate of the maximum amplitude of the transient response which is of about 20 000 in the example. Along with the convergence rate, the proposed approach permits obtaining an estimate of the maximum transient amplitude. For this, the value of $k(\varepsilon)$ is to be minimized with respect to ε. This can be done analytically, since the transformation matrix is known, or it can be easily performed by numerical optimization. Fig. 2 shows the graphical representation of $k(\varepsilon)$. It can be appreciated that the function possesses a minimum at about $\varepsilon \approx 1.74$ corresponding to a value of $k(\varepsilon) \approx 2520$. Hence, the maximum amplitude for the state for any switching law (which would lead to a corresponding maximum amplitude in the output in this example) corresponds to 2520. Note that the maximum amplitude for the output in the example is smaller that the presented bound. In conclusion, the proposed novel mathematical approach provides simultaneously an explicit, easy-to-calculate, Lyapunov function, the estimate of the worst case convergence rate and an estimate of the maximum amplitude of the output. Note that the minimization is performed on $\varepsilon \in [0, \infty)$. This can be done because the selection of ε does not threaten stability, which implies that the maximum is still at $t = 0$, while Remark 3 holds.

6. Conclusions

In this note, an explicit calculation of a common quadratic Lyapunov function for a continuous-time switched system is developed. The calculation is based on a new approach and consequently allows calculating not only the convergence rate, but also an estimate of the maximum amplitude of the transient response in the worst case (for arbitrary switching signal). Finally, the proposed method is extended to the presence of parametric uncertainties.

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