Best approximation by integer-valued functions

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Abstract

Given an integer function $f$, the problem is to find its best uniform approximation from a set $K$ of integer-valued bounded functions. Under certain conditions on $K$, the best extremal (maximal or minimal) approximation is identified. Furthermore, the operator mapping $f$ to its extremal best approximation is shown to be Lipschitzian with some constant $C$ or optimal Lipschitzian having the smallest $C$ among all such operators. The results are applied to approximation problems.

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1. Introduction

Let $S$ be any set and let $B$ be the Banach space of real-valued bounded functions $f$ on $S$ equipped with the uniform norm $\|f\| = \sup\{|f(s)| : s \in S\}$. Let $D \subset B$ be the set of all integer-valued functions on $S$, and $K \subset D$ be any nonempty set. For $f$ in $D$, let $\Delta(f)$ denote the infimum of $\|f - k\|$ for $k$ in $K$. The problem considered is to find $f'$ in $K$ so that

$$\Delta(f) = \|f - f'\| = \inf\{|f - k| : k \in K\}. \quad (1.1)$$

Such an $f'$ is called a best approximation to $f$ from $K$. The set of all best approximations to $f$, denoted by $A_f$, is not necessarily singleton in general. A Lipschitzian selection operator (LSO) $T$ is defined to...
be a selection operator which maps each $f$ in $D$ to an $f'$ in $A_f$ so that for some least number $C(T)$ the following holds:

$$\|T(f) - T(h)\| \leq C(T)\|f - h\| \quad \text{for all } f, h \in D.$$ 

Such a $T$ is called an optimal Lipschitzian selection operator (OLSO) if $C(T) \leq C(T')$ for all LSOs $T'$.

In this article we obtain certain conditions on $K$ so that best approximations and LSOs can be identified. We considered a similar problem on the space $B$ of bounded functions in an earlier article [1]. However, the integer condition imposed on $D$ in the present framework is more restrictive. It will be seen that some of the results of [1] can be extended to the present framework with some changes and modification of proofs. A class of related problems on the space of bounded or continuous functions but without the integer restriction is considered in [2]. Two integer restricted approximation problems are analyzed in [3, 4]. The significance of the integer restriction is explained in [4]. Because of this restriction, any nonempty subset of $D$ is not convex unless it is a singleton. Hence, the classical methods of approximation theory such as those given in [5, 6] cannot be applied directly in the present framework.

We state below three conditions on $K$. Depending upon the case under consideration, only a subset of these conditions will be imposed on $K$.

(i) If $k \in K$, then $k + p \in K$ for all integers $p$.

(ii) If $K' \subset K$ is a set of functions uniformly bounded above on $S$, then the function $k'$, which is the pointwise supremum of functions in $K'$, is in $K$.

(iii) If $K' \subset K$ is a set of functions uniformly bounded below on $S$, then the function $k'$, which is the pointwise infimum of functions in $K'$, is in $K$.

Another related problem of interest is the following. For $f$ in $D$, let $K_f = \{k \in K : k \leq f\}$, and $\overline{\Delta}(f)$ be the infimum of $\|f - k\|$ for $k$ in $K_f$. The problem is to find an $f'$ in $K_f$ so that

$$\overline{\Delta}(f) = \|f - f'\| = \inf\{\|f - k\| : k \in K_f\}. \quad (1.2)$$

We state our main results in the next section. There we give examples of problems for which the above three conditions apply.

2. Main results and applications

For a given $f$ in $D$, let $K_f = \{k \in K : k \leq f\}$ as above, and, in addition, let $K'_f = \{k \in K : k \geq f\}$. If condition (i) holds for $K$, then both $K_f$ and $K'_f$ are nonempty. To see this, let $g \in K$. Then $g - \|f - g\| \leq f$. Since $\|f - g\|$ is an integer, by condition (i), $g - \|f - g\|$ is in $K_f$. A similar proof applies to $K'_f$. Now, for $f$ in $D$, let

$$\overline{f}(s) = \sup\{k(s) : k \in K_f\}, \quad s \in S.$$

$$\underline{f}(s) = \inf\{k(s) : k \in K'_f\}, \quad s \in S.$$ 

Note that if $K$ satisfies condition (ii) (respectively condition (iii)), then $\overline{f}$ (respectively $\underline{f}$) is in $K$. We have, obviously, $\underline{f} \leq f \leq \overline{f}$. These two functions, $\overline{f}$ and $\underline{f}$, are, respectively, called the greatest $K$-minorant and smallest $K$-majorant of $f$.

**Proposition 2.1.** Assume $K$ satisfies conditions (i) and (ii). Then $\|\overline{f} - h\| \leq \|f - h\|$ for all $f, h$ in $D$. Similarly, if $K$ satisfies conditions (i) and (iii), then $\|\underline{f} - h\| \leq \|f - h\|$ for all $f, h$ in $D$. \qed
The proof of this proposition is similar to that of Proposition 2.2 of [1]. To prove the next theorem, we state the following result which holds in broad generality [7, p. 17].

\[ |\Delta(f) - \Delta(h)| \leq \| f - h \|. \tag{2.1} \]

We denote by \( \lceil x \rceil \), the ceiling function of \( x \), i.e., the smallest integer greater than or equal to \( x \).

**Theorem 2.1.** The following applies to Problem (1.1).

(a) Assume \( K \) satisfies conditions (i) and (ii). Then

\[ \Delta(f) = \lceil \| f - \overline{f} \|/2 \rceil, \tag{2.2} \]

and \( f' = \overline{f} + \Delta(f) \) is the maximal best approximation to \( f \). Moreover, if \( f, h \in D \), then

\[ \| f' - h' \| \leq \| f - h \|, \quad \text{if} \ \Delta(f) = \Delta(h), \tag{2.3} \]

and

\[ \| f' - h' \| \leq 2 \| f - h \|. \tag{2.4} \]

The operator \( T : D \to K \) defined by \( T(f) = f' \) is a Lipschitzian selection operator with \( C(T) = 2 \).

(b) Assume \( K \) satisfies conditions (i) and (iii). Then (a) holds with \( \overline{f} \) replaced by \( \underline{f} \) and \( f' = \underline{f} - \Delta(f) \), which is the minimal best approximation to \( f \).

**Proof.** This is a modification of the proof of Proposition 3.1 of [1]. Let \( g \in K \), and \( g_0 = g - \| f - g \| \). Since \( \| f - g \| \) is an integer, by condition (i) on \( K \), we have that \( g_0 \in K \). Now, \( f \geq g_0 \). Hence, \( f \geq \overline{f} \geq g_0 \). This shows that \( f - \overline{f} \leq f - g + \| f - g \| \) or \( \| f - f \|/2 \leq \| f - g \|. \) Since \( \| f - g \| \) is an integer, we must have \( \lceil \| f - f \|/2 \rceil \leq \| f - g \| \). Hence, \( \| f - f \|/2 \leq \Delta(f) \). Again, since \( \lceil \| f - f \|/2 \rceil \) is an integer, by condition (i), \( f' = \overline{f} + \lceil \| f - f \|/2 \rceil \) is in \( K \). It is easy to show that \( \| f - f' \| \leq \lceil \| f - f \|/2 \rceil \). This establishes that \( f' \) is a best approximation and that (2.2) holds. Suppose now that \( g \) is any best approximation. Then \( f \geq g - \Delta(f) \). Consequently, \( f \geq \overline{f} \geq g - \Delta(f) \) and hence \( f' \geq g \). Thus \( f' \) is the maximal best approximation.

Now let \( f' = \overline{f} + \Delta(f) \) and \( h' = \overline{f} + \Delta(h) \) be two best approximations to \( f \) and \( h \) respectively. Then,

\[ \| f' - h' \| \leq \| \overline{f} - \overline{h} \| + |\Delta(f) - \Delta(h)|. \]

From this inequality, (2.1), and Proposition 2.1, both (2.3) and (2.4) follow. By (2.4), we have \( C(T) \leq 2 \).

To show \( C(T) = 2 \), let \( K \) be the set of all integer convex functions on \( S = [0, 1] \). Clearly, each function in \( K \) is constant on \( (0, 1) \) with possible discontinuities at 0 and 1. Let \( f(0) = -1, f(1) = 1 \) on \( (0, 1) \), and \( h(s) = 0 \) on \( [0, 1] \). Then \( f'(s) = 0 \) on \( (0, 1) \), \( f'(1) = 2 \), and \( h'(s) = 0 \) on \([0, 1]\) as may be easily verified. Consequently, \( \| f - h \| = 1 \) and \( \| f' - h' \| = 2 \). Hence \( C(T) = 2 \). The proof of part (b) is similar. □

**Theorem 2.2.** The following applies to Problem (1.2). Assume \( K \) satisfies conditions (i) and (ii). Then \( \overline{f} \) is the maximal best approximation to \( f \) and \( \Delta(f) = \| f - \overline{f} \| \leq 2 \Delta(f) \). The operator \( T : D \to K \) defined by \( T(f) = \overline{f} \) is the unique optimal Lipschitzian selection operator with \( C(T) = 1 \).

**Proof.** The proof of Theorem 3.2 of [1] may be applied by letting the first constant \( c \) in that proof be a positive integer, say 1. The second constant \( c = \| f - \overline{f} \| \) defined there is clearly an integer since both \( f \) and \( \overline{f} \) are integer functions. Hence, \( h = \overline{f} + c \) is in \( K \) by condition (i) since \( \overline{f} \) is in \( K \) by condition (ii). The rest of the proof applies verbatim. □
We now consider some applications of the problem. A function $k$ defined on a convex set $S \subset \mathbb{R}^n$ is said to be quasi-convex if $k(\lambda s + (1-\lambda)t) \leq \max\{k(s), k(t)\}$, for all $s$, $t$ in $S$ and all $0 \leq \lambda \leq 1$ [8]. If $S$ is not convex, for example, when it is a finite set, we define $k$ on $S$ to be quasi-convex if there exists a quasi-convex function $k'$ on the convex hull $\text{co}(S)$ of $S$ whose restriction to $S$ is $k$. It is easy to see that conditions (i) and (ii) hold for the set $K$ of all integer quasi-convex functions on $S$. The results of Theorems 2.1(a) and 2.2 then apply. When $S$ is finite, polynomial algorithms for computation of a best approximation can be developed by methods similar to those given in [9]. For our second example, we consider approximation by integer convex functions on a set $S$. In a manner analogous to the above, we may define a convex function on a domain $S$, which is possibly non-convex, by simply extending its usual definition for a convex domain to a non-convex $S$. Again, it is easy to verify that conditions (i) and (ii) hold for the set $K$ of integer convex functions on $S$. Hence Theorems 2.1(a) and 2.2 apply. If $S$ is convex then an integer convex function on $S$ is necessarily constant in the relative interior of $S$ and may have discontinuities at the points of the relative boundary. If $S$ is not convex, for example, if it is finite, then the set of integer convex functions on $S$ may include non-constant functions. For our third example, let $S$ be a partially ordered set and $K$, the set of all integer isotone functions on $S$. For example, $S$ is a rectangle in $\mathbb{R}^n$ with usual vector ordering. It is easy to show that conditions (i), (ii) and (iii) hold for $K$. Hence, both (a) and (b) of Theorem 2.1 apply. See [4] where stronger results are obtained for such problems on finite sets under weighted uniform norm. If $S$ is a real interval, then $K$ is the set of integer valued monotone non-decreasing functions on $S$. See [3] for a least squares approximation problem involving these functions.

References