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Geometric construction of some families of two-class and three-class association schemes and codes from nondegenerate and degenerate Hermitian varieties

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Abstract

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Taking a nondegenerate Hermitian variety as a projective set in a projective plane $PG(2, s^2)$, Mesner (1967) derived a two-class association scheme on the points of the affine space of dimension 3, for which the projective plane is the plane at infinity.

We generalize his construction in two ways. We show how his construction works both for nondegenerate and degenerate Hermitian varieties in any dimension.

We consider a projective space of dimension N, partitioned into an affine space of dimension N and a hyperplane $\mathscr{H}$ of dimension $N-1$ at infinity.

The points of the hyperplane are next partitioned into 2 or 3 subsets. A pair of points a, b of the affine space is defined to belong to class i if the line $\overline{ab}$ meets the subset i of $\mathscr{H}$.

In the first case, the two subsets of the hyperplane are a nondegenerate Hermitian variety and its complement. In this case, we show that the classification of pairs of affine points defines a family of two-class association schemes. This family of association schemes has the same set of parameters as those derived as restrictions of the Hamming association schemes to two-weight codes defined as linear spans of coordinate vectors of points on a nondegenerate Hermitian variety in a projective space of dimension $N-1$. The relations of these codes to orthogonal arrays and difference sets are described in [5, 6].

In the second case, the three subsets are the singular point of the variety, the regular points of the variety and the complement of the variety defined by a Hermitian form of rank $N-1$. This leads to a family of three-class association schemes on the points of the affine space. A geometric construction is first given for the case $N=3$.

Using a general algebraic method pointed out by the referee, we have also derived the three-class association scheme for general N.

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1. Introduction

The geometry of Hermitian varieties in finite-dimensional projective spaces has been studied by Jordan [11], Dickson [9], Dieudonné [10], and, recently, among others, by Bose [1,2], Segre [13,14], Bose and Chakravarti [3] and Chakravarti [6]. In this paper, however, we have used the results given in [3,6].

If h is any element of a Galois field $GF(s^2)$, where s is a prime or a power of a prime, then $\bar{h}=h^s$ is defined to be conjugate to h. Since $h^{s^2}=h$, h is conjugate to $\bar{h}$. A square matrix $H=(h_{ij})$, $i,j=0,1,\ldots,N$, with elements from $GF(s^2)$ is called Hermitian if $h_{ij}=\bar{h}_{ji}$ for all i,j. The set of all points in $PG(N,s^2)$ whose row vectors $\boldsymbol{x}^{\mathrm{T}}=(x_0,x_1,\ldots,x_N)$ satisfy the equation $\boldsymbol{x}^{\mathrm{T}}H\boldsymbol{x}^{(s)}=0$ are said to form a Hermitian variety V_{N-1}, if H is Hermitian and $\boldsymbol{x}^{(s)}$ is the column vector whose transpose is $(x_0^s,x_1^s,\ldots,x_N^s)$. The variety V_{N-1} is said to be nondegenerate if H has rank $N+1$. The Hermitian form $\boldsymbol{x}^{\mathrm{T}}H\boldsymbol{x}^{(s)}$ where H is of order $N+1$ and rank r can be reduced to the canonical form $y_0\bar{y}_0+\cdots+y_r\bar{y}_r$ by a suitable nonsingular linear transformation $\boldsymbol{x}=A\boldsymbol{y}$. The equation of a nondegenerate Hermitian variety V_{N-1} in $PG(N,s^2)$ can then be taken in the canonical form $x_0^{s+1}+x_1^{s+1}+\cdots+x_N^{s+1}=0$.

Consider a Hermitian variety V_{N-1} in $PG(N,s^2)$ with equation $\boldsymbol{x}^{\mathrm{T}}H\boldsymbol{x}^{(s)}=0$. A point C in $PG(N,s^2)$ with row vector $\boldsymbol{c}^{\mathrm{T}}=(c_0,c_1,\ldots,c_N)$ is called a *singular* point of V_{N-1} if $\boldsymbol{c}^{\mathrm{T}}H=\boldsymbol{0}^{\mathrm{T}}$ or, equivalently, $H\boldsymbol{c}^{(s)}=\boldsymbol{0}$. A point of V_{N-1} which is not singular is called a *regular* point of V_{N-1}. Thus, a nonsingular point is either a regular point of V_{N-1} or a point not on V_{N-1}. It is clear that a nondegenerate V_{N-1} cannot possess a singular point. On the other hand, if V_{N-1} is degenerate and rank $H=r<N+1$, the singular points of V_{N-1} constitute a $(N-r)$-flat called the *singular space* of V_{N-1}.

Let C be a point with row vector $\boldsymbol{c}^{\mathrm{T}}$. Then the *polar space* of C with respect to the Hermitian variety V_{N-1} with equation $\boldsymbol{x}^{\mathrm{T}}H\boldsymbol{x}^{(s)}=0$ is defined to be the set of points of $PG(N,s^2)$ which satisfy $\boldsymbol{x}^{\mathrm{T}}H\boldsymbol{c}^{(s)}=0$.

When C is a singular point of V_{N-1}, the polar space of C is the whole space $PG(N,s^2)$. When, however, C is either a regular point of V_{N-1} or an external point, $\boldsymbol{x}^{\mathrm{T}}H\boldsymbol{c}^{(s)}=0$ is the equation of a hyperplane which is called the *polar hyperplane* of C with respect to V_{N-1}. Let C and D be two points of $PG(N,s^2)$. If the polar hyperplane of C passes through D, then the polar hyperplane of D passes through C. Two such points C and D are said to be *conjugates* to each other with respect to V_{N-1}. Thus, the points lying in the polar hyperplane of C are all the points which are conjugates to C. If C is a *regular* point of V_{N-1}, the polar hyperplane of C passes through C; C is, thus, self-conjugate. In this case, the polar hyperplane is called the *tangent hyperplane* to V_{N-1} at C.

When V_{N-1} is nondegenerate, there is no singular point. To every point, there corresponds a unique polar hyperplane, and, at every point of V_{N-1}, there is a unique tangent hyperplane. If C is an external point, its polar hyperplane will be called a *secant hyperplane.*

The number of points in a nondegenerate Hermitian variety V_{N-1} in $PG(N,s^2)$ is $\Phi(N,s^2)=(s^{N+1}-(-1)^{N+1})(s^N-(-1)^N)/(s^2-1)$.

A polar hyperplane $\mathscr{L}_{N-1}$ of an external point $\mathscr{D}$ (also called a secant hyperplane) in $\mathrm{PG}(N,s^2)$ intersects a nondegenerate Hermitian variety V_{N-1} in a nondegenerate Hermitian variety V_{N-2} of rank N. It has $(s^N-(-1)^N)(s^{N-1}-(-1)^{N-1})/(s^2-1)$ points.

A tangent hyperplane $\mathscr{T}_{N-1}$ to a nondegenerate V_{N-1} at a point C intersects V_{N-1} in a degenerate V_{N-2} of rank $N-1$. The singular space of V_{N-2} consists of the single point C. Every point of V_{N-2} lies on a line joining C to the points of a nondegenerate V_{N-3} lying on an $(N-2)$-dimensional flat disjoint with C.

The number of points in a degenerate Hermitian variety V_{N-1} of rank $r<N+1$ in $\mathrm{PG}(N,s^2)$ is $(s^2-1)f(N-r,s^2)\,\Phi(r-1,s^2)+f(N-r,s^2)+\Phi(r-1,s^2)$, where $f(k,s^2)=(s^{2(k+1)}-1)/(s^2-1)$. Thus, the number of points in a degenerate V_{N-2} of rank $N-1$ is

$$(s^2-1)f(0,s^2)\Phi(N-2,s^2)+f(0,s^2)+\Phi(N-2,s^2)$$
$$=1+(s^{N-1}-(-1)^{N-1})\,(s^{N-2}-(-1)^{N-2})s^2/(s^2-1).$$

For the definition of an association scheme and related results, see [4].

2. Two-class association scheme from a nondegenerate Hermitian variety in PG($N-1,s^2$)

Let V_{N-2} be a nondegenerate Hermitian variety defined by the equation

$$x_0^{s+1}+x_1^{s+1}+\cdots+x_{N-1}^{s+1}=0,$$

in $\mathscr{H}=\mathrm{PG}(N-1,s^2)$. Consider $\mathscr{H}$ as the hyperplane at infinity in a $\mathrm{PG}(N,s^2)$. Then the affine space complementary to $\mathscr{H}$ in $\mathrm{PG}(N,s^2)$ is $\mathrm{EG}(N,s^2)$.

Suppose d_0 is a point on V_{N-2}. The tangent hyperplane $\mathscr{T}(d_0)$ at d_0 intersects V_{N-2} in a degenerate V_{N-3}, with d_0 as the point of singularity. V_{N-3}^0 consists of d_0 and all the points on the lines joining d_0 to the points of a nondegenerate Hermitian variety V_{N-4}. Thus, the number of generator lines through d_0 is the same as the number of points on V_{N-4}, which is

$$(s^{N-2}-(-1)^{N-2})(s^{N-3}-(-1)^{N-3})/(s^2-1).$$

The number of tangent lines through d_0 is equal to the number of lines in $\mathscr{T}(d_0)$ through d_0 minus the number of generator lines through d_0, i.e.

$$(s^{2N-4}-1)/(s^2-1)-(s^{N-2}-(-1)^{N-2})(s^{N-3}-(-1)^{N-3})/(s^2-1)$$
$$=(s^{2N-5}+(-s)^{N-3})/(s+1).$$

The number of secants (lines which are neither tangents nor generators) through d_0 is equal to the number of lines through d_0 in $\mathrm{PG}(N-1,s^2)$ minus the number of lines through d_0 on $\mathscr{T}(d_0)$, i.e.

$$(s^{2N-2}-1)/(s^2-1)-(s^{2N-4}-1)/(s^2-1)=s^{2N-4}$$

Each secant line meets V_{N-2} at $s+1$ points.

Suppose d is an external point of $\mathscr{H}$, that is, a point of $\mathscr{H}$ which is not on V_{N-2}. The polar of d intersects V_{N-2} in a nondegenerate Hermitian variety V_{N-3}. Each one of the points on V_{N-3} is conjugate to d. Hence, the tangent hyperplanes at each one of these points will pass through d. Thus, the number of tangent lines through d is the same as the number of points conjugate to d, which is

$$(s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-1})/(s^2-1).$$

Hence, the number of secant lines through d is

$$\frac{1}{(s^2-1)(s+1)}\{(s^N-(-1)^N)(s^{N-1}-(-1)^{N-1})-(s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})\}$$

$$=s^{N-2}(s^{N-1}-(-1)^{N-1})/(s+1).$$

Define two points a and b of $\mathrm{EG}(N,s^2)$ to be *first associates* if the line $\overline{ab}$ meets $\mathscr{H}$ at a point of V_{N-2}, and *second associates* if the line $\overline{ab}$ meets $\mathscr{H}$ at an external point of $\mathscr{H}$. Then

$$p_{11}^1(a,b)=(s^2-2)+s^2(s^2-1)\frac{(s^{N-2}-(-1)^{N-2})(s^{N-3}-(-1)^{N-3})}{s^2-1}$$

$$+s(s-1)s^{2N-4}$$

$$=s^{2N-2}-(-s)^{N-1}(s-1)-2,$$

which is independent of the pair of points a and b.

Also,

$$p_{11}^2(a,b)=s(s+1)\frac{s^{N-2}(s^{N-1}-(-1)^{N-1})}{s+1}$$

$$=s^{2N-2}-(-s)^{N-1},$$

which is again independent of the pair of points a and b. Thus, this is a two-class association scheme with

$$v=s^{2N},\qquad n_1=(s^N-(-1)^N)(s^{N-1}-(-1)^{N-1}),$$

$$p_{11}^1=s^{2N-2}-(-s)^{N-1}(s-1)-2\quad\text{and}\quad p_{11}^2=s^{2N-2}-(-s)^{N-1}.$$

We now recall how a two-weight code C in s^2 symbols is generated from the nondegenerate Hermitian variety V_{N-2} in $\mathrm{PG}(N-1,s^2)$. This variety has $(s^N-(-1)^N)(s^{N-1}-(-1)^{N-1})/(s^2-1)=n$ (say) points. Consider a matrix $G=(g_{ij})$, $i=0,1,\ldots,N-1, j=1,\ldots,n$, whose columns are the coordinate vectors of the n points on V_{N-2}. The code words of C are $c'G$, where $c'=(c_0,c_1,\ldots,c_{N-1})$, $c_i\in\mathrm{GF}(s^2)$. Then C is a projective linear code $(n,k=N)$. Since a tangent hyperplane meets V_{N-2} at $1+(s^{N-2}-(-1)^{N-2})(s^{N-3}-(-1)^{N-3})s^2/(s^2-1)$ points and a secant hyperplane meets V_{N-2} at $(s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2})/(s^2-1)$ points, the code C has only two distinct nonzero weights $w_1=s^{2N-3}$ and $w_2=s^{2N-3}+(-s)^{N-2}$ with respective frequencies

$$f_{w_1}=(s^N-(-1)^N)(s^{N-1}-(-1)^{N-1}) \quad \text{and} \quad f_{w_2}=(s-1)(s^{2N-1}+(-s)^{N-1}).$$

(For details, see [5, 7].)

Now, from a result due to Delsarte [8], it follows that the projective linear code C in s^2 symbols determines a projective linear code C' in s symbols, with parameters $n'=(s^N-(-1)^N)(s^{N-1}-(-1)^{N-1})/(s-1)$, $k=2N$, $w'_1=s^{2N-2}$, $w'_2=s^{2N-2}-(-s)^{N-1}$, $f_{w'_1}=(s^N-(-1)^N)(s^{N-1}-(-1)^{N-1})$ and $f_{w'_2}=(s-1)(s^{2N-1}+(-s)^{N-1})$.

The graph on s^{2N} vertices corresponding to the code words of C' over GF(s) is strongly regular, that is, it is the graph of a two-class association scheme with parameters [7]

$$v=s^{2N}, \qquad n_1=(s^N-(-1)^N)(s^{N-1}-(-1)^{N-1}),$$

$$p^1_{11}=s^{2N-2}-(-s)^{N-1}(s-1)-2 \quad \text{and} \quad p^2_{11}=s^{2N-2}-(-s)^{N-1}.$$

This is the restriction of the Hamming association scheme $\mathscr{H}_{n'}(s)$ to the code C', which has the same parameters as the one derived earlier by a Mesner-type construction.

3. Three-class association scheme from a degenerate Hermitian variety in PG$(3,s^2)$

Let V_2 be a nondegenerate Hermitian variety in PG$(3,s^2)$. Let V^0_1 denote the degenerate Hermitian variety which is derived as an intersection of V_2 with one of its tangent planes, say, $\mathscr{T}=\text{PG}(2,s^2)$ at the point C on V_2. Then C is the point of singularity of V^0_1. V^0_1 consists of C and the points on lines joining C to the points of a nondegenerate V_0, which has $s+1$ points. Thus, the number of points on V^0_1 is $1+s^2(s+1)=1+s^2+s^3$.

The points on PG$(3,s^2)$ which are not on $\mathscr{T}$ form a 3-dimensional affine space EG$(3,s^2)$, which has s^6 points. Every line of EG$(3,s^2)$ meets $\mathscr{T}$ (the plane at infinity) exactly at one point.

Two points a and b of EG$(3,s^2)$ are defined to be *first associates* if the line $\overline{ab}$ joining a and b meets $\mathscr{T}$ at a *regular point* of V^0_1, *second associates* if the line $\overline{ab}$ meets $\mathscr{T}$ at a point external to V^0_1, and *third associates* if the line $\overline{ab}$ passes through the point of singularity C. To show that this defines a three-class association scheme, we do enumerations and use geometric arguments similar to those of Mesner [12].

Since there are s^3+s^2+1 points on V^0_1, of which only one point C is singular, the remaining s^3+s^2 are regular points. Thus, the number of first associates of a given point is

$$n_1=(s^3+s^2)(s^2-1).$$

Now the number of points on $\mathscr{T}$ which are external to V_1^0 is $(s^4+s^2+1)-(s^3+s^2+1)=s^4-s^3$. Thus, the number of second associates of a given point is

$$n_2=(s^4-s^3)(s^2-1).$$

The number of third associates of a given point a is equal to the number of affine points (other than a) on the line joining a to C; hence, $n_3=s^2-1$.

The following results, which we need for proving the constancy of the parameter $p^i_{jk}(a,b)$, can be found in [3, 6].

(i) There are $s+1$ lines through C, the point of singularity, which are generators, that is, each line intersects V_1^0 at s^2 points other than C. The remaining s^2-s lines on $\mathscr{T}$, passing through C, are tangent lines at C, that is, each line meets V_1^0 only at C.

(ii) Suppose D is a regular point on V_1^0, that is, $D\neq C$. Then there is exactly one generator through D, DC, which meets V_1^0 at s^2+1 points and there are s^2 lines through D and $\mathscr{T}$, which are secants, that is, each line meets V_1^0 at $s+1$ points.

(iii) Suppose D is a point on the plane $\mathscr{T}$, but external to V_1^0. Then DC is a tangent to V_1^0 at C, that is, it meets V_1^0 only at C. The remaining s^2 lines through D on $\mathscr{T}$ are all secants, that is, each line meets V_1^0 at $s+1$ points.

Let $f_D(u)$ denote the number of lines on $\mathscr{T}$ passing through D, each one of which meets V_1^0 at exactly u points, $u=0,1,\ldots,s^2+1$.

Suppose a and b are first associates, that is, the line $\overline{ab}$ meets V_1^0 at a regular point D $(D\neq C)$. Then

$$p^1_{11}(a,b)=s^2-2+(s^2-1)(s^2-2)f_D(s^2+1)+s(s-1)f_D(s+1),$$

where s^2-1 is the number of affine points on the line $\overline{ab}$, $(s^2-1)(s^2-2)$ is the number of ordered pairs of points (e,f) that one can form from the s^2-1 points (other than C and D) on the generator C,D. The intersection of the lines $\overline{ae}$ and $\overline{bf}$ is an affine point which is a first associate of both a and b. Similarly, each secant contributes $s(s-1)$ affine points which are first associates of both a and b. But there is only one generator through D and s^2 secants through D. Thus,

$$f_D(s^2+1)=1 \quad \text{and} \quad f_D(s+1)=s^2.$$

Hence,

$$p^1_{11}(a,b)=s^2-2+(s^2-1)(s^2-2)+s(s-1)s^2$$
$$=s^2(2s^2-s-2),$$

which is independent of a and b.

Suppose now that the line $\overline{ab}$ meets $\mathscr{T}$ at a point D not on V_1^0. Then a and b are second associates. Let us calculate $p^2_{11}(a,b)$. Through D, there are (s^2+1) lines, of which one, namely $\overline{CD}$, is a tangent to V_1^0 and the other s^2 lines are secants to V_1^0, that is, each line meets V_1^0 at $s+1$ points.

Hence,

$$p_{11}^{2}(a,b)=(s+1)s f_D(s+1)=(s+1)s s^2=s^3(s+1),$$

which is independent of a and b.

Now suppose that the line $\overline{ab}$ meets V_1^0 at C. Thus, a and b are third associates. Through C, there are $s+1$ generator lines each one of which meets V_1^0 at s^2+1 points including C. The remaining s^2-s lines through C on $\mathcal{T}$ are tangents. That is, each line meets V_1^0 only at C. Thus,

$$p_{11}^{3}(a,b)=s^2(s^2-1)f_C(s^2+1)=s^2(s^2-1)(s+1).$$

In this manner, using the geometric results quoted before, we have calculated all the $p_{jk}^{i}(a,b)$, $i,j,k=1,2,3$ parameters and these are independent of the pair of points a and b. Hence, this is a three-class association scheme. The parameters are

$$v=s^6, \qquad n_1=(s^3+s^2)(s^2-1), \qquad n_2=(s^4-s^3)(s^2-1), \qquad n_3=s^2-1,$$

$$P_1=(P_{ij}^{1})=\begin{bmatrix} 2s^4-s^3-2s^2 & s^5-s^4 & s^2-1 \\ s^5-s^4 & s^3(s^2-s-1)(s-1) & 0 \\ s^2-1 & 0 & 0 \end{bmatrix},$$

$$P_2=(P_{ij}^{2})=\begin{bmatrix} s^3(s+1) & s^2(s+1)(s^2-s-1) & 0 \\ s^2(s+1)(s^2-s-1) & s^2(s^2-2)+(s^2-s-1)(s^2-s-2)s^2 & s^2-1 \\ 0 & s^2-1 & 0 \end{bmatrix},$$

$$P_3=(P_{ij}^{3})=\begin{bmatrix} s^2(s^2-1)(s+1) & 0 & 0 \\ 0 & s^3(s^2-1)(s-1) & 0 \\ 0 & 0 & s^2-2 \end{bmatrix}.$$

The linear span of the coordinate vectors of the points of the degenerate Hermitian variety V_1^0 provides a three-weight projective code with s^6 code words and $n=1+s^2+s^3$ (see [7]). The questions whether the restriction of the Hamming association scheme $H_n(s)$ to this code provides a three-class association scheme and, if yes, whether the three-class association already derived is related to this code, are still unsettled.

4. Three-class association scheme from a degenerate Hermitian variety in PG($N-1$, s^2)

The association scheme in the previous section, has been derived using geometry. Here we use a general algebraic method pointed out by the referee, to construct a three-class association scheme on the points of EG(N,s^2).

Let A be the set of points $\{a=(x_1,\ldots,x_{N-1})\}$ of EG$(N-1,s^2)$ and B be the elements of GF(s^2). Then $A\times B=\{(a,b)=(x_1,\ldots,x_{N-1};\, x_N)\}$ is the set of points of EG(N,s^2).

The two-class association scheme on A defined in Section 2, is then extended to a three-class scheme on $A \times B$ as follows:

(i) $\{(a,b),(a,b')\}$ are defined to be third associates if $b \neq b'$;

(ii) $\{(a,b),(a',b')\}$ are ith associates if $a \neq a'$ and (a,a') are ith associates, $i=1,2$, in the scheme on A. It is easy to check that this is a three-class association scheme with parameters

$$v=s^{2N}, \qquad n_1=s^2(s^{N-1}-(-1)^{N-1})(s^{N-2}-(-1)^{N-2}),$$

$$n_2=(s-1)s^N(s^{N-1}-(-1)^{N-1}), \qquad n_3=s^2-1,$$

$$p_{11}^1=s^2(s^{2N-4}-(-s)^{N-2}(s-1)-2), \qquad p_{12}^1=(s-1)s^{2N-2},$$

$$p_{13}^1=s^2-1, \qquad p_{22}^1=(s-1)(s^{2N-1}-s^{2N-2}+(-s)^N),$$

$$p_{23}^1=0=p_{33}^1,$$

$$p_{11}^2=s^{2N-2}-(-s)^N, \qquad p_{12}^2=s^2(1-(-s)^{N-2})((-s)^{N-1}+(-s)^{N-2}-1),$$

$$p_{22}^2=s^2[((-s)^{N-1}-2)+((-s)^{N-1}+(-s)^{N-2}-1)((-s)^{N-1}+(-s)^{N-2}-2)],$$

$$p_{13}^2=0, \qquad p_{23}^2=s^2-1, \qquad p_{33}^2=0,$$

$$p_{11}^3=n_1, \qquad p_{22}^3=n_2, \qquad p_{33}^3=s^2-1,$$

$$p_{12}^3=p_{13}^3=p_{23}^3=0.$$

It is not difficult to verify that this scheme is the same as the one that will be obtained by Mesner-type construction applied to a Hermitian variety defined by a Hermitian form of rank $N-1$ in a $PG(N-1,s^2)$. Three-weight codes C and C' in s^2 and s symbols, respectively, derived from this degenerate Hermitian variety are given in [7].

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