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Threshold and generic type I behaviors for a supercritical nonlinear heat equation

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Abstract

We study blow-up of radially symmetric solutions of the nonlinear heat equation $u_t = \Delta u + |u|^{p-1}u$ either on $\mathbf{R}^N$ or on a finite ball under the Dirichlet boundary conditions. We assume that $N \geqslant 3$ and $p > p_S := \frac{N+2}{N-2}$. Our first goal is to analyze a threshold behavior for solutions with initial data $u_0 = \lambda v$, where $v \in C \cap H^1$ and $v \geqslant 0$, $v \not\equiv 0$. It is known that there exists $\lambda^* > 0$ such that the solution converges to 0 as $t \to \infty$ if $0 < \lambda < \lambda^*$, while it blows up in finite time if $\lambda \geqslant \lambda^*$. We show that there exist at most finitely many exceptional values $\lambda_1 = \lambda^* < \lambda_2 < \cdots < \lambda_k$ such that, for all $\lambda > \lambda^*$ with $\lambda \neq \lambda_j$ $(j = 1, 2, \ldots, k)$, the blow-up is complete and of type I with a flat local profile. Our method is based on a combination of the zero-number principle and energy estimates. In the second part of the paper, we employ the very same idea to show that the constant solution κ attains the smallest rescaled energy among all non-zero stationary solutions of the rescaled equation. Using this result, we derive a sharp criterion for no blow-up.

Keywords: Blow-up

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1. Introduction

We consider the nonlinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & (x \in \Omega,\ t > 0), \\ u(x,0) = u_0(x) & (x \in \Omega), \end{cases} \tag{1.1}$$

where either $\Omega = \mathbf{R}^N$ or $\Omega = B_R := \{x \in \mathbf{R}^N \mid |x| < R\}$, and $u_0 \in L^\infty(\Omega) \cap C(\overline{\Omega})$. If $\Omega = B_R$, we impose the Dirichlet boundary condition

$$u(x,t) = 0 \quad (x \in \partial\Omega,\ t > 0). \tag{1.2}$$

The exponent p is supercritical in the Sobolev sense, that is,

$$p > p_S := \frac{N+2}{N-2}, \quad N \geqslant 3.$$

For the threshold result, we consider an initial data of the form

$$u_0(x) = \lambda v(x), \tag{1.3}$$

where $\lambda > 0$ is a parameter and

$$v \geqslant 0, \qquad v \not\equiv 0, \qquad v \in L^\infty(\Omega) \cap C(\overline{\Omega}). \tag{1.4}$$

In the case where $\Omega = \mathbf{R}^N$, we further assume that

$$v \in H^1(\mathbf{R}^N) \quad \text{or} \quad \lim_{|x|\to\infty} r^{\frac{2}{p-1}} v(x) = 0. \tag{1.5}$$

We denote by u^λ the solution of (1.1)–(1.3). Throughout this paper we deal with radially symmetric solutions. We use such notation as $U(r,t)$, $U_0(r)$, $V(r)$ that are defined by

$$u(x,t) = U(|x|,t), \qquad u_0(x) = U_0(|x|), \qquad v(x) = V(|x|).$$

By a *blow-up* we mean an L^∞ blow-up, that is, there exists $T \in (0,\infty)$ such that $u(x,t)$ is bounded and smooth for $0 < t < T$ and that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \to T.$$

This value T is called the *blow-up time* of u. Given a solution u that blows up at $t = T$, we say that $a \in \Omega$ is a *blow-up point* of u (or that u blows up at $x = a$) if there exists no neighborhood of a on which u remains bounded as $t \to T$.

Now we recall the notion of type I and type II blow-ups.

Definition 1.1. We say that the blow-up is of **type I** if $(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_{L^\infty}$ remains bounded as $t \to T$. The blow-up is of **type II** if it is not of type I.

The existence of type II blow-up solutions for (1.1) with $\Omega = \mathbf{R}^N$ was discovered by Herrero and Velázquez [11,12] for the range $p > p_{JL}$, where

$$p_{JL} := \begin{cases} \infty & \text{if } 1 \leqslant N \leqslant 10, \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{if } N \geqslant 11. \end{cases}$$

On the other hand, in the range $p_S < p < p_{JL}$, no type II blow-up can occur as far as radially symmetric solutions are concerned; see [16] and also [17, Theorems 3.7–3.9]. As we have shown in [17, Theorem 3.2], any type II blow-up satisfies

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} \left\| u(\cdot, t) \right\|_{L^\infty} = \infty.$$

Furthermore, as one can easily see from [16, Theorem 3.1], type II blow-ups can occur only at $r = 0$ (see also [17, Remark 3.4]).

Let $u(x,t)$ be a solution that blows up at $x = a$ and $t = T$. Then the rescaled solution at a is defined by

$$w_a(y,s) := e^{-\frac{s}{p-1}} u\big(a + e^{-\frac{s}{2}} y, T - e^{-s}\big) = (T-t)^{\frac{1}{p-1}} u(a + \sqrt{T-t}\, y, t), \tag{1.6}$$

where $s := \log \frac{1}{T-t}$ and $y = \frac{x-a}{\sqrt{T-t}}$. The function w_a solves the rescaled equation

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p-1} w + |w|^{p-1} w. \tag{1.7}$$

As far as radially symmetric solutions are concerned, the limit

$$w_a^*(y) := \lim_{s \to \infty} w_a(y,s) = \lim_{t \to T} (T-t)^{\frac{1}{p-1}} u(a + \sqrt{T-t}\, y, t) \tag{1.8}$$

exists. We call w_a^* the *local blow-up profile* of u at $x = a$. More precisely, if the blow-up occurs at $a \neq 0$, then the blow-up is always of type I and

$$w_a^* = \kappa := (p-1)^{-\frac{1}{p-1}} \quad \text{or} \quad w_a^* = -\kappa.$$

See [16, Section 6.2] and the references therein. It is known that any blow-up outside $x = 0$ is complete; see, for example, [8, Section 8.1] (for positive solutions) and [17, Proposition 5.13] (for possibly sign-changing solutions). Here, by a "complete blow-up" we mean that the minimal extension of the solution becomes $+\infty$ for a.e. $x \in \Omega$ and all $t > T$, see [1,8]; see also [17] for generalization to sign-changing solutions.

On the other hand, if the blow-up occurs at $x = 0$, the local blow-up profile at $x = 0$, which we denote by $w_0^*(y) = W_0^*(|y|)$, is a radially symmetric stationary solution – either regular or singular – of (1.7); hence $\Psi = W_0^*(r)$ solves the equation

$$\Psi'' + \frac{N-1}{r}\Psi' - \frac{r}{2}\Psi' - \frac{1}{p-1}\Psi + |\Psi|^{p-1}\Psi = 0 \quad \text{for } 0 < r < \infty. \tag{1.9}$$

Furthermore, as we see in [17, Theorem 3.1], the blow-up is of type I if and only if $w_0^*(r)$ is a bounded stationary solution of (1.7), while it is of type II if and only if $w_0^*(y) = \pm\varphi^*(y) := \pm\Phi^*(|y|)$, where $\Phi^*(r)$ is the singular solution of (1.9) given by:

$$\Phi^*(r) = c^* r^{-\frac{2}{p-1}}, \quad \text{where } \left(c^*\right)^{p-1} = \frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right). \tag{1.10}$$

Next we recall the notion of single-intersection blow-up, which we introduced in our previous paper [17]. Let $u(x,t) = U(|x|,t)$ be a solution of (1.1)–(1.2) that blows up at $t = T$. We say that the blow-up is a *single-intersection blow-up* if it occurs at $r = 0$ and if there exist $r_0 > 0$ and $t_0 \in [0, T)$ such that

$$\mathcal{Z}_{(0,r_0]}\left[\left|U(\cdot,t)\right| - \Phi^*\right] \leqslant 1, \quad \text{for } t_0 < t < T. \tag{1.11}$$

Here $\mathcal{Z}_I[w]$ denotes the number of zeros of a function $w(r)$ in the interval I. If $r_1(t)$, $r_2(t)$ denote the smallest and the second smallest zeros of $|U(r,t)| - \Phi^*(r)$, then the blow-up occurs at $r = 0$ if and only if

$$\liminf_{t\to T} r_1(t) = 0$$

(see, for example, [17, Lemma 3.13]), and (1.11) holds if and only if

$$\liminf_{t\to T} r_2(t) > 0,$$

where we set $r_2(t) = \infty$ if the second zero does not exist. We say that the blow-up is a *multi-intersection blow-up* if

$$\liminf_{t\to T} r_2(t) = 0.$$

We recall that, so far as $p \neq p_{JL}$, any single-intersection blow-up is of type I and $W_0^* = \kappa$ or $-\kappa$. Consequently the blow-up is complete; see [17, Theorem 5.28]. Conversely, if a type I blow-up occurs at $r = 0$ and if $W_0^* = \pm\kappa$, then it is a single-intersection blow-up by virtue of Corollary 4.8 of [17]. Summarizing, we have the following picture:

$$\text{Blow-up at } x = 0 \quad \begin{cases} \text{single-intersection} \Rightarrow \text{type I},\ w_0^* = \pm\kappa,\ \text{complete}, \\ \text{multi-intersection} \Rightarrow \begin{cases} w_0^* \neq \pm\kappa,\ \pm\varphi^* & \text{(if type I)}, \\ w_0^* = \pm\varphi^* & \text{(if type II)}, \end{cases} \end{cases}$$

$$\text{Blow-up at } x = a \neq 0 \quad \Rightarrow \quad \text{type I}, \quad w_a^* = \pm\kappa, \quad \text{complete}.$$

Now we return to the problem (1.1)–(1.3). As regards the solution u^λ of this problem, the following is known to hold:

Threshold behavior. *Let $p_S < p < \infty$ and assume* (1.4)–(1.5). *Then there exists a $\lambda^* > 0$ such that*:

$$\lambda \in (0, \lambda^*) \quad \Rightarrow \quad u^\lambda \text{ is globally classical and } \|u^\lambda(\cdot, t)\|_{L^\infty} \to 0 \text{ as } t \to \infty,$$
$$\lambda \in [\lambda^*, \infty) \quad \Rightarrow \quad u^\lambda \text{ blows up in finite time}.$$

Furthermore, if $\lambda = \lambda^*$, the following hold:

(i) *u^{λ^*} blows up in finite time, say at $t = T^*$, but can be continued as a weak solution (L^1 solution) for all $t \geqslant T^*$.*
(ii) *Blow-up can occur only at $x = 0$, and there exists $T_1 \in [T^*, \infty)$ such that $\tilde{u}^{\lambda^*}_{min}$ is smooth for $t > T_1$, where $\tilde{u}^{\lambda^*}_{min}$ denotes the minimal extension of u^{λ^*}. The same holds for any limit L^1 continuation $\tilde{u}^{\lambda^*}$ of u^{λ^*}.*
(iii) *$\|\tilde{u}^{\lambda^*}(\cdot, t)\|_{L^\infty} \to 0$ as $t \to \infty$.*
(iv) *If $p \neq p_{JL}$, then there is a sequence $T = t_1 < t_2 < \cdots < t_k$ with $1 \leqslant k < \infty$ such that $\tilde{u}^{\lambda^*}_{min}$ is smooth in the space–time region*

$$\{(x, t) \in \Omega \times (0, \infty) \mid (x, t) \neq (0, t_j),\ j = 1, 2, \ldots, k\}. \tag{1.12}$$

The same holds for any limit L^1 continuation $\tilde{u}^{\lambda^*}$ if $p_S < p < p_{JL}$.

See [5] and [17, Section 5] for the definition of "limit L^1 continuation". Roughly speaking, a limit L^1 continuation is an L^1 solution that equals a given solution u until its blow-up time and can be expressed as a limit of classical solutions. The uniqueness of limit L^1 continuation of a given solution is not known, but its minimal element coincides with the minimal (or proper) solution introduced in [1,8], except that the former is defined only until the complete blow-up time of the latter; see [17, Proposition 5.5].

Statement (i) above was first established in [8] for $\Omega = B_R$, $p_S < p < p_L$, where

$$p_L := \begin{cases} \infty & \text{if } 1 \leqslant N \leqslant 10, \\ 1 + \frac{6}{N-10} & \text{if } N \geqslant 11. \end{cases}$$

A similar result was obtained in [16] for $\Omega = \mathbf{R}^N$, $p_S < p < p_L$, under the assumption that v is compactly supported. Later [20] proved (i) for the range $p > p_{JL}$ by a different argument. See [17, Theorem 5.15] for a simpler proof of statements (i)–(iii) for $p_S < p < \infty$ under the assumption $v \in H^1$. We note that [4] proves statements similar to (i)–(iii) without assuming radial symmetry on a bounded convex domain. As for statement (iv), the proof differs between the case $p_S < p < p_{JL}$ and the case $p > p_{JL}$, but in both cases the assertion follows immediately from known results found in [17] and partly in [5]. We will give a brief proof in Section 2.4.

It is worth noting that u^{λ^*} can blow up only at $x = 0$, no matter where $v(x)$ attains its maximum (see (ii) above). This is a peculiarity of the threshold solution u^{λ^*}, and does not necessarily hold for u^λ with $\lambda > \lambda^*$, as seen in Lemma 2.9.

What is also worth emphasizing is that u^λ can never converge to a positive stationary solution, which exists if $\Omega = \mathbf{R}^N$ and $p > p_S$. We have either blow-up or convergence to 0. Positive stationary solutions are unreachable from initial data satisfying (1.5).

Our first main result is concerned with the blow-up behavior for $\lambda > \lambda^*$:

Theorem 1.2 *(Type of blow-up above the threshold). Let $p_S < p < \infty$. Denote by u^λ the solution of* (1.1)–(1.3) *and assume* (1.4)–(1.5). *Then there exist at most finitely many exceptional values $\lambda_1, \dots, \lambda_k$ with*

$$\lambda^* = \lambda_1 < \lambda_2 < \cdots < \lambda_k$$

such that, for any $\lambda \in [\lambda^*, \infty) \setminus \{\lambda_1, \dots, \lambda_k\}$, u^λ exhibits either a single-intersection blow-up or a blow-up outside $x = 0$. Consequently the blow-up is of type I and complete, and its local blow-up profile is κ for these values of λ.

Remark 1.3. In the special case where v is radially decreasing (that is, $V'(r) \leqslant 0, \not\equiv 0$), we have $U_r^\lambda(r,t) < 0$ for $t > 0$, $r \neq 0$, hence blow-up always occurs at $r = 0$. Therefore u^λ exhibits a single-intersection blow-up for almost all $\lambda \in (\lambda^*, \infty)$.

The above theorem, in particular, implies that incomplete or type II blow-up is a highly non-generic phenomenon. We remark that, as far as generic complete blow-up of u^λ is concerned, there is a rather simple argument to prove it, as shown in [22]. Indeed, if T_λ denotes the blow-up time of u^λ for $\lambda > \lambda^*$, then T_λ is a decreasing function of λ and, by [14, Theorem 2], it is discontinuous whenever the blow-up is incomplete. Since a monotone function can have at most countably many discontinuities, one finds that u^λ can exhibit an incomplete blow-up for at most countably many values of λ. Compared with this simple observation, our Theorem 1.2 provides much more detailed information about the exceptional values of λ and the nature of generic blow-up.

Our next theorem is concerned with the rescaled energy. Let

$$\mathcal{E} = \text{the set of all bounded solutions of (1.9)}, \tag{1.13}$$

$$\mathcal{E}_+ = \text{the set of all bounded positive solutions of (1.9)}. \tag{1.14}$$

We slightly abuse the notation, so that $\mathcal{E}_+$ (resp. $\mathcal{E}$) will also denote the set of functions of the form $\psi(x) = \Psi(|x|)$ with $\Psi \in \mathcal{E}_+$ (resp. $\Psi \in \mathcal{E}$), in other words, the set of radially symmetric bounded positive solutions (resp. bounded solutions) of

$$\Delta\psi - \frac{1}{2} y \cdot \nabla\psi - \frac{1}{p-1}\psi + |\psi|^{p-1}\psi = 0, \quad y \in \mathbf{R}^N. \tag{1.15}$$

We note that, in the subcritical case $1 < p < p_S$, we have $\mathcal{E} = \{0, \pm\kappa\}$ as shown in [9], but it is known that $\mathcal{E}$ contains other elements if $p_S < p < p_L$ [13,15].

Theorem 1.4. *Let $p_S < p < \infty$ and let E denote the rescaled energy defined in* (3.1). *Then*

$$E(\psi) \geqslant E(\kappa) \quad \text{for any } \psi \in \mathcal{E} \cup \{\pm\varphi^*\} \setminus \{0\}. \tag{1.16}$$

Furthermore, the equality holds if and only if $\psi = \kappa$ or $-\kappa$.

Remark 1.5. We will in fact prove the following stronger version of the above result:

$$\inf_{\psi \in \mathcal{E} \cup \{\pm\varphi^*\} \setminus \{0, \pm\kappa\}} E(\psi) > E(\kappa). \tag{1.17}$$

Our proof of the above theorem is "parabolic" and is based on the same zero-number argument as in the proof of Theorem 1.2. From the above theorem one can derive the following sharp criterion for no blow-up.

Corollary 1.6 *(Non-blow-up criterion). Let u be a radially symmetric solution of* (1.1)–(1.2) *with $u_0 \in L^\infty(\Omega) \cap C(\overline{\Omega})$ that blows up at $t = T$, and w_a be the rescaled solution at $x = a$ as defined in* (1.6). *Suppose that $E(w_a(\cdot\,; s)) < E(\kappa)$ for some $s \geqslant s_0 = -\log T$. Then u cannot blow up at $x = a$.*

A result similar to Corollary 1.6 is well known in the subcritical case $1 < p < p_S$ (see [10, Remark 3.7]), but in the supercritical case $p > p_S$, such a sharp non-blow-up criterion has not been known. The next corollary follows easily from Corollary 1.6.

Corollary 1.7. *Let v be a radially symmetric function satisfying $v \in L^\infty(\Omega) \cap C(\overline{\Omega})$, $v \not\equiv 0$, and assume that v is uniformly continuous if $\Omega = \mathbf{R}^N$. Let u^λ be the solution of* (1.1)–(1.2) *with initial data* (1.3). *Fix $\varepsilon > 0$ arbitrarily and set*

$$D^\varepsilon = \Big\{ x \in \Omega \;\Big|\; \sup_{|z-x|<\varepsilon} \big|v(z)\big| > (1-\varepsilon)\|v\|_{L^\infty} \Big\}.$$

Then, for all sufficiently large $\lambda > 0$, u^λ blows up in finite time and its blow-up points are contained in D^ε.

By obvious rescaling, the conclusion of Corollary 1.7 remains true for the problem

$$u_t = \sigma \Delta u + |u|^{p-1}u \quad (\sigma \ll 1), \qquad u(x,0) = v(x),$$

where the initial data is fixed and the diffusion coefficients σ tend to zero. This case has been studied previously by a number of authors using super/subsolution methods; see [7] and the references therein. Those previous results allow v to be non-radial but assume $v \geqslant 0$. Our Corollary 1.7, on the other hand, is limited to radial solutions but it allows v to change sign. As our method is based on energy estimates, it does not rely on the sign of the solution.

Before concluding this section, we remark that the following proposition can easily be derived from our earlier results found in [16,17].

Proposition 1.8 *(Type of blow-up at the threshold). Let u^λ and λ^* be as in Theorem* 1.2. *Then the following holds concerning the blow-up of u^{λ^*}:*

(i) *if $p \in (p_S, p_{JL})$, the blow-up is of type I, with $w_0^* \not\equiv \kappa, 0$;*
(ii) *if $p \in (p^{**}, \infty)$, the blow-up is of type II,*

where

$$p^{**} := \sup\{p > p_S \mid \mathcal{E}_+ \supsetneq \{\kappa\}\}$$

and $\mathcal{E}_+$ *is as in* (1.14).

We remark that the only element of $\mathcal{E}_+$ satisfying $\mathcal{Z}_{(0,\infty)}[\Psi - \Phi^*] = 1$ is $\Psi = \kappa$ by virtue of [2, Lemma 3.29] (see also Lemma A.1 of the present paper), therefore

$$p^{**} = \sup\{p > p_S \mid \mathcal{Z}_{(0,\infty)}[\Psi - \Phi^*] \geqslant 2 \text{ for some } \Psi \in \mathcal{E}_+\}.$$

It is shown in [15] that $p^{**} \geqslant p_L := 1 + \frac{6}{N-10}$ $(> p_{JL})$, while $p^{**} \leqslant 1 + \frac{7}{N-11}$ by [19]. A more recent result of Mizoguchi [21] shows that $p^{**} = p_L$.

Proof of Proposition 1.8. Statement (i) follows from the fact that no type II blow-up occurs if $p_S < p < p_{JL}$; see [16, Theorem 1.5] or [17, Theorem 3.7] for the case $\Omega = B_R$, and [17, Theorems 3.8–3.9] (also partly [16, Theorem 1.6]) for the case $\Omega = \mathbf{R}^N$. To prove statement (ii), suppose the contrary. Then the blow-up is of type I, therefore, $w_0^* = \kappa$ by the assumption on p. This, however, is impossible since the local profile $w_0^*(y)$ for an incomplete blow-up must satisfy the estimate $w_0^*(y) \leqslant C|y|^{-\frac{2}{p-1}}$ by Proposition 5.13 of [17] or by Corollary 3.13 of [16]. This completes the proof of the proposition. □

The above result gives a simple proof of the existence of type II blow-up for the range $p > p^{**} = p_L$. The argument is totally different from the direct proof of [11,12], which is based on a fine asymptotic analysis and technically delicate calculations. Note that [20, Theorem 1.2] also shows that u^{λ^*} exhibits a type II blow-up for the range $p > p^{**}$ with $\Omega = B_R$, under the additional assumption that v is radially decreasing and possesses certain intersection properties. We do not need such additional assumptions in the statement (ii) above. In fact, the same proof even shows that any incomplete blow-up (in the range $p > p^{**}$) is of type II even if the solution is sign-changing.

This paper is organized as follows. We prove Theorem 1.2 in Section 2, and Theorem 1.4 and its corollaries in Section 3. In Appendix A, we recall the result of [2] concerning the stationary solution with a single intersection with Φ^* and give a slightly simpler proof for the self-containedness of the present paper. In Appendix B, we present some general results on the solutions of (1.1) with singular initial data. In Appendix C, we prove uniform spatial decay estimates for solutions of (1.1) on $\mathbf{R}^N$. In Appendix D, we prove a lemma concerning the comparison of blow-up time for an ordered pair of solutions, which is an extension of a result in [14] to sign-changing solutions.

2. Proof of Theorem 1.2

The theorem follows from the following three lemmas. Note that the assumption (1.5) is not needed for Lemma 2.3.

Lemma 2.1. *For any $\lambda_0 \in (\lambda^*, \infty)$, there exists $\varepsilon > 0$ such that, for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, the solution u^λ exhibits a single-intersection blow-up if the blow-up occurs at $r = 0$.*

Lemma 2.2. *For any $\lambda_0 \in (\lambda^*, \infty)$, there exists $\varepsilon > 0$ such that, for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, the solution u^λ exhibits a single-intersection blow-up if the blow-up occurs at $r = 0$.*

Lemma 2.3. *Assume $v \in L^\infty(\Omega) \cap C(\overline{\Omega})$, $v \not\equiv 0$. Then there exists $M > 0$ such that, for any $\lambda > M$, the solution u^λ blows up in finite time. Furthermore, the blow-up is of type I and, if it occurs at $r = 0$, then it is a single-intersection blow-up.*

Remark 2.4. As is clear from its proof, the conclusion of Lemma 2.1 holds true in a much more general setting. First, the condition (1.5) is not necessary, and v can even change sign. Moreover u_0 need not be in the form (1.3). All we have to assume is:

(1) $u^\lambda(x, 0)$ depends on λ continuously and is monotone increasing in λ;
(2) u^{λ_0} blows up in finite time, say at T_{λ_0} and $\max_x u^\lambda(x,t) \to +\infty$ as $t \to T_{\lambda_0}$;
(3) the smallest zero of $U^{\lambda_0}(r,t) - \Phi^*(r)$ remains simple for t sufficiently close to T_{λ_0}.

Proof of Theorem 1.2. Denote by Y the subset of $[\lambda^*, \infty)$ consisting of all λ for which u^λ does not exhibit a single-intersection blow-up nor blow-up outside $r = 0$. What we have to show is that Y is a finite set. Lemmas 2.1 and 2.2 imply that Y has no accumulation point, while Lemma 2.3 shows that $Y \subset [\lambda^*, M]$ for some $M > 0$. Therefore Y has to be a finite set. The theorem is proved. □

2.1. Proof of Lemma 2.1

Let us first prove the following general lemma:

Lemma 2.5. *Let $\Omega = \mathbf{R}^N$ and let $u(x,t) = U(|x|, t)$ be a solution of (1.1) with $u_0 \in C(\mathbf{R}^N) \cap H^1(\mathbf{R}^N)$. Let $T > 0$ denote the blow-up time of u if u blows up in finite time; otherwise let $T > 0$ be arbitrary. Then there exist constants $C > 0$, $r_1 > 0$ and $t_1 \in [0, T)$ such that*

$$\left|U(r,t)\right| \leqslant C r^{-(N-2)/2} \quad \text{for } r \geqslant r_1,\ t \in [t_1, T). \tag{2.1}$$

Consequently, $|U(r,t)| < \Phi^*(r)$ for all large r and $t \in [t_1, \infty)$.

Proof. By [17, Proposition 2.16], there exists $r_1 > 0$ such that

$$\limsup_{t \to T} \int_{|x| \geqslant r_1} \left|\nabla u(x,t)\right|^2 dx < \infty.$$

Thus there exist $M > 0$ and $t_1 \in [0, T)$ such that

$$\int_{r_1}^{\infty} |U_r|^2 r^{N-1}\, dr \leqslant M \quad \text{for } t \in [t_1, T).$$

Consequently

$$\left|U(r,t)\right| \leqslant \int_r^\infty |U_r|\,dr \leqslant \left(\int_r^\infty (U_r)^2 r^{N-1}\,dr\right)^{1/2}\left(\int_r^\infty \frac{1}{r^{N-1}}\,dr\right)^{1/2} \leqslant Cr^{-\frac{N-2}{2}}$$

for all $t \in [t_1, T)$, where $C > 0$ is some constant. Since $p > p_S$, we have $\frac{N-2}{2} > \frac{2}{p-1}$, hence $|U(r,t)| = o(r^{-\frac{2}{p-1}})$. This proves the last statement of the lemma. □

Remark 2.6. The estimate $u(x,t) = o(|x|^{-\frac{2}{p-1}})$ $(0 \leqslant t < T)$ holds also under the second condition in (1.5). In fact, the following more general estimates hold for solutions of (1.1):

$$u(x,0) = o\left(|x|^{-\alpha}\right) \quad \text{for some } \alpha \geqslant 0 \quad \Rightarrow \quad u(x,t) = o\left(|x|^{-\alpha}\right), \tag{2.2}$$

$$u(x,0) = O\left(|x|^{-\alpha}\right) \quad \text{for some } \alpha > 0 \quad \Rightarrow \quad u(x,t) = O\left(|x|^{-\alpha}\right) \tag{2.3}$$

uniformly in $t \in [0,T)$, where T is the blow-up time of u. See Appendix C for details.

Proof of Lemma 2.1. In what follows, T_λ will denote the blow-up time of U^λ for each $\lambda \geqslant \lambda^*$. Then, by Lemma D.1,

$$T_{\lambda_1} < T_{\lambda_2} \quad \text{for any } \lambda^* \leqslant \lambda_1 < \lambda_2 < \infty. \tag{2.4}$$

Next, in the case where $\Omega = B_R$, we see from [3] that

$$\mathcal{Z}_I\left[U^{\lambda_0}(\cdot,t) - \Phi^*\right] < \infty$$

with $I = (0,R)$ for all $t \in (0, T_{\lambda_0})$. The same holds true with $I = (0,\infty)$ for all $t \in (t_{\lambda_0}, T_{\lambda_0})$ if $\Omega = \mathbf{R}^N$, where t_λ is as in Lemma 2.5. Thus, no matter whether $\Omega = B_R$ or $\Omega = \mathbf{R}^N$, the value of $\mathcal{Z}_I[U^{\lambda_0}(\cdot,t) - \Phi^*]$ becomes eventually constant, since it is monotone non-increasing in t and is a nonnegative integer. Once it becomes constant, we see, again from [3], that all the zeros of $U^{\lambda_0}(r,t) - \Phi^*(r)$ are simple. Thus we can choose $\tau_0 \in (0, T_{\lambda_0})$ such that $U^{\lambda_0}(r,\tau_0) - \Phi^*(r)$ have only finitely many zeros in I, all of which are simple, and that the set of all the zeros of $U^{\lambda_0}(r,t) - \Phi^*(r)$ is given in the form

$$r_1^{\lambda_0}(t) < r_2^{\lambda_0}(t) < \cdots < r_m^{\lambda_0}(t) \tag{2.5}$$

for every t in $[\tau_0, T_{\lambda_0})$. Here m is an even integer, since $U^{\lambda_0}(r,t) < \Phi^*(r)$ for $0 < r < r_1^{\lambda_0}(t)$ and for $r > r_m^{\lambda_0}(t)$, the latter being a consequence of Lemma 2.5. By the implicit function theorem, each $r_j^{\lambda_0}(t)$ $(j = 1,2,\ldots,m)$ is a smooth function of t.

Now we choose $\varepsilon > 0$ sufficiently small, so that, for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon]$, the zeros of the function $U^\lambda(r,\tau_0) - \Phi^*(r)$ are all simple and are given in the form

$$\rho_1^\lambda < \rho_2^\lambda < \cdots < \rho_m^\lambda.$$

Since $U^\lambda(r,t) > U^{\lambda_0}(r,t)$ for $0 < t < T_\lambda$ by the strong comparison principle, we have

$$\rho_1^\lambda < r_1^{\lambda_0}(\tau_0) < r_2^{\lambda_0}(\tau_0) < \rho_2^\lambda.$$

In particular,

$$\mathcal{Z}_{(0,\bar r_1(\tau_0))}\big[U^\lambda(\cdot,\tau_0) - \Phi^*\big] = 1,$$

where we have put $\bar r_1(t) := r_1^{\lambda_0}(t)$. Again by the strong comparison principle,

$$U^\lambda\big(\bar r_1(t), t\big) - \Phi\big(\bar r_1(t)\big) > U^{\lambda_0}\big(\bar r_1(t), t\big) - \Phi\big(\bar r_1(t)\big) = 0 \quad \text{for } t \in [\tau_0, T_\lambda),$$

hence

$$\mathcal{Z}_{(0,\bar r_1(t))}\big[U^\lambda(\cdot,t) - \Phi^*\big] \leqslant \mathcal{Z}_{(0,\bar r_1(\tau_0))}\big[U^\lambda(\cdot,\tau_0) - \Phi^*\big] = 1 \quad \text{for } t \in [\tau_0, T_\lambda).$$

Furthermore, by (2.4) we have $T_\lambda < T_{\lambda_0}$, which implies that there exists a constant $\delta > 0$ such that $\bar r_1(t) \geqslant \delta$ for all $t \in [\tau_0, T_\lambda)$. Consequently

$$\mathcal{Z}_{(0,\delta]}\big[U^\lambda(\cdot,t) - \Phi^*\big] \leqslant 1 \quad \text{for } t \in [\tau_0, T_\lambda).$$

Thus U^λ exhibits a single-intersection blow-up if the blow-up occurs at $r = 0$. The proof of the lemma is complete. □

2.2. *Proof of Lemma 2.2*

Let τ_0 and $r_j^{\lambda_0}(t)$ $(j = 1, 2, \dots, m)$ be as in the proof of Lemma 2.1. Choose $\varepsilon > 0$ small enough, so that, for any $\lambda \in [\lambda_0 - \varepsilon, \lambda_0)$, the zeros of the function $U^\lambda(r,\tau_0) - \Phi^*(r)$ are all simple and are given in the form

$$\rho_1^\lambda < \rho_2^\lambda < \dots < \rho_m^\lambda.$$

By the strong comparison principle, we have $U^\lambda(r,t) < U^{\lambda_0}(r,t)$ for $0 < t < T_{\lambda_0}$, hence

$$r_{2j-1}^{\lambda_0}(\tau_0) < \rho_{2j-1}^\lambda < \rho_{2j}^\lambda < r_{2j}^{\lambda_0}(\tau_0) \quad \text{for } j = 1, 2, \dots, m/2.$$

Now, for each $\lambda \in [\lambda_0 - \varepsilon, \lambda_0]$, we set

$$A_+^\lambda = \big\{(r,t) \in (0,\infty) \times [\tau_0, T_\lambda) \bigm| U^\lambda(r,t) - \Phi^*(r) > 0\big\}.$$

Then the inequality $U^\lambda < U^{\lambda_0}$ yields

$$A_+^\lambda \cap \{\tau_0 \leqslant t < T_{\lambda_0}\} \subset A_+^{\lambda_0} \quad \text{for } \lambda \in [\lambda_0 - \varepsilon, \lambda_0].$$

Let $J \subset [\lambda_0 - \varepsilon, \lambda_0)$ be the set of λ's such that U^λ exhibits a multi-intersection blow-up. We will show that the number of elements of J does not exceed $m/2$.

Choose $\lambda \in J$ arbitrarily. Then, just as we have shown for λ_0 above, there exists $\tau_\lambda \in [\tau_0, T_\lambda)$ such that all the zeros of $U^\lambda(r,t) - \Phi^*(r)$ are simple for each $t \in [\tau_\lambda, T_\lambda)$ and therefore can be expressed as smooth functions of t. We denote by $r_1^\lambda(t) < r_2^\lambda(t)$ the first and the second zeros of $U^\lambda(r,t) - \Phi^*(r)$ in the range $t \in [\tau_\lambda, T_\lambda)$. Then clearly

$$U^\lambda(r,t) - \Phi^*(r) > 0 \quad \text{for } (r,t) \in \bigcup_{t\in[\tau_\lambda, T_\lambda)} \bigl(r_1^\lambda(t), r_2^\lambda(t)\bigr) =: D^\lambda.$$

Denote by B^λ the connected component of A_+^λ containing D^λ. Then $B^\lambda \cap \{\tau_0 < t < T_\lambda\}$ is an open connected subset of $\{r > 0,\ \tau_0 < t < T_\lambda\}$, and the maximum principle implies

$$B^\lambda \cap \{t = \tau_0\} \neq \emptyset. \tag{2.6}$$

Next let Γ^λ denote the connected component of $\partial B^\lambda \cap \{\tau_0 < t < T_\lambda\}$ containing the curve $(r_2^\lambda(t), t)$, $\tau_\lambda \leqslant t < T_\lambda$. Then it is easily seen from the maximum principle that Γ^λ does not have a vertical turning point nor does it contain a horizontal line segment. Consequently, Γ^λ can be expressed as a graph of a continuous function $\rho^\lambda(t)$ $(\tau_0 < t < T_\lambda)$ which coincides with $r_2^\lambda(t)$ for t close to T_λ. This and (2.6) imply that

$$\liminf_{t\to T_\lambda} \rho^\lambda(t) = 0, \qquad \lim_{t\to\tau_0+0} \rho^\lambda(t) = \rho^\lambda_{i(\lambda)+1} \tag{2.7}$$

for some even integer $2 \leqslant i(\lambda) \leqslant m$.

Now suppose that there exist $\lambda', \lambda \in J$ with $\lambda' < \lambda$. Then, since $U^{\lambda'} < U^\lambda$, we have $T_{\lambda'} > T_\lambda$ by (2.4), and that $\Gamma^\lambda \cap \Gamma^{\lambda'} = \emptyset$. Combining this and (2.7), we see that

$$i\bigl(\lambda'\bigr) > i(\lambda).$$

Since $i(\lambda)$ can take values in the set $\{2, 4, \dots, m\}$ as λ varies in J, we conclude that J consists of at most $m/2$ elements. The proof of the lemma is complete.

2.3. *Proof of Lemma 2.3*

In this subsection we simply assume $v \in L^\infty(\Omega) \cap C(\overline{\Omega})$, $v \not\equiv 0$. We rescale U^λ as

$$\hat{U}^\lambda(r,t) := \lambda^{-1} U^\lambda\bigl(\lambda^{-\frac{p-1}{2}} r,\ \lambda^{-(p-1)} t\bigr).$$

Then $\hat{u}^\lambda(x,t) := \hat{U}^\lambda(|x|, t)$ satisfies (1.1) on the domain $\hat{\Omega}^\lambda := \{\lambda^{\frac{p-1}{2}} x \mid x \in \Omega\}$, and

$$\hat{u}^\lambda(x,0) = v\bigl(\lambda^{-\frac{p-1}{2}} x\bigr) =: v^\lambda(x) \quad \text{for } x \in \hat{\Omega}^\lambda.$$

If u^λ blows up in finite time, say at $t = T_\lambda$, then so does $\hat{u}^\lambda$, and its blow-up time $\hat{T}_\lambda$ is given by

$$\hat{T}_\lambda = \lambda^{p-1} T_\lambda.$$

Clearly u^λ exhibits a single-intersection blow-up (or a blow-up outside $x = 0$) if and only if $\hat{u}^\lambda$ has the same property. So we will prove Lemma 2.3 for $\hat{u}^\lambda$ instead of u^λ.

Lemma 2.7. *For all sufficiently large $\lambda > 0$, $\hat{u}^\lambda$ blows up in finite time, and its blow-up time $\hat{T}_\lambda$ satisfies*

$$\lim_{\lambda\to\infty} \hat{T}_\lambda = \frac{1}{p-1}\big(\|v\|_{L^\infty}\big)^{-(p-1)}. \tag{2.8}$$

Proof. For simplicity we assume that $|v(x)|$ attains its maximum in Ω (which is always the case if Ω is bounded). The general case can be treated similarly with minor modification. Let $a \in \Omega$ be the point where $|v|$ attains its maximum. Without loss of generality, we may assume that $v(a) > 0$.

If $\hat{u}^\lambda$ blows up in finite time, we let $\hat{T}_\lambda$ be its blow-up time; otherwise we set $\hat{T}_\lambda = \infty$. Denote by $g(t;\alpha)$ the solution of the problem

$$\frac{dg}{dt} = g^p \quad (t > 0), \qquad g(0) = \alpha > 0.$$

Then by the comparison principle, we have

$$\big|\hat{u}^\lambda(x,t)\big| \leqslant g\big(t; \|v\|_{L^\infty}\big) = \big(\|v\|_{L^\infty}^{-(p-1)} - (p-1)t\big)^{-\frac{1}{p-1}}.$$

Consequently,

$$\hat{T}_\lambda \geqslant \frac{1}{p-1}\big(\|v\|_{L^\infty}\big)^{-(p-1)}. \tag{2.9}$$

Next, choose a constant $T_1 > \frac{1}{p-1}(\|v\|_{L^\infty})^{-(p-1)}$ arbitrarily and define

$$w^\lambda(y,s) := e^{-\frac{s}{p-1}}\hat{u}^\lambda\big(a + e^{-\frac{s}{2}}y, T_1 - e^{-s}\big) = (T_1 - t)^{\frac{1}{p-1}}\hat{u}^\lambda(a + \sqrt{T_1 - t}\,y, t),$$

where $s := \log\frac{1}{T_1 - t}$ and $y = \frac{x-a}{\sqrt{T_1 - t}}$. The function $w(y,s)$ satisfies Eq. (1.7) for $s > s_0 := -\log T_1$. Note that

$$\lim_{\lambda\to\infty} w^\lambda(y,s_0) = T_1^{\frac{1}{p-1}} v(a) = T_1^{\frac{1}{p-1}}\|v\|_{L^\infty} > \kappa$$

locally uniformly in $\mathbf{R}^N$. Furthermore, $w^\lambda(y,s_0)$ is uniformly bounded on $\mathbf{R}^N$ as λ varies. Fix s_1 with $s_1 > s_0$. Then $w^\lambda(y,s_1)$ converges to some constant $\kappa_1 \in (\kappa,\infty)$ locally uniformly in the C^1 sense as $\lambda \to \infty$, while both $w^\lambda(y,s_1)$ and $\nabla_y w^\lambda(y,s_1)$ remain uniformly bounded on $\mathbf{R}^N$ as $\lambda \to \infty$. It follows that

$$\lim_{\lambda\to\infty} E\big(w^\lambda(\cdot, y_1)\big) = E(\kappa_1),$$

where E denotes the rescaled energy defined in (3.1). Since $\kappa_1 > \kappa$, we have $E(\kappa_1) < (\frac{1}{2} - \frac{1}{p+1})\kappa_1^{p+1} = (\frac{1}{2} - \frac{1}{p+1})(\int_{\mathbf{R}^N}\kappa_1^2\rho(y)\,dy)^{\frac{p+1}{2}}$. Consequently,

$$E\big(w^\lambda(\cdot,s_1)\big) < \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\int_{\mathbf{R}^N} \left|w^\lambda(y,s_1)\right|^2 \rho(y)\,dy\right)^{\frac{p+1}{2}}$$

for all sufficiently large λ. By the blow-up criterion of [18] (see also [16, Lemma 2.5]), the above inequality implies that w^λ blows up in finite time. This means that the blow-up time of $\hat{u}^\lambda$ is smaller than T_1, hence

$$\lim_{\lambda\to\infty} \hat{T}_\lambda < T_1 \quad \text{for any } T_1 > \frac{1}{p-1}\big(\|v\|_{L^\infty}\big)^{-(p-1)}.$$

The conclusion of the lemma now follows from this and (2.9). □

Lemma 2.8. *Suppose that $v(0) > 0$. Then for all sufficiently large $\lambda > 0$, there exist $r_0 > 0$ and $t_0 \in [0, \hat{T}_\lambda)$ such that (1.11) holds for $U = \hat{U}^\lambda$ and $T = \hat{T}_\lambda$. Consequently, the blow-up is a single-intersection blow-up if it occurs at $r = 0$.*

Proof. Let T_∞ denote the right-hand side of (2.8). Fix any $t_0 \in (0, T_\infty)$. Since $v^\lambda(x)$ is uniformly bounded and converges to $v(0)$ as $\lambda \to \infty$ locally uniformly, we have

$$\hat{U}^\lambda(r,t_0) \to g\big(t_0, v(0)\big), \qquad \hat{U}_r^\lambda(r,t_0) \to 0 \quad \text{locally uniformly in } r \in [0,\infty)$$

as $\lambda \to \infty$. Choose $r_0 > 0$ such that

$$\Phi^*(r_0) < g\big(t_0, v(0)\big).$$

Then from the above observation we have

$$\mathcal{Z}_{(0,r_0]}\big[\hat{U}^\lambda(\cdot,t_0) - \Phi^*\big] = 1 \tag{2.10}$$

for all large λ. Next let $q^\lambda(x,t) = Q^\lambda(|x|,t)$ be the solution of the following problem:

$$\begin{cases} q_t^\lambda = \Delta q^\lambda & (x \in \Omega_\lambda,\ t > t_0), \\ q^\lambda(x,t_0) = \hat{u}^\lambda(x,t_0) & (x \in \Omega_\lambda). \end{cases}$$

Here we impose the Dirichlet boundary condition on $\partial\Omega_\lambda$ if Ω is a ball. Then

$$\hat{u}^\lambda(x,t) \geqslant q^\lambda(x,t) \quad \text{for } x \in \Omega^\lambda,\ t \in [t_0, \hat{T}_\lambda).$$

Since $\hat{u}^\lambda(x,t_0)$ remains bounded and converges to the constant $g(t_0, v(0))$ locally uniformly as $\lambda \to \infty$, we have

$$Q^\lambda(r,t) \to g\big(t_0, v(0)\big) \quad \text{uniformly in } (r,t) \in [0,r_0] \times [t_0, T_1]$$

as $\lambda \to \infty$, where T_1 is a fixed constant satisfying $T_1 > T_\infty$ (hence $T_1 > \hat{T}_\lambda$ for all large λ). Since $\Phi^*(r_0) < g(t_0, v(0))$, we have $Q^\lambda(r_0,t) > \Phi^*(r_0)$ for all large λ and $t \in [t_0, T_1]$; hence

$$\hat{U}^\lambda(r_0,t) > \Phi^*(r_0) \quad \text{for all } t \in [t_0, \hat{T}_\lambda).$$

Combining this and (2.10), we obtain

$$\mathcal{Z}_{(0,r_0]}\big[\hat{U}^{\lambda}(\cdot,t)-\Phi^*\big]\leqslant 1 \quad \text{for all } t\in[t_0,\hat{T}_\lambda).$$

The lemma is proved. □

Lemma 2.9. *Suppose that $v(0)=0$, or more generally that v does not attain its maximum at $x=0$. Then for all large $\lambda>\lambda^*$, $x=0$ is not a blow-up point of $\hat{u}^{\lambda}$.*

Since the above lemma is a special case of Corollary 1.6, which is to be proved in Section 3, we omit the proof here. Now the conclusion of Lemma 2.3 for $\hat{u}^{\lambda}$ follows immediately from Lemmas 2.8 and 2.9. This completes the proof of Lemma 2.3.

2.4. Finiteness of the number of blow-up times

Here we prove statement (iv) of the threshold result given in the Introduction. More precisely, we will show that the minimal extension $\tilde{u}^{\lambda^*}_{min}$ of u^{λ^*} is smooth in a space–time region of the form (1.12) if $p_S<p<\infty$, $p\neq p_{JL}$, and the same holds for any limit L^1 continuation $\tilde{u}^{\lambda^*}$ if $p_S<p<p_{JL}$.

By virtue of statement (ii), we see that $\tilde{u}^{\lambda^*}$ (hence $\tilde{u}^{\lambda^*}_{min}$) has no singularity in the region $t>T_1$ and also for $x\neq 0$, therefore what needs to be shown is that $\tilde{u}^{\lambda^*}$ (or $\tilde{u}^{\lambda^*}_{min}$) can develop singularities only at discrete time moments.

If $p_S<p<p_{JL}$, the blow-up is automatically of type I by Theorems 3.9–3.11 of [17] (also by [16, Theorem 1.5] in the case $\Omega=B_R$), therefore the above claim follows from [17, Theorem 5.19] on the immediate regularization of type I blow-ups. If we only consider the minimal extension $\tilde{u}^{\lambda^*}_{min}$ instead of a more general limit L^1 continuation, the same claim follows also from [5, Theorem 3.1] provided that $\Omega=B_R$.

In the range $p>p_{JL}$, both type I and type II blow-ups can occur. The above claim for $\tilde{u}^{\lambda^*}_{min}$ follows by combining Theorem 5.19 of [17] for type I blow-ups and Theorem 5.21 for type II blow-ups. Note that the assumption $\tilde{u}^{\lambda^*}(x,t)\not\equiv\varphi^*(x)$ in [17, Theorem 5.21] is fulfilled by virtue of Lemma 2.5 and Remark 2.6. This completes the proof of statement (iv) of the threshold result.

3. Minimality of the energy $E(\kappa)$

In this section we prove Theorem 1.4 and its corollaries. We recall that the rescaled energy of a solution of Eq. (1.7) is defined by

$$E(w)=\int_{\mathbf{R}^N}\left(\frac{1}{2}|\nabla w|^2+\frac{1}{2(p-1)}|w|^2-\frac{1}{p+1}|w|^{p+1}\right)\rho(y)\,dy, \tag{3.1}$$

where

$$\rho(y):=(4\pi)^{-\frac{N}{2}}\exp\left(-\frac{|y|^2}{4}\right), \tag{3.2}$$

and the following identity holds for any solution $w(y,s)$ of (1.7):

$$\frac{d}{ds}E\big(w(\cdot,s)\big) = -\int_{\mathbf{R}^N} w_s^2(y,s)\,ds. \tag{3.3}$$

If ψ is a solution of (1.15), Green's formula yields

$$E(\psi) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int \rho|\psi|^{p+1}\,dy.$$

In particular, $E(\kappa) = (\frac{1}{2} - \frac{1}{p+1})\kappa^{p+1}$.

Given a solution u of (1.1)–(1.2) that blows up at $t = T$, let $w_a(y,s)$ be the rescaled solution introduced in (1.6). This function is defined for y such that $a + e^{-s/2}y \in \Omega$, but for notational simplicity we use the expression (3.1) even if $\Omega \neq \mathbf{R}^N$, by setting $w_a = 0$ outside its domain of definition. The integral (3.1) will be understood in this way when we discuss the energy of w_a for $\Omega = B_R$.

3.1. Basic lemma

We start with the following preliminary lemma:

Lemma 3.1. *Let u be a solution of (1.1)–(1.2) that blows up at $t = T$, and let $w_0(y,s)$ and $w_0^*(y)$ be as defined in (1.6), (1.8) with $a = 0$. Assume that u_0 and ∇u_0 are both bounded. Then*

$$E\big(w_0^*(\cdot + b)\big) \leqslant E\big(w_0(\cdot\,;s_0)\big) \quad \textit{for } b \in \mathbf{R}^N, \tag{3.4}$$

where $s_0 = -\log T$. The same holds if s_0 is replaced by any $s \geqslant s_0$.

Proof. Choose $b \in \mathbf{R}^N$ arbitrarily. Then a simple change of variable shows

$$w_{e^{-s/2}b}(y,s) = w_0(y+b,s).$$

Since $E(w_0(\cdot\,;s))$ is monotone non-increasing, we obtain

$$E\big(w_0(\cdot + b, s)\big) \leqslant E\big(w_{e^{-s/2}b}(\cdot\,, s_0)\big),$$

the right-hand side being equal to

$$T^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{\mathbf{R}^N} \left(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{2(p-1)T}|u_0|^2 - \frac{1}{p+1}|u_0|^{p+1}\right)\rho\left(\frac{x - e^{-s/2}b}{\sqrt{T}}\right)dx.$$

Letting $s \to \infty$, we obtain

$$\lim_{s\to\infty} E\big(w_0(\cdot + b, s)\big) \leqslant E\big(w_0(\cdot\,, s_0)\big). \tag{3.5}$$

Next, given $\delta > 0$, we recall that the following estimates hold for $|y| > 0$, $s \geqslant s_0 + \delta$:

$$\big|w_0(y,s)\big| \leqslant C\big(1 + |y|^{-\frac{2}{p-1}}\big), \qquad \big|\nabla w_0(y,s)\big| \leqslant C\big(1 + |y|^{-\frac{p+1}{p-1}}\big),$$

where $C > 0$ is a constant that depends on the initial data u_0, as well as on p, N, δ (see [16, Theorem 3.1] or [17, Proposition 2.5]). Since $|y|^{-\frac{2(p+1)}{p-1}}\rho(y)$ is integrable on $\mathbf{R}^N$ by the assumption $p > p_S$, we see by the Lebesgue convergence theorem that

$$E\big(w_0^*(\cdot + b)\big) = E\Big(\lim_{s\to\infty} w_0(\cdot + b, s)\Big) = \lim_{s\to\infty} E\big(w_0(\cdot + b, s)\big).$$

Combining this and (3.5), we obtain the desired estimate. $\square$

3.2. Proof of Theorem 1.4 (Part 1)

Here we prove the first part of Theorem 1.4, namely (1.16). For Corollaries 1.6 and 1.7, we only need this part. One can split (1.16) as follows:

$$E(\psi) \geqslant E(\kappa) \quad \text{for any } \psi \in \mathcal{E} \setminus \{0\}, \tag{3.6}$$

$$E\big(\varphi^*\big) \geqslant E(\kappa). \tag{3.7}$$

Let us begin with (3.6).

Proof of (3.6). Let $\psi(y) = \Psi(|y|)$ be any element of $\psi \in \mathcal{E} \setminus \{\pm\kappa, 0\}$. Without loss of generality, we may assume that $\psi(0) > 0$. It is known that $\psi(y)$ decays with the order $|y|^{-\frac{2}{p-1}}$ and that

$$0 < \lim_{|y|\to\infty} |y|^{\frac{2}{p-1}}\psi(y) \neq \lim_{|y|\to\infty} |y|^{\frac{2}{p-1}}\varphi^*(y).$$

See, for example, [16, Lemma A.2] or [17, Lemma A.1] and the references therein. Consequently, $\Psi(r)$ intersects $\Phi^*(r)$ only finitely many times. Next we show that

$$E\big(\psi(\cdot + b)\big) \leqslant E(\psi) \quad \text{for } b \in \mathbf{R}^N. \tag{3.8}$$

To see this, let $u(x,t)$ be the solution of (1.1) in $\mathbf{R}^N$ with $u_0(x) = \psi(x)$. Then

$$u(x,t) = (1-t)^{-\frac{1}{p-1}}\psi\left(\frac{x}{\sqrt{1-t}}\right). \tag{3.9}$$

The corresponding rescaled solution at $x = 0$ is given by

$$w_0(y,s) = \psi(y), \qquad s = -\log(1-t), \qquad y = \frac{x}{\sqrt{1-t}}.$$

Applying Lemma 3.1 to this w_0, we obtain (3.8).

Now, for each $\varepsilon \geqslant 0$, let $u^\varepsilon(x,t)$ be the solution of (1.1) in $\mathbf{R}^N$ with the following initial data:

$$u^\varepsilon(x,0) = \begin{cases} (1+\varepsilon)\psi(x) & \text{where } \psi(x) \geqslant 0, \\ (1-\varepsilon)\psi(x) & \text{where } \psi(x) < 0. \end{cases} \tag{3.10}$$

Then u^0 is given by (3.9), and $u^\varepsilon \geqslant u^0$ for $\varepsilon > 0$. Since $u^0(0,t) \to +\infty$ as $t \to 1$, u^ε blows up in finite time. Let T^ε be the blow-up time of u^ε. Clearly $T^\varepsilon \leqslant T^0 = 1$. Furthermore, the well-posedness of (1.1) implies that T^ε is lower semi-continuous in ε. Hence

$$T^\varepsilon \to T^0 = 1 \quad \text{as } \varepsilon \to +0. \tag{3.11}$$

Let $a^\varepsilon \in \mathbf{R}^N$ be a blow-up point of u^ε and denote by $w^\varepsilon(y,s)$ the corresponding rescaled solution at $x = a^\varepsilon$. By Lemma 2.1 and Remark 2.4, for all small $\varepsilon > 0$, u^ε exhibits either a single-intersection blow-up at $x = 0$ or a blow-up outside $x = 0$. In either case,

$$\lim_{s\to\infty} w^\varepsilon(y,s) = \kappa.$$

Consequently,

$$E\big(w^\varepsilon\big(\cdot, s_0^\varepsilon\big)\big) \geqslant E(\kappa), \tag{3.12}$$

where $s_0^\varepsilon = -\log T^\varepsilon$ and

$$w^\varepsilon\big(y, s_0^\varepsilon\big) = \big(T^\varepsilon\big)^{\frac{1}{p-1}}(1 \pm \varepsilon)\psi\big(\sqrt{T^\varepsilon}\, y + a^\varepsilon\big).$$

If $|a^{\varepsilon_k}| \to \infty$ for some sequence $\varepsilon_1 > \varepsilon_2 > \cdots \to 0$, then $w^\varepsilon(y, s_0^\varepsilon) \to 0$ locally uniformly in the C^1 sense along this sequence while $|\nabla w^\varepsilon| + |w^\varepsilon|$ stays uniformly bounded; hence $\liminf_{\varepsilon\to 0} E(w^\varepsilon(\cdot, s_0^\varepsilon)) = 0$, which, however, is impossible by (3.12). Therefore $|a_\varepsilon|$ remains bounded as $\varepsilon \to 0$. Choose a sequence $\varepsilon_k \to 0$ such that $a^{\varepsilon_k} \to a^*$ for some $a^* \in \mathbf{R}^N$. Letting $\varepsilon \to 0$ in (3.12) along this sequence gives

$$E\big(\psi\big(\cdot + a^*\big)\big) \geqslant E(\kappa).$$

Combining this and (3.8), we obtain the inequality (3.6). □

3.3. Proof of Theorem 1.4 (Part 1) continued

We next prove (3.7). It should be possible to derive this inequality from the expression

$$\frac{E(\varphi^*)}{E(\kappa)} = \left(2\left(N - 2 - \frac{2}{p-1}\right)\right)^{\frac{p+1}{p-1}} \frac{\int_0^\infty e^{-\frac{r^2}{4}} r^{N-1-\frac{2(p+1)}{p-1}}\, dr}{\int_0^\infty e^{-\frac{r^2}{4}} r^{N-1}\, dr},$$

but we will give a more "parabolic proof" without using this explicit formula. We start with the following lemma which is for the most part well known:

Lemma 3.2. *Let $p > p_S$ and let μ^* be the supremum of all $\mu > 0$ such that the minimal solution of the initial value problem*

$$\begin{cases} u_t = \Delta u + u^p & (x \in \mathbf{R}^N,\ t > 0), \\ u(x,0) = \mu\varphi^*(x) & (x \in \mathbf{R}^N \setminus \{0\}) \end{cases} \tag{3.13}$$

is globally classical for $t > 0$. Then

(i) $\mu^* = 1$ *if* $p \geqslant p_{JL}$, *while* $1 < \mu^* < \infty$ *if* $p_S < p < p_{JL}$;
(ii) *for any* $\mu > \mu^*$, (3.13) *has no solution*; *more precisely, the minimal solution of this problem equals* $+\infty$ *for a.e.* $x \in \mathbf{R}^N$, $t > 0$;
(iii) *if* $p_S < p < p_{JL}$, *the solution for* $\mu = \mu^*$ *is classical for* $t > 0$.

Proof. It is well known that $\mu^* \geqslant 1$, since for any $\mu \in (0,1)$ the problem (3.13) possesses a smooth forward self-similar solution of the form

$$u(x,t) = t^{-\frac{1}{p-1}} \tilde{\Psi}\left(\frac{|x|}{\sqrt{t}}\right), \tag{3.14}$$

where $\tilde{\Psi}(y)$ is a bounded positive solution of the equation

$$\tilde{\Psi}'' + \frac{N-1}{r}\tilde{\Psi}' + \frac{r}{2}\tilde{\Psi}' + \frac{1}{p-1}\tilde{\Psi} + |\tilde{\Psi}|^{p-1}\tilde{\Psi} = 0 \quad \text{for } 0 < r < \infty. \tag{3.15}$$

See [8, Lemma 10.3] for details. Furthermore, if $p \geqslant p_{JL}$, the solution of (3.13) blows up instantaneously for any $\mu > 1$ [8, Theorem 10.4], which means that $\mu^* = 1$ for this range of p. On the other hand, if $p_S < p < p_{JL}$, there exist smooth forward self-similar solutions of (3.13) for some finite range of $\mu > 1$ (see [23]); hence $\mu^* > 1$. It is also easily seen by a Kaplan type estimate that the solution of (3.13) cannot be globally smooth for $t > 0$ if μ is sufficiently large; hence $\mu^* < \infty$. This proves (i). The statement (ii) follows immediately from Lemma 3.3 below and the definition of μ^*. To prove (iii), choose a sequence $0 < \mu_k \nearrow \mu^*$ and let u_k be the solution of (3.13) for $\mu = \mu_k$ ($k = 1, 2, \ldots$). Then the solutions u_k are globally classical for $t > 0$ and $u_k \nearrow u$. Arguing as in the proof of Proposition B.1, we obtain an estimate of the form (B.4) with $T_1^* = \infty$. Hence the same estimate holds for u. Consequently, by Lemma 3.3 below, u is a smooth forward self-similar solution. The proof of the lemma is complete. □

Lemma 3.3. *Let $p > p_S$ and let u be the minimal solution of* (3.13) *with $\mu > 0$, $\mu \neq 1$. Then u is either a smooth forward self-similar solution of the form* (3.14) *or equal to $+\infty$ for a.e. $x \in \mathbf{R}^N$, $t > 0$.*

Proof. Suppose $u \not\equiv +\infty$ ($x \in \mathbf{R}^N,\ t > 0$). Then by Proposition B.1 in Appendix B, there exists $T^* \in (0, \infty]$ such that u is C^2 in $x \in \mathbf{R}^N \setminus \{0\}$, $0 < t < T^*$, and that (B.2) holds. Since $\lambda^{\frac{1}{p-1}} u(\sqrt{\lambda}x, \lambda t)$ is a minimal solution of (1.1)–(1.2) for the initial data $\lambda^{\frac{1}{p-1}} u(\sqrt{\lambda}x, 0)$, and since $\mu\varphi^*$ is invariant under the rescaling $v(x) \mapsto \lambda^{\frac{1}{p-1}} v(\sqrt{\lambda}x)$, we have

$$\lambda^{\frac{1}{p-1}} u(\sqrt{\lambda}x, \lambda t) \equiv u(x,t) \quad \text{for any } \lambda > 0.$$

Hence u can be written in the form (3.14), and the smoothness of u in $(\mathbf{R}^N \setminus \{0\}) \times (0, T^*)$ implies that $\tilde{\Psi}(r)$ is C^2 in $r > 0$. Hence $\tilde{\Psi}$ satisfies (3.15). From this expression it is also clear that $T^* = \infty$. Note that (B.2) implies

$$\tilde{\Psi}(r) \leqslant C\bigl(1 + r^{-\frac{2}{p-1}}\bigr)$$

for some constant $C > 0$. It is known that any positive solutions of (3.15) satisfying the above estimate is either Φ^* or a bounded solution. This can be shown, for example, by the same argument as in the proof of [16, Proposition A.1] for Eq. (1.9). If $\Psi = \Phi^*$, then $u(x,t) \equiv \varphi^*(x)$, but this contradicts the assumption that $\mu \neq 1$. Therefore Ψ is a bounded smooth solution of (3.15). The proof of the lemma is complete. □

Now we are ready to prove (3.7).

Proof of (3.7). For each $\varepsilon > 0$, denote by $u^\varepsilon(x,t) = U^\varepsilon(|x|,t)$ the solution of

$$\begin{cases} u_t = \Delta u + u^p & \bigl(x \in \mathbf{R}^N,\ t > 0\bigr), \\ u(x,0) = \min\bigl\{(1+\varepsilon)\mu^*\varphi^*(x), 1\bigr\} & \bigl(x \in \mathbf{R}^N\bigr), \end{cases} \tag{3.16}$$

where μ^* is as in Lemma 3.2. We first show that u^ε blows up in finite time. For that purpose, we define, for each $M > 0$,

$$u^{\varepsilon,M}(x,t) := M^{\frac{2}{p-1}} u^\varepsilon\bigl(Mx, M^2 t\bigr).$$

Then $u^{\varepsilon,M}$ satisfies the same equation as in (3.16) and

$$u^{\varepsilon,M}(x,0) = \min\bigl\{(1+\varepsilon)\mu^*\varphi^*(x), M^{\frac{2}{p-1}}\bigr\}.$$

Suppose that u^ε does not blow up in finite time. Then its rescaling $u^{\varepsilon,M}$ does not blow up either. Since $u^{\varepsilon,M}(x,0) \to (1+\varepsilon)\mu^*\varphi^*(x)$ as $M \to \infty$, the same argument as in the proof of Lemma 3.2(iii) shows that the solution of (3.13) with $\mu = (1+\varepsilon)\mu^*$ satisfies the estimate (B.2) with $T^* = \infty$, but this is impossible by Lemma 3.2(ii). This contradiction shows that u^ε blows up in finite time. We denote its blow-up time by T_ε.

Next we show that

$$T_\varepsilon \to \infty \quad \text{as } \varepsilon \to 0. \tag{3.17}$$

It suffices to verify that the solution of (3.16) with $\varepsilon = 0$ is globally classical for $t > 0$. In the case where $p \geqslant p_{JL}$, we have $\mu^* = 1$, hence $u^0(x,0) \leqslant \varphi^*(x)$, $u^0(x,0) \not\equiv \varphi^*(x)$. Consequently, by [8, Theorem 10.4(ii)], $u^0(x,t)$ is classical for $0 < t < \infty$. In the case where $p_S < p < p_{JL}$, the solution of (3.13) with $\mu = \mu^*$ is classical for $t > 0$ by Lemma 3.2(iii), therefore the same is true of u^0.

Now we denote by $w_0^\varepsilon(y,s)$ the rescaled solution corresponding to u^ε:

$$w_0^\varepsilon(y,s) := e^{-\frac{s}{p-1}} u^\varepsilon\bigl(e^{-\frac{s}{2}} y, T_\varepsilon - e^{-s}\bigr) = (T_\varepsilon - t)^{\frac{1}{p-1}} u^\varepsilon\bigl(\sqrt{T_\varepsilon - t}\, y, t\bigr),$$

where $s = -\log(T_\varepsilon - t)$. Since $U^\varepsilon(r,0)$ intersects $\Phi^*(r)$ only once, and since $U^\varepsilon(r,0)$ is monotone non-increasing, the blow-up occurs at $r=0$ and it is a single-intersection blow-up. Consequently,

$$\lim_{s\to\infty} w_0^\varepsilon(y,s) = \kappa.$$

It follows that

$$E\big(w_0^\varepsilon\big(\cdot, s_0^\varepsilon\big)\big) \geqslant E(\kappa). \tag{3.18}$$

Here $s_0^\varepsilon = -\log T_\varepsilon$ and

$$w_0^\varepsilon\big(y, s_0^\varepsilon\big) = (T_\varepsilon)^{\frac{1}{p-1}} u^\varepsilon(\sqrt{T_\varepsilon}\, y) = \min\big\{(1+\varepsilon)\mu^*\varphi^*(y), T_\varepsilon^{\frac{1}{p-1}}\big\},$$

where $\mu^* = 1$ for $p \in [p_{JL}, \infty)$ and $\mu^* > 1$ for $p \in (p_S, p_{JL})$. Letting $\varepsilon \to 0$ in (3.18) and recalling (3.17), we obtain

$$E\big(\mu^*\varphi^*\big) \geqslant E(\kappa). \tag{3.19}$$

Since the quantity

$$E\big(\mu\varphi^*\big) = \left(\frac{|\mu|^2}{2} - \frac{|\mu|^{p+1}}{p+1}\right)\int \rho\big(\varphi^*\big)^{p+1}\,dy$$

attains its maximum at $\mu = \pm 1$, we have $E(\varphi^*) \geqslant E(\mu^*\varphi^*) \geqslant E(\kappa)$. The proof of (3.7) is complete. □

3.4. Proof of Theorem 1.4 (Part 2)

It remains to prove the second part of Theorem 1.4, namely the assertion $E(\psi) = E(\kappa) \Rightarrow \psi = \pm\kappa$. We will instead prove the stronger statement (1.17).

Lemma 3.4. *The constant solution κ is isolated in $\mathcal{E}$. More precisely,*

$$\inf_{\psi\in\mathcal{E}\setminus\{\kappa\}} \big|\psi(0) - \kappa\big| > 0. \tag{3.20}$$

Proof. Suppose that (3.20) does not hold. Then there exists a sequence of bounded solutions $\psi_k(y) := \Psi_k(|y|) \not\equiv \kappa$ of (1.15) such that $\Psi_k(0) \to \kappa$ as $k \to \infty$. Since $\Psi_k'(0) = 0$, $\Psi_k(r)$ converges to κ locally uniformly in the C^1 sense in the region $r \geqslant 0$ as $k \to \infty$. Define

$$J_k(r) = \frac{1}{2}\big(\Psi_k'\big)^2 - \frac{1}{2(p-1)}\Psi_k^2 + \frac{1}{p+1}|\Psi_k|^{p+1}.$$

Then

$$J_k'(r) = -\left(\frac{N-1}{r} - \frac{r}{2}\right)\big(\Psi_k'\big)^2.$$

Therefore J_k is increasing in the interval $\sqrt{2(N-1)} \leqslant r < \infty$. Furthermore it is strictly increasing since Ψ_k' does not vanish on any interval of positive size. Since it is known that $\Psi_k(r) \to 0$ as $r \to \infty$, we have $J_k(r) \to 0$ as $r \to \infty$. Combining these, we obtain $J_k(r) < 0$ for $\sqrt{2(N-1)} \leqslant r < \infty$; hence

$$\Psi_k(r) \neq 0 \quad \text{for } \sqrt{2(N-1)} \leqslant r < \infty. \tag{3.21}$$

Next let $V_k(r) := r^{\frac{2}{p-1}} \Psi_k(r)$. Then V_k satisfies Eq. (A.2) in Appendix A. Define

$$\tilde{J}_k = \frac{1}{2}\big(r V_k'\big)^2 + G(V_k),$$

where

$$G(V) := -\frac{(c^*)^{p-1}}{2} V^2 + \frac{1}{p+1} |V|^{p+1}.$$

Then

$$\tilde{J}_k' = -\left(N - 2 - \frac{4}{p-1} - \frac{r^2}{2}\right) r \big(V_k'\big)^2.$$

Therefore $\tilde{J}_k$ is increasing in the interval $\sqrt{2(N-1)} \leqslant r < \infty$. Note that

$$G(0) = G(M) = 0, \qquad G(V) < 0 \quad (0 < v < M), \qquad G(V) > 0 \quad (M < v < \infty),$$

where $M = (\frac{p+1}{2})^{\frac{1}{p-1}} c^*$. Choose $r_0 \in [\sqrt{2(N-1)}, \infty)$ such that $\kappa r_0^{\frac{2}{p-1}} > M$. Since $\Psi_k \to \kappa$ in the C^1 sense,

$$V_k(r_0) > M, \qquad V_k'(r) > 0 \quad (0 < r \leqslant r_0) \quad \text{for all large } k. \tag{3.22}$$

Fix such k. As we see in the proof of Lemma A.1 in Appendix A, the function $\frac{r}{2}\Psi_k' + \frac{1}{p-1}\Psi_k$ changes sign; hence so does V_k'. Denote by $r_1 > 0$ the first zero of V_k'. Then, by (3.22), $r_1 > r_0$. Consequently $V_k(r_1) > M$; hence, by (A.2), $V_k''(r_1) < 0$.

Now suppose that V_k' vanishes for some $r > r_1$ and let r_2 be the smallest such r. Then, since $V_k > 0$ for $r \geqslant r_1$ by (3.21),

$$V_k'(r) < 0 \quad \text{for } r \in (r_1, r_2), \qquad 0 < V_k(r_2) < V_k(r_1).$$

Note that $G(V)$ is monotone increasing for $V \geqslant M$ and $G(V) < 0$ for $0 < V < M$. Therefore

$$\tilde{J}_k(r_1) = G\big(V_k(r_1)\big) > G\big(V_k(r_2)\big) = \tilde{J}_k(r_2),$$

but this is impossible by the monotonicity of $\tilde{J}_k(r)$. Hence V_k' does not vanish. Consequently V_k is strictly decreasing in the interval $[r_1, \infty)$ and it converges to some value α with $0 \leqslant \alpha < V_k(r_1)$. This convergence implies $\liminf_{r\to\infty} r|V_k'(r)| = 0$, therefore

$$\tilde{J}_k(r) \to G(\alpha) < G\big(V_k(r_1)\big) = \tilde{J}_k(r_1) \quad \text{as } r \to \infty,$$

which again contradicts the monotonicity of $\tilde{J}_k(r)$. The lemma is proved. □

Lemma 3.5. *For any $M > 0$ there exists $\delta_M > 0$ such that*

$$\int\limits_{|y|\leqslant 1} \big|\psi(y) - \kappa\big|^2\,dy \geqslant \delta_M \quad \text{for any } \psi \in \mathcal{E} \cup \big\{\pm\varphi^*\big\} \setminus \{\kappa\} \text{ with } E(\psi) \leqslant M. \tag{3.23}$$

Proof. Suppose that (3.23) does not hold for any some $M > 0$ and any $\delta_M > 0$. Then there exists a sequence $\psi_k \in \mathcal{E}$ $(k = 1, 2, 3, \dots)$ such that

$$E(\psi_k) \leqslant M \quad (k = 1, 2, 3, \dots), \qquad \lim_{k\to\infty} \int\limits_{|y|\leqslant 1} \big|\psi_k(y) - \kappa\big|^2\,dy = 0.$$

Since $E(\psi_k)$ is uniformly bounded, we have, by Theorem 3.1 and Corollary 3.2 of [16] (or Proposition 2.5 and Corollary 2.6 of [17]), that

$$\big|\nabla^j \psi_k(y)\big| \leqslant C\big(|y|^{-\frac{2}{p-1}-j} + 1\big) \quad \text{for } j = 0, 1, 2, 3,\ k = 1, 2, 3, \dots$$

for some constant $C > 0$. Hence by the Ascoli–Arzelà theorem, ψ_k converges to κ locally uniformly in $0 < |y| < 1$, while it remains uniformly bounded in $|y| \geqslant 1$. Applying the "no-needle lemma" (Lemma 2.14 of [17]) – or, more precisely, the argument used in the proof of this lemma – we see that the derivatives of ψ_k are uniformly bounded in $\mathbf{R}^N$. Consequently ψ_k converges to κ uniformly in $|y| \leqslant 1$. This contradicts Lemma 3.4, and the lemma is proved. □

Now we are ready to complete the proof of Theorem 1.4 by showing (1.17).

Proof of (1.17). Suppose that (1.17) does not hold. Then either of the following holds:

(a) $E(\varphi^*) = E(\kappa)$;
(b) there exists a sequence $\psi_k \in \mathcal{E} \setminus \{\pm\kappa, 0\}$ such that $E(\psi_k) \to E(\kappa)$ as $k \to \infty$.

Let us first consider the case (b) and derive a contradiction. The case (a) can be treated similarly with minor modification. Without loss of generality, we may assume that $\psi_k(0) > 0$ for all k, since $E(\psi) = E(-\psi)$. Define

$$M := \sup_{k\geqslant 1} E(\psi_k) + 1. \tag{3.24}$$

Next, for each k, we denote by $u_k(x,t)$ the solution of (1.1) on $\mathbf{R}^N$ whose initial data is given by (3.10) with $\psi = \psi_k$ and $\varepsilon = \varepsilon_k > 0$, where ε_k is chosen sufficiently small so that

$$M \geqslant E\big(w_k(\cdot, s_{0,k})\big) \to E(\kappa) \quad \text{as } k \to \infty. \tag{3.25}$$

Here $w_k(y,s)$ denotes the rescaled solution for u_k, namely

$$w_k(y,s) = e^{-\frac{s}{p-1}} u_k\big(e^{-\frac{s}{2}} y, T_k - e^{-s}\big),$$

where T_k is the blow-up time of u_k, and $s_{0,k} := -\log T_k$. In view of (3.23), we can choose ε_k small enough so that it satisfies, in addition to (3.25),

$$\int\limits_{|y|\leqslant 1} \big|w_k(\cdot, s_{0,k}) - \kappa\big|^2 \, dy \geqslant \frac{\delta_M}{2},$$

where δ_M is as in (3.23) with M as in (3.24). Now choose s_k^* such that

$$\int\limits_{|y|\leqslant 1} \big|w_k\big(y, s_k^*\big) - \kappa\big|^2 \, dy = \frac{\delta_M}{4}. \tag{3.26}$$

Such $s_k^* \in (s_{0,k}, \infty)$ exists since w_k converges to κ as $s \to \infty$ locally uniformly in $\mathbf{R}^N$ as shown in Section 3.2. Furthermore, since each ψ_k is a stationary solution, w_k stays close to ψ_k for a long time if ε_k is chosen to be very small. Therefore we may assume that

$$s_k^* \geqslant s_{0,k} + 1 \quad \text{for all } k = 1, 2, 3, \ldots. \tag{3.27}$$

Thus, by Theorem 3.1 and Corollary 3.2 of [16], we have, for all $j = 0, 1, 2, 3$ and $k = 1, 2, 3, \ldots,$

$$\big|\nabla_y^j w_k(y,s)\big| \leqslant C\big(|y|^{-\frac{2}{p-1} - j} + 1\big) \quad \text{for } y \in \mathbf{R}^N \setminus \{0\},\ s \geqslant s_k^*. \tag{3.28}$$

Hence, by the Ascoli–Arzelà theorem, we can choose a subsequence of $\{w_k(y, s + s_k^*)\}$, which we denote by $\{\tilde{w}_k(y,s)\}$, that converges to some function $\tilde{w}_\infty(y,s)$ on $(\mathbf{R}^N \setminus \{0\}) \times [0,\infty)$ locally uniformly in the C^2 sense in y and in the C^1 sense in s. Consequently, $\tilde{w}_\infty(y,s)$ satisfies Eq. (1.7) in $(\mathbf{R}^N \setminus \{0\}) \times [0,\infty)$.

Since (3.28) also implies that the sequence $\{\tilde{w}_k(y,0)\}$ is bounded in $H^1(\{|y| < 1\})$, hence compact in $L^2(\{|y| \leqslant 1\})$, the convergence $\tilde{w}_k(y,0) \to \tilde{w}_\infty(y,0)$ takes place in $L^2(\{|y| \leqslant 1\})$. This, together with (3.26), implies

$$\int\limits_{|y|\leqslant 1} \big|\tilde{w}_\infty(y,0) - \kappa\big|^2 \, dy = \frac{\delta_M}{4}. \tag{3.29}$$

Next we recall that $\tilde{w}_k(y,s) \to \kappa$ as $s \to \infty$ locally uniformly in $\mathbf{R}^N$. Since $\tilde{w}_k$ remains bounded along with its derivatives, we have $E(\tilde{w}_k(\cdot,s)) \to E(\kappa)$ as $s \to \infty$. This and (3.3) yield

$$\int\limits_0^\infty \int\limits_{\mathbf{R}^N} (\partial_s \tilde{w}_k)^2(y,s) \, dy \, ds = E\big(\tilde{w}_k(\cdot,0)\big) - E(\kappa).$$

Combining this and (3.25), and using Fatou's lemma, we obtain

$$\int_0^\infty \int_{\mathbf{R}^N} (\partial_s \tilde{w}_\infty)^2(y,s)\,dy\,ds = 0.$$

This implies that $\tilde{w}_\infty(y,s)$ is a stationary solution of (1.7) in $(\mathbf{R}^N \setminus \{0\}) \times [0,\infty)$. Hence $\psi(y) := \tilde{w}_\infty(y,0)$ satisfies Eq. (1.15) for $y \neq 0$ along with the estimate $|\tilde{w}_\infty(y,0)| \leqslant C(|y|^{-\frac{2}{p-1}} + 1)$. Thus, by Proposition A.1 of [16], $\tilde{w}_\infty(y,0)$ is either a bounded solution of (1.15) on $\mathbf{R}^N$ or $\tilde{w}_\infty(y,0) = \pm\varphi^*(y)$. But this is impossible by virtue of Lemma 3.5 and (3.26). This contradiction shows that the case (b) above does not occur.

It remains to show that the case (a) does not occur. This follows immediately from (3.19) if $p_S < p < p_{JL}$, since in this case $\mu^* > 1$, hence $E(\varphi^*) > E(\mu^*\varphi^*) \geqslant E(\kappa)$. In the range $p_{JL} \leqslant p < \infty$, we have $\mu^* = 1$. In this case, basically the same argument as for the case (b) above applies, by setting $\psi_k = \varphi^*$ ($k = 1, 2, 3, \ldots$) and replacing (3.10) by (3.16) in the definition of u_k. The only major difference is that φ^* is a singular stationary solution that does not belong to the space where Eq. (1.7) is well-posed, therefore deriving (3.27) is not as straightforward as in the case (b), where ψ_k is a smooth stationary solution. Nonetheless, by a simple subsolution argument, one can easily derive an inequality of the form

$$s_k^* \geqslant s_{0,k} + \delta \quad \text{for all } k = 1, 2, 3, \ldots$$

for some constant $\delta > 0$, and this is sufficient to derive the estimate (3.28). This shows that the case (a) above does not occur. The proof of the theorem is complete. □

3.5. Proof of corollaries

Proof of Corollary 1.6. By Lemma 3.1 and the assumption, we have

$$E(w_a^*) < E(\kappa).$$

In the case where $a \neq 0$, the above inequality immediately implies no blow-up at $x = a$, since we would have $w_a^* = \kappa$ if a is a blow-up point. On the other hand, if $a = 0$, we recall that w_0^* belongs to $\mathcal{E} \cup \{\pm\varphi^*\}$ by [17, Theorem 3.1]. Therefore, by Theorem 1.4, $w_0^* = 0$. As we see in the proof of [17, Theorem 4.1], $w_0^* = 0$ implies no blow-up at $x = 0$. This completes the proof of the corollary. □

Proof of Corollary 1.7. Let a be an arbitrary point of $\Omega \setminus D^\varepsilon$. Then

$$\bigl|v(x)\bigr| \leqslant \lambda(1-\varepsilon)\|v\|_{L^\infty} \quad \text{for } |x-a| \leqslant \varepsilon. \tag{3.30}$$

From (2.8) we see that

$$T_\lambda \approx \frac{\lambda^{-(p-1)}}{p-1}\bigl(\|v\|_{L^\infty}\bigr)^{-(p-1)}.$$

Denote by $w_a^\lambda(y,s)$ the rescaled solution at $x=a$, namely

$$w_a^\lambda(y,s)=(T_\lambda-t)^{\frac{1}{p-1}}u^\lambda(a+\sqrt{T_\lambda-t}\,y,t),$$

where $s=-\log(T_\lambda-t)$. Its initial data is given by

$$w_a^\lambda(y,-\log T_\lambda)=T_\lambda^{\frac{1}{p-1}}u^\lambda(a+\sqrt{T_\lambda}\,y,0)=\lambda T_\lambda^{\frac{1}{p-1}}v(a+\sqrt{T_\lambda}\,y).$$

Consequently

$$w_a^\lambda(y,-\log T_\lambda)\to\frac{v(a)\kappa}{\|v\|_{L^\infty}}\leqslant(1-\varepsilon)\kappa\quad\text{as }\lambda\to\infty$$

locally uniformly in $y\in\mathbf{R}^N$. Furthermore, by (3.30),

$$\left|\frac{v(a)\kappa}{\|v\|_{L^\infty}}\right|\leqslant(1-\varepsilon)\kappa.$$

Thus, for each fixed $\delta>0$,

$$\limsup_{\lambda\to\infty}\bigl|w_a^\lambda(y,-\log T_\lambda+\delta)\bigr|\leqslant\eta(\delta)\quad\text{locally uniformly},$$

where $\eta(t)$ is the solution of the following initial value problem:

$$\frac{d\eta}{dt}=-\frac{1}{p-1}\eta+\eta^p,\qquad\eta(0)=(1-\varepsilon)\kappa.$$

Since $0\leqslant\eta(0)<\kappa$ by the assumption, we have $0\leqslant\eta(\delta)<\kappa$, hence

$$\limsup_{\lambda\to\infty}E\bigl(w_a^\lambda(\cdot,-\log T_\lambda+\delta)\bigr)\leqslant E\bigl(\eta(\delta)\bigr)<E(\kappa).\tag{3.31}$$

Consequently $E(w_a^\lambda(\cdot,-\log T_\lambda+\delta))<E(\kappa)$ for all large λ, which, by Corollary 1.6, implies that u^λ cannot blow up at $x=a$. The above estimates are all uniform in $a\in\Omega\setminus D^\varepsilon$, therefore we obtain the desired conclusion. The corollary is proved. □

Appendix A. Stationary solution with a single intersection

In this appendix, we prove the following lemma due to [2]. We have used this lemma to show that the local profile of a single-intersection blow-up is $\pm\kappa$. Since this lemma is of central importance in the proof of Theorem 1.4, and since part of the arguments in [2] rely on extra assumptions which we do not assume in the present paper, we give our own proof for the convenience of the reader.

Lemma A.1. *(See [2, Lemma 3.28].) Let $p_S < p < \infty$. Then the only bounded positive solution of* (1.9) *that satisfies $\mathcal{Z}_{(0,\infty)}[\Psi - \Phi^*] = 1$ is κ.*

Proof. Let $\Psi(r)$ be a bounded solution of (1.9) satisfying $\mathcal{Z}_{(0,\infty)}[\Psi - \Phi^*] = 1$ and set

$$Q(r) := \frac{r}{2}\Psi'(r) + \frac{1}{p-1}\Psi(r).$$

Then, as shown in [2], Q satisfies

$$Q'' + \frac{N-1}{r}Q' - \frac{r}{2}Q' + \big(f'(\Psi) - 1\big)Q = 0 \quad (0 < r < \infty),$$

along with the boundary condition $Q(0) > 0$, $Q'(0) = 0$, where $f(u) = -\frac{1}{p-1}u + |u|^{p-1}u$. Equivalently, the function $q(y) := Q(|y|)$ is a bounded radially symmetric solution of

$$\rho^{-1}\nabla \cdot (\rho \nabla q) + \big(f'(\psi) - 1\big)q = 0 \quad \big(y \in \mathbf{R}^N\big),$$

where ρ is as in (3.2). Since $\Psi \not\equiv \Phi^*$, we have $Q \not\equiv 0$. Therefore q is an eigenfunction of the self-adjoint operator

$$\mathcal{L}\varphi := -\rho^{-1}\nabla \cdot (\rho \nabla \varphi) - \big(f'(\psi) - 1\big)\varphi$$

in the weighted space $L^2_\rho(\mathbf{R}^N)$ corresponding to the eigenvalue 0. The first eigenvalue of this operator is given by

$$\mu_1 = \inf_{\varphi \in H^1_\rho \setminus \{0\}} \frac{\mathcal{H}_\psi(\varphi)}{\int_{\mathbf{R}^N} \rho\varphi^2\,dy}, \qquad \mathcal{H}_\psi(\varphi) := \int_{\mathbf{R}^N} \rho\big(|\nabla\varphi|^2 - \big(f'(\psi) - 1\big)\varphi^2\big)\,dy.$$

Here we note that the spectrum of $\mathcal{L}$ consists of eigenvalues, which can be easily seen by using the orthonormal basis of the Hilbert space L^2_ρ consisting of Hermite polynomials – or, more precisely, products of Hermite polynomials of the form $H_1(x_1)H_2(x_2)\cdots H_N(x_n)$ (see, for example, [6]).

Now let us suppose that $\psi \not\equiv \kappa$. Then

$$\mathcal{H}_\psi(\psi) = \int_{\mathbf{R}^N} \rho\left(|\nabla\psi|^2 - p|\psi|^{p+1} + \frac{p}{p-1}\psi^2\right)dy = -(p-1)\int_{\mathbf{R}^N} \rho|\nabla\psi|^2\,dy < 0,$$

which implies $\mu_1 < 0$. Thus 0 is not the principal eigenvalue of $\mathcal{L}$. Consequently q (hence Q) changes sign. Denote by $r_0 > 0$ the first zero of $Q(r)$. Then

$$Q(0) > 0 \quad (0 \leqslant r < r_0), \qquad Q(r_0) = 0. \tag{A.1}$$

Let $V(r) := r^{\frac{2}{p-1}}\Psi(r)$. Then V satisfies

$$V'' + \left(N - 1 - \frac{4}{p-1} - \frac{r^2}{2}\right)\frac{1}{r}V' + \frac{1}{r^2}\left(-\left(c^*\right)^{p-1}V + V^p\right) = 0 \quad (0 < r < \infty), \tag{A.2}$$

where c^* is as in (1.10). Since $V' = 2r^{\frac{2}{p-1}-1}Q$, (A.1) implies $V' > 0$ $(0 < r < r_0)$ and $V'(r_0) = 0$. Thus $V''(r_0) \leqslant 0$. Since V is not a constant solution of (A.2), V' and V'' cannot vanish simultaneously, therefore $V''(r_0) < 0$. This and (A.2) yield

$$-\left(c^*\right)^{p-1}V(r_0) + V^p(r_0) > 0,$$

hence $V(r_0) > c^*$. Since $\mathcal{Z}_{(0,\infty)}[V - c^*] = \mathcal{Z}_{(0,\infty)}[\Psi - \Phi^*] = 1$ by the assumption, and since V intersects c^* once in the interval $0 < r < r_0$, we see that $V(r) > c^*$ for $r \geqslant r_0$. Thus

$$-\left(c^*\right)^{p-1}V + V^p > 0 \quad \text{for } r \geqslant r_0.$$

This and (A.2) imply

$$V'' + \left(N - 1 - \frac{4}{p-1} - \frac{r^2}{2}\right)\frac{1}{r}V' < 0 \quad \text{for } r \geqslant r_0. \tag{A.3}$$

Since $V'(r_0) = 0$ and $V''(r_0) < 0$, we have $V'(r) < 0$ on some interval $(r_0, r_0 + \varepsilon)$. This and (A.3) imply that $V'(r) < 0$ for $r > r_0$. Hence, again by (A.3), V is concave and monotone decreasing for all large r. However, this is impossible since $V > c^*$ for $r > r_0$. This contradiction shows that $\psi \equiv \kappa$. The lemma is proved. □

Appendix B. Solutions with singular initial data

The following lemma deals with solutions with singular initial data. This lemma is an immediate consequence of the basic estimates found in our earlier papers [16,17]. We have used it in the proof of Lemma 3.2 of the present paper.

Proposition B.1. *Let $p_S < p < \infty$, and let $u_0 \in H^1_{loc}(\Omega)$ be a radially symmetric function satisfying*

$$u_0 \geqslant 0, \qquad \int_\Omega e^{-\alpha|x|^2}\left(|\nabla u_0|^2 + u_0^2\right)dx < \infty \tag{B.1}$$

for some $\alpha \geqslant 0$. Then one of the following holds for the minimal solution of (1.1)–(1.2).

(a) *$u(x,t) = +\infty$ for a.e. $x \in \mathbf{R}^N$, $t > 0$ (instantaneous blow-up);*
(b) *there exists $0 < T^* < \infty$ such that $u(\cdot,t) \in C^2(\Omega \setminus \{0\})$ for $0 < t < T^*$ and that*

$$0 \leqslant u(x,t) \leqslant C_t\left(1 + |x|^{-\frac{2}{p-1}}\right) \quad \text{for } x \in \Omega \setminus \{0\},\ 0 < t < T^*, \tag{B.2}$$

where C_t is a constant depending on t and u_0. Furthermore, $u(x,t) = +\infty$ for a.e. $x \in \mathbf{R}^N$, $t > T^*$;

(c) *$u(\cdot,t) \in C^2(\Omega \setminus \{0\})$ for $0 < t < \infty$ and* (B.2) *holds with $T^* = \infty$. If, in addition, $u_0 \in H^1(\Omega)$, then u is smooth for all large t and $\|u(\cdot,t)\|_{L^\infty} \to 0$ as $t \to \infty$.*

Proof. Let $u_k(x,t)$ $(k = 1,2,3,\ldots)$ be the solution of (1.1)–(1.2) for the initial data

$$u_k(x,0) = u_{k,0}(x) := \min\{u_0(x), k\}.$$

Denote by $T(u_k)$ the blow-up time of u_k. Here we set $T(u_k) = \infty$ if u_k does not blow up. We use the same symbol u_k for the minimal extension of u_k beyond $t = T(u_k)$. Let $T_c(u_k)$ denote the complete blow-up time of u_k (here, again, we set $T_c(u_k) = \infty$ if u_k never becomes identically $+\infty$ a.e. over Ω). By [14, Theorem 2], we have

$$T(u_1) > T_c(u_2) \geqslant T(u_2) > T_c(u_3) \geqslant T(u_3) > T_c(u_4) \geqslant \cdots.$$

Hence the following limit exists:

$$T^* = \lim_{k\to\infty} T(u_k) \Big(= \lim_{k\to\infty} T_c(u_k) \Big).$$

If $T^* < \infty$, then, since $u \geqslant u^k \equiv +\infty$ for a.e. $x \in \Omega$, $t > T_c(u^k)$, we have

$$u(x,t) = +\infty \quad \text{a.e. } x \in \Omega,\ t > T^*. \tag{B.3}$$

Thus the alternative (a) holds if $T^* = 0$. In what follows we assume $T^* > 0$.

Let α be the constant in (B.1), and set

$$T_1^* := \min\{T^*, \alpha^{-2}\},$$

where we set $T_1^* = T^*$ if $\alpha = 0$. Define a rescaled solution $w_k(y,s)$ by

$$w_k(y,s) = \big(T_1^* - t\big)^{\frac{1}{p-1}} u_k\big(\sqrt{T_1^* - t}\, y, t\big),$$

where $s = -\log(T_1^* - t)$. The initial data of w_k is written in the form

$$w_k\big(y, -\log T_1^*\big) = \big(T_1^*\big)^{\frac{1}{p-1}} u_{k,0}\big(\sqrt{T_1^*}\, y\big),$$

hence its initial energy is given by

$$\begin{aligned} &E\big(w_k\big(\cdot, -\log T_1^*\big)\big) \\ &\quad = \big(T_1^*\big)^{\frac{p+1}{p-1} - \frac{N}{2}} \int\limits_{\mathbf{R}^N} \left(\frac{1}{2}|\nabla u_{k,0}|^2 + \frac{1}{2(p-1)T_1^*}|u_{k,0}|^2 - \frac{1}{p+1}|u_{k,0}|^{p+1} \right) \rho\left(\frac{x}{\sqrt{T_1^*}} \right) dx. \end{aligned}$$

Thus the assumption (B.1) implies that

$$E\big(w_k\big(\cdot, -\log T_1^*\big)\big) \leqslant C \quad (k = 1,2,3,\ldots),$$

where C is a constant independent of k. In view of this and the fact that u_k is defined for all $t \in [0, T_1^*)$, we see from Theorem 3.1 and Remark 3.5 of [16] that there exists a constant C_t – depending on t but independent of k – such that

$$0 \leqslant u_k(x,t) \leqslant C_t\bigl(1 + |x|^{-\frac{2}{p-1}}\bigr) \quad \text{for } x \in \Omega \setminus \{0\},\ 0 < t < T_1^*. \tag{B.4}$$

Letting $k \to \infty$, we obtain (B.2) for the range $0 < t < T_1^*$. Now we fix $t_0 \in (0, T_1^*)$ arbitrarily. Then by (B.2) and the assumption $p > p_S$,

$$\int_\Omega e^{-\beta|x|^2}\bigl(\bigl|\nabla u(x,t_0)\bigr|^2 + u^2(x,t_0)\bigr)\,dx < \infty$$

for any $\beta > 0$. Replacing $u_0(x)$ by $u(x,t_0)$ and repeating the above argument, we see that (B.2) holds for all $t \in (0, T^*)$. The C^2 smoothness in $x \in \Omega \setminus \{0\}$ follows from parabolic estimates. This and (B.3) confirm (b) as well as the first part of (c). The second part of (c) follows from the same argument as in the proof of [17, Theorem 5.14]. More precisely, assume $u \in H^1(\Omega)$ and $T^* = \infty$. Then, by [16, Remark 3.5], the constant C_t in (B.2) tends to 0 as $t \to \infty$. Hence, by [17, Lemma 2.9], $\|u(\cdot,t)\|_{L^\infty} \to 0$. The proof of the proposition is complete. □

Appendix C. Uniform decay at infinity

Here we make a more precise statement of the estimates mentioned in Remark 2.6. The proof follows easily by combining known estimates.

Proposition C.1. *Let u be a radially symmetric solution of* (1.1) *that blows up at $t = T$. Assume that*

$$u_0(x) = o\bigl(|x|^{-\alpha}\bigr) \quad \textit{for some } \alpha \geqslant 0 \quad \bigl(\textit{resp. } u_0(x) = O\bigl(|x|^{-\alpha}\bigr) \textit{ for some } \alpha > 0\bigr) \tag{C.1}$$

as $|x| \to \infty$. Then

$$u(x,t) = o\bigl(|x|^{-\alpha}\bigr) \quad \bigl(\textit{resp. } u(x,t) = O\bigl(|x|^{-\alpha}\bigr)\bigr) \quad \textit{as } |x| \to \infty \tag{C.2}$$

uniformly in $t \in [0, T)$

Proof. We only prove the former estimate $o(|x|^{-\alpha})$ as the latter can be shown similarly. By the decay estimate of the heat kernel as $|x| \to \infty$, there exists $t_0 \in (0, T)$ such that

$$\begin{cases} \lim\limits_{|x|\to\infty} |x|^\alpha u(x,t) = 0, \\ \lim\limits_{|x|\to\infty} |x|^{\alpha+1}\bigl|\nabla_x u(x,t)\bigr| = 0 \end{cases} \quad \text{uniformly in } t \in [t_0, T). \tag{C.3}$$

Now we set $t_1 = t_0/2$ and take $u(x,t_1)$ as the initial data for (1.1) (with blow-up time $T - t_1$), and denote by $w_a(y,s)$ the corresponding rescaled solution at $x = a$ as defined in (1.6). Then

(C.3) implies

$$E\big(w_a(\cdot,s_1)\big)=o\big(|a|^{-2\alpha}\big) \quad \text{as } |a|\to\infty,$$

where $s_1:=-\log(T-t_1)$. By Lemma 2.2 of [17], the above estimate yields

$$\big|w_a(\cdot,s)\big|=O\big(e^{-\frac{s}{p-1}}E\big(w_a(\cdot,s_1)\big)^{\frac{1}{p+1}}\big)=o\big(e^{-\frac{s}{p-1}}|a|^{-\frac{2\alpha}{p+1}}\big)$$

uniformly in $s\in[s_0,\infty)$, where $s_0=-\log(T-t_0)>s_1$. This implies

$$\big|u(a,t)\big|=o\big(|a|^{-\frac{2\alpha}{p+1}}\big) \quad \text{uniformly in } t\in[t_0,T). \tag{C.4}$$

Consequently, $u(x,t)$ is uniformly bounded in $|x|\geqslant R$, $0\leqslant t<T$ for some $R>0$. Combining this and (C.3) and using again the decay estimate of the heat kernel as $|x|\to\infty$, we easily obtain (C.2). □

Appendix D. Comparison of the blow-up time

The following lemma is used in Section 2.1 to show (2.4). This is essentially the same result as Lemma 1 of [14], in which the positivity of the solution is assumed. Since we do not assume positivity, we give a proof to this lemma for the self-containedness of the present paper. Since the lemma is of interest in its own right, we present it in a rather general setting without assuming radial symmetry.

Lemma D.1. *Let Ω be a domain in $\mathbf{R}^N$ with smooth boundary, and let u, $\tilde{u}$ be solutions of* (1.1)–(1.2) *with $u_0,\tilde{u}_0\in L^\infty(\Omega)\cap C(\overline{\Omega})$, $u_0(x)\leqslant\tilde{u}_0(x)$, $u_0\not\equiv\tilde{u}_0$. Suppose that u blows up at $t=T$ and that its blow-up points are confined in a bounded set. Furthermore, assume that*

$$\lim_{t\to T}\max_{x\in\Omega}u(x,t)=+\infty. \tag{D.1}$$

Then $\tilde{u}$ blows up at some $\tilde{T}$ with $0<\tilde{T}<T$. (Here u_0, $\tilde{u}_0$ need not be radial.)

Proof. We modify the proof of [14, Lemma 1] to allow sign-changing solutions. By the assumption, there exists a finite ball B such that u is uniformly bounded in $\Omega\setminus B$ as $t\to T$. We choose a slightly larger ball B' such that $\overline{B}\subset B'$. For each $0\leqslant t<T$, define

$$D(t):=\big\{x\in B' \,\big|\, u(x,t)>0\big\}.$$

By the comparison principle we have $u\leqslant\tilde{u}$, therefore, by (D.1), $\tilde{u}$ blows up at some $\tilde{T}$ with $0<\tilde{T}\leqslant T$. Let

$$v(x,t):=\frac{1}{u^p(x,t)}, \qquad \tilde{v}(x,t)=\frac{1}{\tilde{u}^p(x,t)}, \qquad z:=v-\tilde{v}$$

for $x\in D(t)$, $0\leqslant t<\tilde{T}$. A simple calculation shows

$$z_t = \Delta z - \frac{p}{p-1}\frac{(\nabla v + \nabla \tilde{v})}{v}\cdot \nabla z + \frac{p}{p-1}\frac{|\nabla \tilde{v}|^2}{v\tilde{v}} z$$
$$\geqslant \Delta z - \frac{p}{p-1}\frac{(\nabla v + \nabla \tilde{v})}{v}\cdot \nabla z. \tag{D.2}$$

Thus, by the strong maximum principle, we have $u(x,t) < \tilde{u}(x,t)$ for $x \in \Omega$, $0 < t < \tilde{T}$. Hence $z > 0$ in $x \in D(t)$, $0 < t < \tilde{T}$.

Now fix $T_1 \in (0, \tilde{T})$ arbitrarily. Since $\tilde{u} - u$ satisfies

$$(\tilde{u} - u)_t = \Delta(\tilde{u} - u) + |\tilde{u}|^{p-1}\tilde{u} - |u|^{p-1}u \geqslant \Delta(\tilde{u} - u)$$

along with the Dirichlet boundary condition on $\partial\Omega$, the Hopf boundary lemma implies

$$\tilde{u}(x,t) - u(x,t) \geqslant C\, d(x) \quad \text{for } x \in \Omega \cap B',\ T_1 \leqslant t < \tilde{T}$$

for some constant $C > 0$, where $d(x)$ denotes the distance from x to $\partial\Omega$. This yields an estimate of the form

$$z(x,t) \geqslant C' u^{-p}(x,t)\, d(x) \quad \text{for } x \in D(t),\ T_1 \leqslant t < \tilde{T} \tag{D.3}$$

for some constant $C' > 0$.

Let x_0 be an arbitrary point on $\partial D(t)$. Then one of the following holds: $x_0 \in \Omega$, $u(x_0,t) = 0$ or $x_0 \in \partial\Omega \cap \partial D(t)$ or $x_0 \in \Omega \cap \partial B'$. In the first case, since $0 = u(x_0,t) < \tilde{u}(x_0,t)$, we have

$$z(x,t) \to \infty \quad \text{as } x \to x_0,\ x \in D(t). \tag{D.4}$$

In the second case, we again have (D.4) for each $t \in [T_1, \tilde{T})$, by virtue of (D.3) and the fact that $u(x,t) = O(d(x))$. In the third case, since u is uniformly bounded in $B' \setminus B$, the estimate (D.3) implies

$$\liminf_{x \to x_0,\, x \in D(t)} z(x,t) \geqslant \delta > 0$$

for some constant $\delta > 0$ that is independent of $t \in [T_1, \tilde{T})$. Combining these, we obtain

$$\liminf_{x \to \partial D(t)} z(x,t) \geqslant \delta \quad \text{for } t \in [T_1, \tilde{T}). \tag{D.5}$$

Since $z(x,T_1) > 0$ in $D(T_1)$, this and (D.5) imply that $z(x,T_1) \geqslant \delta'$ in $D(T_1)$ for some constant $0 < \delta' \leqslant \delta$. In view of this and (D.5), and recalling that z satisfies (D.2), we see by the maximum principle that

$$z(x,t) \geqslant \delta' \quad \text{for } x \in D(t),\ T_1 \leqslant t < \tilde{T}.$$

It follows that $u^{p-1} \leqslant (\delta')^{-1}$ for $T_1 \leqslant t < \tilde{T}$, hence u cannot blow up at $t = \tilde{T}$. Consequently $T > \tilde{T}$. The proof of the lemma is complete. □

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