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An Extension of a Theorem of L. O'Raifeartaigh*

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1. INTRODUCTION AND PHYSICAL BACKGROUND

In a recent article [6], a classical method in the theory of Lie algebras has been applied to obtain a strong apparent limitation on the possible derivation of mass differences through the use of a model in which the Poincaré group and an internal symmetry group are integrated in comparatively unrestricted fashion. Since this type of model is an ultimate form of that indicated by the metaphysics associated with recent developments in particle classification, such as the use of $SU(6)$, it suggests the rather important general conclusion that mass differences, as between the baryons, cannot be theoretically derived from the exploration of any such model.

On the other hand, there are two types of serious limitation on this conclusion. First, physically, as emphasized already by O'Raifeartaigh, there are a great many slight modifications and alternative interpretations which could, so far as is now established, lead to theoretical determinations of apparent mass differences. It is not the purpose of this article to develop the physics of the subject, but a few preliminary remarks on the matter may be helpful. It must be freely admitted that there are, at the present quite phenomenological stage of our understanding of the elementary particles, many and diverse theoretical frameworks for their classification, none of which can be definitively discounted. We mention only a few possibilities in this direction.

(i) the observed particles may be mathematically represented not by isolated points in the spectrum of the mass operator, which are dealt with by the O'Raifeartaigh theorem, but perhaps by packets in the continuous spectrum of a sufficiently short mass-spread to be empirically indistinguishable from states of exact mass.

(ii) the Poincaré group should perhaps be replaced by another

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group of which it is a "limit" in the sense of [*7*] (i.e., a partially degenerate form, or "contraction" in the terminology later introduced by Wigner and Inonu); this means that the total symmetry group might contain a set of generators whose commutation relations differ arbitrarily little from those of the generators of the Poincaré group; in the examples described at the end of [*10*], the mass spectrum may indeed be discrete, but the "geometrical" group merely approximates the Poincaré group.

(iii) quite another theoretical definition of mass might be appropriate. The idea that mass splitting arises from a broken symmetry can, e.g., be formulated as follows. The full collection of single-particle observables may be defined as the ring of operators generated by the given group representation; the "geometrical" observables form surely a subring; this subring is usually taken implicitly as that generated by the restriction of the representation to the Poincaré group (or its universal covering group); however, the central element of the universal covering group has no nontrivial action on Minkowski space, and conceivably there exist other operators without a direct connection with Minkowski space, which nevertheless are "geometrical," in the sense that they naturally cohere with the action of the Poincaré group; unlike the central element in question, they might affect the energy spectrum. More specifically, given a particular presumptive "geometrical" subring of the entire ring of single-particle observables, the corresponding mass spectrum is the set of all infimums of the eigenvalues of the restrictions of the energy operator to the invariant subspaces under this subring. This represents a mathematical transcription of the physical idea that the mass is the energy level of a particle at rest ("rest" being defined as that state in which the energy is least) relative to the kinematics in question; these kinematics are here specified by the designation of the "geometrical subring." In the conventional case, in which the subring is derived from the Poincaré group in the fashion indicated, the usual relativistic definition of mass is naturally reproduced; but in general the mass would not necessarily correspond directly to any particular element of the enveloping algebra of the Lie algebra of the symmetry group.

(iv) conceivably, a nonunitary representation might be involved; the proof of the O'Raifeartaigh theorem uses the unitarity of the representation; in part, this condition can be relaxed, but the unitarity of the restriction of the representation to the space-time translation subgroup, or some condition in this direction, seems essential for the validity of the conclusion.

Secondly, on the mathematical side, what is actually established in [*6*] is infinitesimal invariance of the eigenspace associated with a given point in the mass spectrum, under the infinitesimal Hermitian generators, under the finite unitary action of the global symmetry group. What is however physically most relevant is such global invariance. Conceivably, there could exist no nonzero vector in the eigenspace in question which is in the domains of the unbounded Hermitian generators; or, a dense invariant submanifold of such regular vectors might exist in the eigenspace, while under the finite unitary action of the group, the eigenspace is nevertheless not invariant. Certainly many infinitesimal results on Lie algebras of Hermitian operators in Hilbert spaces are known, for which formally analogous conclusions about the global action of the associated groups are simply quite false. Indeed, such difficulties with the treatment of unbounded operators in the O'Raifeartaigh Theorem have led to some doubts about its effective validity [*2*].

Partly despite and partly because of these qualifications regarding the O'Raifeartaigh Theorem and its scientific context, it remains of interest from a physical and a mathematical viewpoint. We aim here to treat the corresponding global question in a rigorous mathematical way. We show actually that the eigenspaces of the mass operator, in the situation considered by O'Raifeartaigh, are indeed globally invariant, at least when the eigenvalue in question is isolated. Indeed the Theorem is valid in considerably strengthened and generalized form, under a certain regularity hypothesis; the Poincaré group may be replaced by an arbitrary Lie subgroup, and the mass operator by an arbitrary central element of the enveloping algebra of the Lie algebra of the subgroup, which acts in a similarly nilpotent fashion on the enveloping algebra of the full group. This is encouraging from a purely mathematical standpoint by its provision of further evidence that Lie-theoretical arguments can in some relatively complex cases be exponentiated to conclusions regarding continuous infinite-dimensional representations of general Lie groups.

Physically, the theorem provides a speck of theoretical certainty in a vast uncharted sea of uncertainty. It is reassuring, at least, that it is quite consistent with what has been suggested repeatedly during the past forty years, namely that radically new notions of space and time are needed; in particular, the assumption that the Poincaré group is the geometrical group of particle theory may well be valid only as an approximation similar to that of the relativistic quantum theory by the nonrelativistic one. The O'Raifeartaigh theorem depends primarily on the essential nilpotency of the action of the linear momenta on any group containing the Poincaré group; such nilpotency is quite unstable with respect to relatively small changes in the geometrical group or algebra.

The proof depends crucially on the essential normality of the "mass" and other operators, and on the existence of suitably regular vectors in the representation space. Such matters are considered under general circumstances, with increasing degrees of specialization. Theorem 1, below, applies to any representation of a Lie group in a Banach space which together with its contragredient is continuous; Theorem 2 deals with the case of a unitary representation in a Hilbert space.

The situation is then specialized to a context including, in addition to a continuous representation U of a Lie group G in a Hilbert space (considered in the preceding section), a Lie subgroup G' of G and a central element M of the (universal) enveloping algebra $\mathbf{E}(G')$ of the Lie algebra $\mathscr{G}$ of G'. The crucial hypothesis here is that of the nilpotent action of ad $\mathscr{G}'$ on $\mathbf{E}(G)$; in addition, our analytical methods require the assumption of the unitary of the restriction of U to G', and of a type of fairly mild regularity for the eigenspace in question of the operator representing M. Theorem 3 asserts that under hypotheses such as those indicated, the eigenspace is invariant not only under $U(G')$ but under all of $U(G)$. Further specialization to the case in which G' is Abelian leads to Theorem 4, essentially a corollary to Theorem 3. In case G' is the vector-displacement subgroup of a pseudo-Euclidean group G'', assumed nested between G' and G: $G' \subset G'' \subset G$, the nilpotency and regularity conditions hold automatically for an isolated eigenvalue, as stated in Corollary 4.1. When G'' is specialized to the Poincaré group and the polynomial in the generators of the vector displacements is taken as the conventional mass operator, the global form of the O'Raifeartaigh theorem is obtained.

2. Unbounded Operators Associated with Representations of Lie Groups

We begin by establishing notation and recalling some known results. For any Lie group G, we denote the Lie algebra (of all right- or left-invariant vector fields on the manifold of G) by the corresponding script letter, as: $\mathscr{G}$. Let V denote an arbitrary strongly (or equivalently, weakly) continuous representation of G by invertible linear operators on the Banach space $\mathbf{B}$. If $[g(t) : t \text{ real}]$ is any continuous one-parameter subgroup of G, then $[V(g(t)) : t \text{ real}]$ is a strongly continuous one-parameter group of operators on **B** whose infinitesimal generator (cf. [*3*]) will be denoted $v'(X)$, where X is the element of $\mathscr{G}$ which generates $[g(t) : t \text{ real}]$. We recall that the domain of $v'(X)$ is dense, and includes all vectors of the form $\int V(a)xf(a)\,da$, where x is an arbitrary vector in **G**, f is any element of the collection $\mathbf{C}_0^\infty$ of all infinitely differentiable functions on G of compact support, da is the element of left-invariant (Haar) measure on G, and the integral may be taken in either the strong or the weak sense. Writing $L_f = \int_G V(a)f(a)\,da$, the products of $v'(X)$ with L_f are readily computed as

$$v'(X)L_f = L_{l(X)f}\,, \quad L_f v'(X) = -L_{r(X)f}\,,$$

where $l(X)$ and $r(X)$ are respectively the right- and left-invariant vector fields on G corresponding to the element X of $\mathscr{G}$ [note that $r(X)$ is the transform of $l(X)$ under the induced action on vector fields of the transformation $a \to a^{-1}$ on G, and that $r(X)_e = -\,l(X)_e$, where e is the group unit]. The "maximal domain" **D** for V is defined as the maximal linear subset of **B** contained in the domains of all the $v'(X)$, $X \in \mathscr{G}$, and left invariant by all of the $v'(X)$; this domain is dense, since it includes all the vectors of the form $L_f x$ for f arbitrary in **G** and x arbitrary in **B**. The restriction of $v'(X)$ to **D** will be denoted $v(X)$; v extends uniquely to a homomorphism, also denoted v, of $\mathbf{E}(G)$ into the algebra of linear transformations on **D**.

We refer to v as the "infinitesimal representation" determined by the "finite" or "global" representation V, and unless otherwise indicated, always take the domains of the operators of the infinitesimal representation as the maximal domain **D**. The representations l and r of $\mathscr{G}$ extend uniquely to isomorphisms of $\mathbf{E}(G)$, which will be denoted by the same letters, into linear operators on C^∞, the set of all infinitely differentiable functions on G; the relation $v(Z)L_f = L_{l(Z)f}$ is valid for arbitrary $Z \in \mathbf{E}(G)$ and $f \in \mathbf{C}_0^\infty$. In terms of the exponential mapping $X \to e^X$ of G into G, l and r may be described as follows:

$$l(X)f(a) = \left(\frac{d}{dt}\right) f(e^{-tX}a)\Big|_{t=0}; \qquad r(X)f(a) = \left(\frac{d}{dt}\right) f(ae^{tX})\Big|_{t=0}.$$

For the foundations of Lie group theory, we refer to the well-known books by C. Chevalley, P. Cohn, and L. Pontrjagin; concerning the relation with Banach space representations (cf. [*3*]) and concerning group representations (cf. [*1*], [*7*], [*11*], and cited literature therein). Now denoting by T^*, for any densely defined operator T in **B**, the operator on the dual space $\mathbf{B}^*$ whose domain consists of all vectors f

for which there exists an f' in $\mathbf{B}^*$ such that $\langle Tx,f\rangle = \langle x,f'\rangle$, and which is defined as the mapping $f \to f'$ on this domain, then the mapping $V^* : a \to V(a^{-1})^*$ is again a representation of G in a Banach space, the so-called contragredient representation to V. If V^* is continuous (as is automatically the case when $\mathbf{B}$ is reflexive), if $\mathbf{D}^*$ denotes the maximal domain for V^*, if $x \in \mathbf{D}$, and $f \in \mathbf{D}^*$, then it is easily seen that

$$\langle v(Z)\,x,f\rangle = \langle x, v^*(Z^*)f\rangle$$

for any element Z in $\mathbf{E}(G)$, where the operation $*$ on the enveloping algebra is the unique adjunction operation which carries each element A of $\mathcal{G}$ into $-A$. Thus $u(Z)^*$ extends $u^*(Z^*)$, and is in particular densely defined, so that $u(Z)^{**}$ exists; noting that T^* is always closed, it follows that $\overline{u(Z)}$ exists.

We recall that an operator in a Hilbert space is essentially self-adjoint (or normal) if the operator has a closure, and this closure is self-adjoint (or normal); such an operator is said to have property "P," defined initially only for self-adjoint (or normal) operators in case its closure has this property. Criteria for essential self-adjointness or normality play a crucial part in dealing with questions similar to those involved in the O'Raifeartaigh Theorem. The following results extend parts of some earlier ones in related directions (cf. especially [*11*] and [*7*]).

THEOREM 1. *Let U be any representation of the Lie group G on the Banach space $\mathbf{B}$ which together with its contragredient is continuous; let Z be any central element of the enveloping algebra $\mathbf{E}(G)$; then the closure of the restriction of $u(Z)$ to any dense domain which is invariant under the $U(a)$, $a \in G$, is the same as $\overline{u(Z)}$.*

Let $\mathbf{D}$ denote the maximal domain, and let $\mathbf{D}_0$ be any invariant dense domain; let T_0 denote the restriction of $u(Z)$ to $\mathbf{D}_0$; it must be shown that $\overline{T_0} = \overline{u(Z)}$.

LEMMA 1.1. *Let x be arbitrary in $\mathbf{D}$, let d be a given positive integer, and let f be arbitrary in $\mathbf{C}_0^\infty$; then there exists a sequence $\{L_n\}$ of operators on $\mathbf{B}$, each of which is a finite linear combination of the $U(a)$, such that $u(Z')\,L_n x \to u(Z')\,L_f x$ for all central $Z' \in \mathbf{E}(\mathcal{G})$ whose degree does not exceed d.*

From the central character of Z' it follows (cf. [*9*]) that $U(a)\,u(Z') = u(Z')\,U(a)$ for all a in G. The conclusion of Lemma 1 is therefore equivalent to the relation: $L_n u(Z')\,x \to L_f u(Z')\,x$ for all central Z' of degree not exceeding d. The set of all such Z' is finite-dimensional, so it suffices to show that L_n of the indicated type exist for which $L_n y_j \to L_f y_j$, where $y_1, y_2, \ldots, y_n$ is a given finite set of vectors. By the definition of the strong operator topology, $U(a)\, x$ is a continuous function of a; consequently, for any compact set S containing the support $S(f)$ of f in its interior, $U(a)\, x$ is uniformly continuous as a function of a in S; in particular, for any $\epsilon > 0$ there exists a neighborhood N of 0 such that $\| U(a)\, y_j - U(b)\, y_j \| < \epsilon$, for all j, provided $a - b \in N$ and a and b are both in S. By compactness, S is covered by a finite number of the $b + N$, say by the $b_k + N$, where k ranges over an interval of integers. Setting $N_k = (b_k + N) - \bigcup_{k'<k} (b_{k'} + N)$, the N_k are mutually disjoint Baire sets whose union contains $S(f)$. Now setting

$$L^{(\epsilon)} = \sum_k U(b_k) \left[\int_{N_k} f(a)\, da \right],$$

$L^{(\epsilon)} y - L_f y = \sum_k \int_{N_k} [U(b_k) - U(a))\, y] f(a)\, da$ for any vector y, from which it follows that $\| L^{(\epsilon)} y_j - L_f y_j \| \leq \epsilon \| f \|_1$ (where here and henceforth, the notation $\| f \|_p$ indicates the norm of f in the space L_p, i.e., $[\int | f(a)^p\, da]^{(1/p)}$). Taking $L_n = L^{(\epsilon')}$ with $\epsilon' = \min(1, (n \| f \|_1)^{-1})$, the stated conclusion is readily deduced.

LEMMA 1.2. *If x is in $\mathbf{D}_0$ and f is in $\mathbf{C}_0^\infty$, then $L_f x$ is in the domain of T_0, and $T_0 L_f x = u(Z)\, L_f x$.*

Let $\{L_n\}$ be as in Lemma 1, for degrees up to and including that of Z. Then $L_n x \in \mathbf{D}_0$ by the invariance of $\mathbf{D}_0$, and $L_n x \to L_f x$ (taking Z' as the unit, of degree zero), while at the same time, $T_0 L_n x \to u(Z)\, L_f x$ by Lemma 1.1. The conclusion of Lemma 1.2 now follows from the definition of closure.

LEMMA 1.3. *If x is in the domain of $\overline{u(Z)}$ and $f \in \mathbf{C}_0^\infty$, then $u(Z) L_f x = L_f \overline{u(Z)} x$.*

Let $x_n \to x$ with $x_n \in \mathbf{D}$ and $u(Z)\, x_n \to \overline{u(Z)}\, x$. Since Z is central, $u(Z)\, L_f x_n = L_f u(Z)\, x_n$ for $n = 1, 2, \ldots$. Noting that $u(Z)\, L_f$ is a bounded operator and passing to the limit $n \to \infty$, it follows that $u(Z)\, L_f x = L_f \overline{u(Z)}\, x$.

Proof of Theorem. Since T_0 is evidently extended by $\overline{u(Z)}$, it is only necessary to show the inclusion $\overline{u(Z)} \subset T_0$. Let x be arbitrary in the domain of $\overline{u(Z)}$; let $\{x_n\}$ be a sequence of elements of $\mathbf{D}_0$ such that $x_n \to x$. Then $L_f x_n \to L_f x$ for any element f in $\mathbf{C}_0^\infty$; on the other hand, $T_0 L_f x_n = u(Z)\, L_f x_n$ by Lemma 1.2, and $u(Z)\, L_f = L_{l(Z)f}$ so that

$$u(Z)\, L_f x_n \to L_{l(Z)f} x = u(Z)\, L_f x.$$

Thus $L_f x$ is in the domain $\mathbf{D}_1$ of T_0, and $T_0 L_f x = u(Z) L_f x$. Now replacing f by a sequence $\{f_n\}$ such that $L_{f_n} x \to x$ for all x, then by Lemma 1.3 $u(Z) L_{f_n} x = L_{f_n} \overline{u(Z)}\, x$, so that $T_0 L_{f_n} x \to \overline{u(Z)}\, x$, showing that $x \in \mathbf{D}_1$ and that $T_0 x = u(Z)\, x$.

In the case of a unitary representation in a Hilbert space, a sharper result than that given by Theorem 1 follows with the aid of a method used for the case of a formally Hermitian element in [*11*].

THEOREM 2. *Let U be a continuous unitary representation of the Lie group G on the Hilbert space* $\mathbf{H}$; *let Z be any central element of the enveloping algebra* $\mathbf{E}(G)$; *then the restriction of $u(Z)$ to any dense domain which is invariant under all the $U(a)$, $a \in G$, is essentially normal.*

Let $\mathbf{R}$ be the ring of operators, in the sense of Murray-von Neumann, generated by the $U(a)$ for a in G,—i.e., the minimal weakly closed self-adjoint algebra of operators containing the $U(a)$. By virtue of Theorem 1 it suffices to show that $u(Z)$ is essentially normal. Now $u(Z)$ commutes with the $U(a)$ for all in G, and hence does its closure T. It follows that the partially isometric and self-adjoint constituents in the canonical polar decomposition likewise commute with all the $U(a)$. The partially isometric constituent, being bounded, thereby commutes with all operators in the ring $\mathbf{R}$. The spectral projections for the self-adjoint constituent commute with all $U(a)$, and so for the same reason commute with all operators in the ring $\mathbf{R}$. It follows that the self-adjoint and the partially isometric constituents of T, and hence also T itself, commutes with all unitary operators in $\mathbf{R}$. This means that T is affiliated with (or "belongs to," in the Murray-von Neumann terminology) the commuting ring $\mathbf{R}'$.

Now let S denote an arbitrary unitary operator in $\mathbf{R}'$; then in particular, S commutes with each one-parameter unitary subgroup representing a one-parameter subgroup of G. It follows that S leaves invariant the maximal domain $\mathbf{D}$ of the infinitesimal representation u. The same argument shows that S commutes with $u(Z')$ for all Z' in $\mathbf{E}(G)$, and so with $u(Z)$; and hence, ultimately, with the closure T of $u(Z)$. This means that T is affiliated with the double commutor $\mathbf{R}'' = \mathbf{R}$.

Thus T is affiliated with both $\mathbf{R}$ and $\mathbf{R}'$. Its partially isometric and self-adjoint constituents (and ultimately the spectral projections of the latter) commute with all unitary operators in $\mathbf{R}$ and with all unitary operators in $\mathbf{R}'$, and hence with all unitary operators in the ring $\mathbf{R} \vee \mathbf{R}'$ generated by $\mathbf{R}$ and $\mathbf{R}'$. This means that T is affiliated with the commutor of this ring, i.e., $\mathbf{R}' \wedge \mathbf{R}'' = \mathbf{R}' \wedge \mathbf{R} =$ the center of $\mathbf{R}$. It is known however that any closed, densely defined, operator which is affiliated with any abelian ring of operators, is normal (see [*8*]).

A classical difficulty in the theory of infinite-dimensional representations of Lie groups is that a linear submanifold of **D** may be invariant under all the $u(X)$ without its closure being invariant under the $U(a)$. To overcome this type of difficulty, the concept of a "well-behaved" or "analytic" vector was introduced by Harish-Chandra, and its theory developed by him, Cartier and Dixmier [*1*], Nelson, Gårding, and others. What is basically involved here is an apparently less-restricted type of vector, as follows: a vector x in the representation space **B** of a continuous representation V of a Lie group G is said to be *pseudo-analytic* in case for all a in G, $V(a)\, x'$ is in the closed linear span of $v(\mathbf{E}(G))\, x'$ for all $x' \in v(\mathbf{E}(G))\, x$. The following simple result is surely well-known, but for completeness and in the absence of a known reference, we give its proof.

SCHOLIUM 1. *An analytic vector is pseudo-analytic.*

From the definition of "analytic vector" (q.v.) it is clear that if z is analytic, then $U(e^{tY})\, z$ is in the closed linear span of the $u(Y)^n\, z$ ($n = 0, 1, 2,...; u(Y)^0 = I$) if t is sufficiently small. To show that $U(a)\, z$ is in this closed linear span **M**, it suffices to show that $U(e^Y)$ leaves **M** invariant for all elements Y o $\mathscr{G}$, for in a connected group G every element a is the product of a finite number of elements of the form e^Y, and any product of operators leaving **M** invariant will again leave **M** invariant. As the basis of an indirect proof, suppose there exists an element Y in $\mathscr{G}$ such that $U(e^{tY})\, z$ is not in **M** for all real t. Let t_0 be the supremum of the positive real numbers t such that $U(e^{t'Y})\, z \in \mathbf{M}$ for $0 < t' \leqslant t$. Let $t_n \uparrow t_0$, $t_n < t_0$; then $U(e^{t_nY})\, z \in \mathbf{M}$, and $U(e^{t_nY})\, z \to U(e^{t_0Y})\, z$, so that $U(e^{t_0Y})\, z \in \mathbf{M}$. Now $U(e^{t_0Y})\, z$ is readily seen to be again an analytic vector, so that for all sufficiently small s, $U(e^{sY})\, U(e^{t_0Y})\, z$ is in the closed linear manifold spanned by the $u(Y)^n\, U(e^{t_0Y})\, z$ $(n = 0, 1, 2,...)$. Since $u(Y)^n\, U(e^{t_0Y}) = U(e^{t_0Y})\, u(Y)^n$ for all n, this closed linear manifold is contained in **M**. Thus, whenever $0 \leqslant t \leqslant t_0 + \epsilon$, where ϵ is a sufficiently small positive number, $U(e^{t'Y})\, z \in \mathbf{M}$, contradicting the assumption that t_0 is finite. Now if z is analytic, so also are all vectors in $u(\mathbf{E}(G))\, z$, and the Scholium follows.

It was shown in [*1*] that the analytic vectors form a dense subset of the representation space, for any continuous unitary representation in Hilbert space of a connected Lie group; it follows from the preceding Scholium that the same is true of the pseudo-analytic vectors.

3. The Generalized O'Raifeartaigh Theorem

Let G' be a Lie subgroup of the Lie group F; G' is called *relatively unipotent* in case for every element Z of $\mathbf{E}(G')$, *ad* Z acts nilpotently on $\mathscr{G}$. (The index of nilpotency will be in general vary; there is otherwise no difficulty in passing from the infinitesimal to the global form of the O'Raifeartaigh theorem.)

If m is any regular measure of finite total variation on $\mathscr{G}$, the integral $\int U(e^X)\,dm(X)$ exists in the strong or weak sense, U being any continuous bounded Banach representation of G, and will be denoted as $L^{(m)}$, when it is clear from the context which representation U is involved. Such operators $L^{(m)}$ are easily seen to form a dense subset of the weakly closed ring $R(G)$ generated by the $U(a)$, $a \in G$. A subspace $\mathbf{K}$ of $\mathbf{H}$ will be called *mildly accessible* (relative to U) in case there exists a sequence $\{m_n\}$ of measures on $\mathscr{G}$, each of which has moments of all orders (i.e., the integrals $\int |p(X)|\,|dm_n(X)|$ are convergent for all polynomials p and all n) such that the ranges of the L^{m_n} are contained, and have union dense, in $\mathbf{K}$.

The main result of this section is:

Theorem 3. *Let U be a continuous unitary representation of the connected Lie group G on a Hilbert space* $\mathbf{H}$; *let G' be a Lie subgroup which acts unipotently relative to G; let M be any central element of the universal enveloping algebra of the Lie algebra of G'. Then the eigenspace associated with any eigenvalue of the (necessarily essentially normal) operator $u(M)$ is invariant under all of $U(G)$, provided this eigenspace is mildly accessible relative to $U(G')$, the restriction of U to G'.*

The first lemma makes Lemma II of [6] precise and extends it to normal operators.

Lemma 3.1. *If T is a normal operator in a Hilbert space, and if for some integer n, x is in the domain of T^n and $T^n x = 0$, then $Tx = 0$.*

A normal operator is precisely one which is unitarily equivalent to the operation of multiplication by a measurable function, acting on the domain in a space $L_2(M)$ of all square-integrable functions on a suitable abstract Lebesgue-measure space M, of all elements for which the product is again in $L_2(M)$. This means that x may be assumed to have the form $f(a)$, where f is a square-integrable function of a, and T is assumed to consist of the operation of multiplication by the fixed measurable function $k(a)$ acting on the domain of all elements g of $L_2(M)$ such that $k(a)\,g(a)$ is again in $L_2(M)$. To say that $T^n x = 0$ is then to say that $k(a)^n f(a) = 0$ a.e.; this implies that, for almost all a, either $k(a) = 0$ or $f(a) = 0$; in either case, $k(a)f(a) = 0$, which means that $Tx = 0$.

The argument used to prove the next lemma is identical with that used to conclude the proof of the Theorem in [6].

LEMMA 3.2. *Let Z be any element of $\mathbf{E}(G)$ such that $u(Z)$ is essentially normal, and such that ad Z acts nilpotently on $\mathscr{G}$. Let $\mathbf{K}$ denote the nullspace of $u(Z)$. Then $u(\mathscr{G})$ leaves invariant $\mathbf{D} \cap \mathbf{K}$.*

Set $S = u(Z)$; if $x \in \mathbf{D}$ and $Sx = 0$, then it is easily verified by induction on m ($= 1, 2, \ldots$) that, for any element Y of $\mathscr{G}$,

$$S^m u(Y)\, x = [S, [S, \cdots [S, u(Y)] \cdots]]\, x,$$

where there are m commutators on the right-hand side. By the hypothesis and the homomorphism property of u, the right-hand side vanishes if m is sufficiently large. It follows from Lemma 3.1 that $Su(Y)\, x = 0$, which means that $u(Y)\, x \in \mathbf{K}$. Since $u(Y)\, x$ is evidently again in $\mathbf{D}$, the proof is complete.

A typical analytical difficulty in the proof is dealt with by

LEMMA 3.3. *Let f be an integrable function of class C^1 on G, such that $l(Y)f$ is integrable, where Y is a given element of $\mathscr{G}$. Then for any vector x in $\mathbf{H}$, $L_f x$ is in the domain of $\overline{u(Y)}$, and $\overline{u(Y)}\, L_f x = L_{l(Y)f} x$.*

This is a corollary to the known result that $u(Y)$ is essentially skew-adjoint on the domain $\mathbf{D}'$ spanned by the vectors of the form $L_g y$, with g in $\mathbf{C}_0^\infty$ and y arbitrary in $\mathbf{H}$; this result, first proved in [7], is also an immediate corollary to Theorem 2, in view of the invariance of $\mathbf{D}'$ under the $U(e^{tY})$, t real. The conclusion of the Lemma is thereby equivalent to the equality

$$\langle u(Y) L_g y, L_f x \rangle = - \langle L_g y, L_{l(X)f} x \rangle,$$

for all g and y of the indicated types.

To establish this equation, note first that, for arbitrary integrable functions h and k on G and for vectors p and q in $\mathbf{H}$,

$$\langle L_h p, L_k q \rangle = \iint \langle U(a)\, p,\, U(b)\, q \rangle\, h(a)\, k(b)\, da\, db,$$

by a simple application of the Fubini theorem; on replacing a by the variable $c = b^{-1}a$, which leaves the measure invariant, it follows that

$$\langle L_h p, L_k q \rangle = \iint \langle U(c)\, p,\, q \rangle\, h(bc)\, k(b)\, db\, dc.$$

Applying this identity to the left side of the equality in question, while recalling that $u(Y)L_g = L_{l(Y)g}$, the following expression is obtained: $\int\int < U(c)p, q > (l(Y)g)(bc)\bar{f}(b)\, db\, dc$. For the right side, the following is obtained: $-\int\int < U(c)p, q > g(bc)(l(Y)\bar{f}(b)\, db\, dc$. Reducing these double integrals to iterated ones by another use of Fubini's theorem, it follows that it suffices to show that

$$\int (l(Y)g)(bc)f(b)\, db = -\int g(bc)(l(Y)f)(b)\, db.$$

Since right-translation (here by the element c) and the $l(Y)$ commute, it suffices to show that, for an arbitrary function g in $\mathbf{C}_0^\infty$,

$$\int (l(Y)g(b)\bar{f}(b)\, db = -\int g(b)(l(Y)\bar{f}(b))\, db.$$

The verification of the last equation depends on the invariance of Haar measure, for the utilization of which we write the left side of the last equation as $\lim_{t\to 0} t^{-1} \int [g(\exp(tY)\, b) - g(b)]\bar{f}(b)\, db$; this may be justified by dominated convergence, in turn justified by the observation that

$$t^{-1}[g(e^{tY}b) - g(b)] = (l(Y)g)(e^{t'Y}b)$$

for some t' with $|t'| \leqslant |t|$, by the mean-value theorem; the last expression is dominated by the supremum of $l(Y)g$, evidently. Similarly, the right side of the equation now in question may be expressed as

$$-\lim_{t\to 0} \int g(b)\, t^{-1}[\bar{f}(e^{tY}b) - \bar{f}(b)]\, db,$$

again using dominated convergence, the difference quotient under the integral sign being dominated by the supremum over the support of g of $l(Y)f$. On the other hand, the change of variables: $b \to e^{tY}b$, in the first of these integrals, leads to the second, except for a change in the sign of t which is precisely compensated by the minus sign outside of the integral.

Remark 1. It is interesting to note that this appears to be a basically Hilbert-space result, in the sense that it is doubtful whether the corresponding result is true in $L_1(G)$ itself, i.e., whether

$$\lim_{t\to 0} t^{-1}[f(e^{-tY}b) - f(b)]$$

exists as a limit in $L_1(G)$. Such a result would, it is easily seen, supersede Lemma 3.3 and avoid the use of spectral theory. However, despite its plausibility at first glance, it does not follow, due to the problem of showing suitable domination.

The next two lemmas establish conditions under which the hypotheses of Lemma 3.3 are satisfied.

LEMMA 3.4. *If m is any bounded regular measure on $\mathcal{G}'$ and if g is in $\mathbf{C}_0^\infty(G)$, then the function h, where*

$$h(a) = \int_{G'} g(e^Y a)\, dm(Y),$$

is infinitely differentiable and $r(Z)\, h$ is integrable for all Z in $\mathbf{E}(G)$.

The integral defining h evidently exists; a simple application of the Fubini theorem shows that $\| h \|_1 \leqslant \| g \|_1 \| m \|$. By a dominated convergence argument similar to those employed in the proof of Lemma 3.3, it follows that, for any element X in G,

$$\lim_{t \to 0} t^{-1}[h(ae^{tX}) - h(a)] = \int (r(X)\, g)\, (e^Y a)\, dm(Y).$$

The integral on the right is of the form $\int k(e^Y a)\, dm(Y)$ where k is a continuous function of compact support on G; such an integral represents a continuous function of a, by the uniform continuity of k relative to left translations, and a familiar elementary estimate. It follows that the indicated limit is a continuous function of a, for every $X \in \mathcal{G}$, which means that h is of class C^1 on G. At the same time it has been shown that $r(X)\, h$ is given by an integral of the same form as that defining h; it results that $r(X)\, h$ is of class C^1 also; it follows by induction that h is of class C^∞ on G. The estimate given above for $\| h \|_1$ is applicable to $r(Z)\, h$ for arbitrary Z, with g replaced by $r(Z)\, g$, showing that $r(Z)\, h$ is integrable for all Z.

LEMMA 3.5. *With the notation and hypotheses of Lemma 3.4, and the additional hypotheses* (i) *G' acts unipotently on G,* (ii) *m has moments of all finite order, then: $l(Z)\, h$ is integrable for all Z in $\mathbf{E}(G)$.*

Let X be an arbitrary element of $\mathcal{G}$. Consider the difference quotient

$$t^{-1}[h(e^{-tX}b) - h(b)] = \int_{G'} t^{-1}[g(e^Y e^{-tX} b) - g(e^Y b)]\, dm(Y).$$

We recall from the general theory of Lie groups that

$$e^Y e^{tX} e^{-Y} = e^{tX'}, \qquad X' = e^{\delta(Y)} X,$$

and note that by virtue of the nilpotency assumption, the power series expansion for $e^{\delta(Y)}$ terminates after a finite number of terms, so that X' has the form $\sum_k p_k(Y)\, X_k$, where the p_k are polynomials on $\mathscr{G}'$ and the X_k form a basis for $\mathscr{G}$. Writing $g(e^Y e^{tX} b) = g(e^{tX'} e^Y b)$, and substituting in the difference quotient under the integral sign, it follows that

$$t^{-1}[g(e^Y e^{-tX} b) - g(e^Y b)] \to l(X')\, g(e^Y b);$$

the difference quotient is bounded by the supremum on G of $l(X')\, g$; in view of the form of X' just derived, this is, in turn, bounded by const. $\times \sum_k | p_k(Y) |$. By virtue of the assumption of finite moments for m, dominated convergence is now applicable for the passage to the limit $t = 0$, and it results that the indicated difference quotient for h has the limit

$$\int \Big(\sum_k p_k(Y)\Big)\, (l(X_k)\, g)\, (e^Y b)\; dm(Y) = \sum_k \int g_k(e^Y b)\; dm_k(Y),$$

with $g_k = l(X_k)\, g$ and $dm_k(Y) = p_k(Y)\, dm(Y)$.

This is a finite sum of integrals of the same form as those which define h, and it follows that $l(Y)\, h$ is integrable. Furthermore, the same argument may now be applied to each of the integrals in the expression for $l(h)$, and it follows by induction that $l(Z)\, h$ is integrable for all $Z \in \mathbf{E}(G)$.

Remark 2. Lemma 3.5 is conceivably valid without the nilpotency assumption; our original form of the Lemma aimed in this direction and assumed only the suitably dominated convergence of the series for $e^{\delta(Y)} g$. This series is trivial in the nilpotent case (an observation made to us by a colleague who ascribed it to R. Jost), and this case suffices for the purposes of the O'Raifeartaigh theorem, so we have eliminated consideration of the more general case. The question of when a suitably regular element of $\mathbf{H}$ (in $\mathbf{D}$, or pseudo-analytic) remains so after smoothing relative to a *subgroup* of G has, *a priori*, little to do with whether the subgroup acts unipotently, and appears of some general mathematical interest.

LEMMA 3.6. *Let l be any eigenvalue of $\overline{u(M)}$ for which the corresponding eigenspace $\mathbf{H}_l$ is mildly accessible. Then $\mathbf{D} \wedge \mathbf{H}_l$ is dense in $\mathbf{H}_l$.*

In accordance with the definition of a mildly accessible subspace, let $\{m_n\}$ be a sequence of measures on $\mathscr{G}'$ having moments of all orders such that the ranges of the $L^{(m_n)}$ are contained and have union dense in $\mathbf{H}_l$. Then for any vector x in $\mathbf{H}$, and element g of $\mathbf{C}_0^\infty$

$y = L^{(m_n)} L_g x$ is in $\mathbf{H}_l$. Now $L^{(m_n)} L_g$ has the form L_h, with h as given in Lemma 3.4 (m being replaced by m_n). By Lemmas 3.4 and 3.5, h is of class C^∞, and $l(Z)\, h$ is integrable for all $Z \in \mathbf{E}(G)$. By Lemma 3.3, this implies that $L_h x$ is in the domain of $\overline{u(X)}$ for all X in $\mathscr{G}$, and $\overline{u(X)}\, L_h x = L_{l(X)h} x$. Now $l(X)\, h$ is given, according to the computation in Lemma 3.5, by a finite sum of integrals of the same form as that representing h. It follows by induction that y is in the domains of all finite products of the $\overline{u(X)}$, which means that y is in $\mathbf{D}$. Thus every such y is in $\mathbf{D} \wedge \mathbf{H}_l$; on the other hand, from the density of the ranges of the $L^{(m_n)}$ in $\mathbf{H}_l$ and from the density of the vectors of the form $L_g x$ in $\mathbf{H}$, it follows easily that the set of all such vectors y is dense in $\mathbf{H}_l$.

It may be noted that the foregoing proof is applicable to any subspace which is affiliated with $\mathbf{R}(G')$ in the sense of Murray and von Neumann.

The following lemma was conjectured by L. O'Raifeartaigh in the particular case under consideration by him.

LEMMA 3.7. *The set of all vectors of the form* $(u(M^*) - \bar{l}I)\, y$, *where* y *is pseudo-analytic, is dense in the orthogonal complement of* $\mathbf{H}_l$.

This lemma may be established by a direct argument, but it is more easily obtained as a Corollary to Theorem 2, as follows. Recall first that $u(M^*)$ and $u(M)$ are adjoint to each other on the maximal domain $\mathbf{D}$, for a unitary representation U; this means that $u(M^*) \subset (u(M))^*$; hence $\overline{u(M^*)}\, (u(M))^{***} = \overline{(u(M))^*}$. By Theorem 2, both $\overline{u(M)^*}$ and $\overline{u(M)}$ are normal; so also is $\overline{u(M)}^*$, as the adjoint of a normal operator; but if one normal operator extends another, then, as is easily seen, the two must be equal. Thus

$$u(M^*)^{**} = u(M)^*, \qquad u(M^*)^* = \overline{u(M)}.$$

Note next that the vectors z of the form $(u(M^*) - lI)\, y$, with $y \in \mathbf{D}_0$, where $\mathbf{D}_0$ denotes the domain of all pseudo-analytic vectors, are actually orthogonal to $\mathbf{H}_l$. For if $v \in \mathbf{H}_l$, then $v \in \mathbf{D}_{\overline{u(M)}}$, and $\overline{u(M)}\, v = lv$; hence

$$\langle (u(M^*) - \bar{l}I)\, y, v\rangle = \langle u(M^*)\, y, v\rangle - \bar{l}\langle y, v\rangle = \langle y, u(M)\, v\rangle - \bar{l}\langle y, v\rangle = 0.$$

Now suppose there exists a vector v orthogonal to $\mathbf{H}_l$ which is orthogonal to all the vectors z;

$$\langle (u(M^*) - \bar{l}I)\, y, v\rangle = 0, \qquad y \in \mathbf{D}_0.$$

Let S_0 denote the restriction of $u(M^*) - lI$ to the domain $\mathbf{D}_0$; the foregoing equation then asserts that $\langle S_0 y, v\rangle = 0$ for all vectors y in the domain of S_0. This means that v is in the domain of S_0^*, and that $S_0^* v = 0$. Now the domain $\mathbf{D}_0$ is invariant under $U(G)$, so that by Theorem 2, $u(M^*)$ is essentially normal on $\mathbf{D}_0$, and in particular, the closure of its restriction to $\mathbf{D}_0$ is $\overline{u(M^*)}$, i.e., $S_0^{**} = \overline{u(M^*)} - lI$. Now $S_0^* = S_0^{***} = (S_0^{**})^* = (\overline{u(M^*)} - lI)^* = \overline{u(M)} - lI$. It follows that v is in $\mathbf{H}_l$, and as it is also orthogonal to $\mathbf{H}_l$, $v = 0$.

The remaining argument was obtained basically in the course of discussions with L. O'Raifeartaigh.

Completion of Proof of Theorem. Since a self-adjoint collection of bounded operators on a Hilbert space leaves invariant a closed linear manifold if and only if it leaves invariant its orthogonal complement, it suffices to show that the $U(a)$ leave invariant the orthocomplement $\mathbf{K}$ of $\mathbf{H}_l$. Now if y is an arbitrary pseudo-analytic vector, then for any element Z of $\mathbf{E}(G)$ $u(Z^m)(T - lI)^* y$ is again pseudo-analytic, where T denotes $u(M)^{**}$. Moreover, this vector is in $\mathbf{K}$ since, for any element y' of $\mathbf{D} \wedge \mathbf{H}_l$,

$$\langle u(Z^m)(T - lI)^* y, y'\rangle = \langle (T - lI)^* y, u(Z^*)^m y'\rangle$$

and the expression on the right vanishes, since by Lemma 3.2, $u(Z^*) y'$ is again in $\mathbf{D} \wedge \mathbf{H}_l$. By the definition of pseudo-analyticity, $U(a)(T - lI)^* y$ is itself in $\mathbf{K}$. Thus each $U(a)$ maps a dence submanifold of $\mathbf{K}$ into itself, and hence leaves $\mathbf{K}$ invariant.

4. Groups Containing the Poincaré Group as a Subgroup

The nilpotency and mild accessibility hypotheses of Theorem 3 are readily verified for a general class of groups including those containing the Poincaré group as a subgroup.

THEOREM 4. *Let A be an abelian analytic subgroup of the connected Lie group G, having the property that $\mathscr{A} \subset [\mathscr{A}, \mathscr{G}]$. Let U be any continuous unitary representation of G on a complex Hilbert space* $\mathbf{H}$; *let u denote the corresponding infinitesimal representation of the universal enveloping algebra* $\mathbf{E}(G)$ *of the Lie algebra of G; let M denote an arbitrary element of the enveloping algebra* $\mathbf{E}(A)$; *let l denote any isolated point in the spectrum of the (necessarily normal) closure of $u(M)$. Then the corresponding eigenspace, i.e., $[x \in \mathbf{H} : \overline{u(M)}\, x = lx]$, is invariant under all of $U(G)$.*

The following lemma is an abstraction of Lemma I of [*6*], and its proof follows similar lines; it is purely Lie-algebra-theoretic.

LEMMA 4.1. *For any element Z of $\mathbf{E}(G)$, let $\delta(Z)$ denote the linear transformation on $\mathbf{E}(G) : Z' \to [Z, Z']$. Then for any Z in $\mathbf{E}(A)$ and Z' in $\mathbf{E}(G)$, there is a positive integer n such that $\delta(Z)^n Z' = 0$.*

The assumption that $\mathscr{A} \subset [\mathscr{A}, \mathscr{G}]$ means that any element A in $\mathscr{A}$ can be written in the form $A = \sum_i [A_i, X_i]$, with $A_i \in \mathscr{A}$ and $X_i \in \mathscr{G}$, and where i ranges over some finite set. Inasmuch as δ is a Lie homomorphism, $\delta(A) = \sum_i [\delta(X_i), \delta(A_i)]$. Multiplying by $\delta(A)^n$ and using the commutativity of $\mathscr{A}$, it results that $\delta(A)^n = \sum_i [\delta(X_i)\, \delta(A)^{n-1}, \delta(A_i)]$. Now each operator $\delta(X)$, $X \in \mathscr{G}$, leaves invariant the subspace of $\mathbf{E}(G)$ consisting of the elements whose degree does not exceed a fixed integer d. On restriction to this subspace, $\delta(A)^n$ is represented as a sum of commutators of transformations leaving the subspace invariant; therefore its trace relative to the subspace is zero. Since this is the case for all n, it follows (e.g., from the evaluation of the van der Monde determinant) that the proper values of the restriction to this subspace of $\delta(A)$ are zero. This restriction is therefore, by the Jordan normal form, a nilpotent operator. From the theorem of Engel (that the enveloping algebra of a Lie algebra of nilpotent operators on a finite-dimensional vector space consists only of nilpotent operators; cf. e.g., [*4*]), it results that for any element Z of $\mathbf{E}(A)$, $\delta(Z)$ has nilpotent restriction to the subspace of elements of $\mathbf{E}(G)$ whose degree does not exceed a given integer. Since every element Z' of $\mathbf{E}(G)$ is contained in such a subspace, it can be concluded that $\delta(Z)^n Z' = 0$ for some positive integer.

The following lemma is surely well-known, but it is almost as brief to give a proof as a reference.

LEMMA 4.2. *The Fourier transform of an infinitely differentiable function of compact support on Euclidean space is integrable.*

Let h denote the function in question. Evidently, h and all of its derivatives are square-integrable. It follows that the Fourier transform $\hat{h}$ and all of its products with polynomials are in L_2 also. Now writing $\hat{h}(x) = (1 + x \cdot x)^{-s} \cdot [(1 + x \cdot x)^s \hat{h}(x)]$ and choosing the positive integer s so large that $(1 + x \cdot x)^{-s} \in L_2$, $\hat{h}$ is represented as a product of two functions in L_2, and so is integrable.

LEMMA 4.3. *If p is any polynomial on Euclidean space and ϵ is any positive number, there exists a sequence $\{f_n\}$ of integrable functions* *having moments of all orders the supports of whose Fourier transforms have union equal to the set where* $|p(x)| < \epsilon$.

The set N just indicated is open, and hence the union of a monotone-increasing sequence C_n of compact subsets, each of which is contained in the interior of the next, by elementary analytical topology. Now it is well-known that if K and L are compact subsets of Euclidean space E such that K is contained in the interior of L, then there exists a function h of class C^∞ on E which have the value one on K and zero in the complement of K. In particular, there exists an infinitely differentiable function h_n which have the value one on C_n and zero outside of C_{n+1}. Now taking f_n as the inverse Fourier transform of h_n, f_n is integrable by Lemma 4.2, as are all of its moments, which are the inverse Fourier transforms of derivatives of h_n, and Lemma 4.3 follows.

LEMMA 4.4. *The eigenspace* $\mathbf{H}_l = [x \in \mathbf{H} : \overline{u(M)}\, x = lx]$ *is mildly accessible.*

Since l is an isolated eigenvalue of $u(M)^{**}$, there is a positive number ϵ such that $\mathbf{H}_l$ is the range of $E(N)$, where E is the spectral measure associated with the Abelian group $U \mid A$:

$$U(e^X) = \int e^{iX\cdot Y}\, dE(Y) \qquad (X \in \mathscr{A};\ Y \in \mathscr{A}' = \text{dual to } \mathscr{A});$$

here N is the set in $\mathscr{A}'$ on which $p(Y) - l$ is properly bounded in absolute value by ϵ, and p is the polynomial which corresponds via the Fourier transform to the differential operator $u(M)$; thus in particular,

$$u(M)^{**} = \int p(Y)\, dE(Y).$$

Now setting $dm_n(X) = f_n(X)\, dX$, where the f_n are as in Lemma 4.3, it is clear that the range of L^{m_n} is contained in $\mathbf{H}_l$. Now the range of L^{m_n} contains the range of $E(C)$ for any compact set C on which $\hat{f}_n$ is bounded away from zero; hence it is dense in $E(S_n)$, where S_n is the support of $\hat{f}_n$; it follows that the union of the ranges of the L^{m_n} is dense in $\mathbf{H}_l$.

Completion of Proof of Theorem. Lemmas 4.1 and 4.4 show that the nilpotency and accessibility hypotheses of Theorem 3 are satisfied; the conclusion of Theorem 4 therefore now follows as a corollary to Theorem 3.

The global counterpart to the theorem of O'Raifeartaigh is the special case of the following Corollary in which M is the conventional mass operator $P_0^2 - P_1^2 - P_2^2 - P_3^2$, and the pseudo-Euclidean group in question is the Poincaré group. Since completing the present manuscript, we have been informed that R. Jost has also obtained a proof for this special case.

COROLLARY 4.1. *Let U be a continuous unitary representation of the connected Lie group G; suppose that G contains an analytic subgroup whose Lie algebra is that of the pseudo-Euclidean group of a vector space equipped with a distinguished nondegenerate quadratic form; let M be any element of the enveloping algebra of the Lie algebra of the group of all vector displacements; if l is an isolated eigenvalue of the (necessarily essentially normal) operator $u(M)^{**}$, then the corresponding eigenspace is invariant under all of $U(G)$.*

In particular, if U is irreducible, then l is the only isolated point in the spectrum of $u(M)$.

For the proof it is only necessary to observe that each generator of a vector displacement may be expressed as the commutator of a homogeneous Euclidean infinitesimal operation with another infinitesimal vector displacement: $P_i = \pm [M_{ij}, P_j]$, where P_i generates displacements of the ith pseudo-Euclidean basis vector (leaving the others fixed), and M_{ij} generates a homogeneous one-parameter group in the plane determined by the ith and jth basis vectors, leaving the others fixed. This means that if A denotes the vector-displacement group, then $\mathscr{A} \subset [\mathscr{A}, \mathscr{G}]$ for any group G containing the pseudo-Euclidean group as a Lie subgroup, so that the hypothesis of Theorem 4 is satisfied.

Remark 3. The question of whether the hypothesis that l be an isolated eigenvalue is strictly necessary can be attacked along similar lines, with the use of a strengthened notion of accessibility, applied to the orthocomplement of $\mathbf{H}_l$ rather than $\mathbf{H}_l$ itself. It requires, however, some development of a theory of generalized vectors in Hilbert space relative to a given operator ring, and we shall not treat it at this time.

Remark 4. Unitarity of the representation has been used quite frequently in the foregoing, but it appears that actually, it is only the unitarity of the restriction to the subgroup G' (or the Abelian subgroup A) that is required. With this assumption and with the assumption that $\| U(e^X) \| = O(\| X \|^n)$ for some n, Theorem 3 is readily extendable by slightly modified arguments to the non-unitary case.

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Notes Added in Proof, March 30, 1967.

(1) The foregoing proofs used the unitarity of therep resentation in question in several respects; it appears, however, that the only really essential respect is in the normality of the invariant operator in question. The key limitation in the extension of the proof is in the state of the general theory of analytic vectors for non-unitary group representations; significant extension of this general theory beyond the present existence results of Nelson and Gårding should make it possible, e.g., to establish Theorem 3 omitting the unitarity assumption on U and the mild accessibility assumption on the eigenspace, while adding the assumption that $u(M)$ be essentially self-adjoint, and that the eigenvalue in question be isolated. Recent ingenious work by O'Raifeartaigh [Mass-splitting theorem for non-unitary group representations (preprint, March, 1967)] for the particular case of the mass operator in the Lorentz group confirms this possibility, and provides a potentially general simplification in the treatment of the accessibility question, in which however some limitation on the spectrum of $u(M)$ must be imposed (e.g., self-adjointness, as assumed by O'Raifeartaigh).

These questions are interesting from a purely mathematical standpoint partly because of the challenge provided by the remarkable analytic difficulty of establishing certain simple equations, which are obvious from an intuitive standpoint, and partly as a proving ground for regularization techniques in group representation theory, which should be useful in extending the general theory to a more intrinsic and coherent class of representations than that of all unitary ones.

(2) The points made in (ii) and (iii) of the Introduction are illustrated by the specific models considered by us [Positive-energy particle models with mass splitting. *Proc. Natl. Acad. Sci. U.S.* **57** (1967), 194-197.]

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