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Existence and comparison principles for general quasilinear variational–hemivariational inequalities

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Abstract

We consider quasilinear elliptic variational–hemivariational inequalities involving convex, lower semicontinuous and locally Lipschitz functionals. We provide a generalization of the fundamental notion of sub- and supersolutions on the basis of which we then develop the sub–supersolution method for variational–hemivariational inequalities, including existence, comparison, compactness and extremality results.

Keywords: Comparison principles; Variational–hemivariational inequalities; Multivalued pseudomonotone operators

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, denote the usual Sobolev spaces with their dual spaces

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V^* and V_0^*, respectively. In this paper we deal with the following quasilinear variational–hemivariational inequality:

$$u \in \mathrm{dom}(\psi) \cap V_0: \quad \langle Au - f, v - u \rangle + \psi(v) - \psi(u) + \int_\Omega j^0(u; v - u)\, dx \geqslant 0, \quad \forall v \in V_0, \tag{1.1}$$

where $j^0(s; r)$ denotes the generalized directional derivative of the locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ at s in the direction r given by

$$j^0(s; r) = \limsup_{y \to s,\, t \downarrow 0} \frac{j(y + tr) - j(y)}{t} \tag{1.2}$$

(cf., e.g., [7, Chapter 2]), $f \in V_0^*$, and $\psi : V \to \mathbb{R} \cup \{+\infty\}$ is a convex, lower semicontinuous function such that $\mathrm{dom}(\psi) \cap V_0 \neq \emptyset$. Here $\mathrm{dom}(\psi)$ stands for the effective domain of ψ defined by $\mathrm{dom}(\psi) = \{v \in V \mid \psi(v) < +\infty\}$. The operator $A : V \to V_0^*$ is a second order quasilinear differential operator in divergence form

$$Au(x) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i\big(x, \nabla u(x)\big), \quad \text{with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right). \tag{1.3}$$

The above problem (1.1) includes various special cases such as, e.g., the following:

(i) For $\psi(u) \equiv 0$ and $j : \mathbb{R} \to \mathbb{R}$ smooth with its derivative $j' : \mathbb{R} \to \mathbb{R}$, (1.1) reduces to the weak formulation of the Dirichlet problem

$$u \in V_0: \quad Au + j'(u) = f \quad \text{in } V_0^*.$$

(ii) For $\psi(u) \equiv 0$, and $j : \mathbb{R} \to \mathbb{R}$ not necessarily smooth, then (1.1) is a hemivariational inequality of the form

$$u \in V_0: \quad \langle Au - f, v - u \rangle + \int_\Omega j^0(u; v - u)\, dx \geqslant 0, \quad \forall v \in V_0.$$

(iii) For $j : \mathbb{R} \to \mathbb{R}$ smooth, (1.1) becomes the variational inequality

$$u \in \mathrm{dom}(\psi) \cap V_0: \quad \big\langle Au + j'(u) - f, v - u \big\rangle + \psi(v) - \psi(u) \geqslant 0, \quad \forall v \in V_0.$$

The main goal of this paper is to develop a general framework for the sub–supersolution method for variational–hemivariational inequalities of the form (1.1) which include, e.g., the above special cases. In particular (1.1) includes constraint hemivariational inequalities as well in case that $\psi := I_K$, where I_K is the indicator function of some closed convex set K. Existence, comparison and compactness results for problem (1.1) are given. In particular, we prove the existence of extremal solutions in the order interval formed by sub- and supersolutions, and provide applications that demonstrate the applicability of the developed theory.

2. Notation and hypotheses

We assume the following hypotheses of Leray–Lions type on the coefficient functions a_i, $i = 1, \dots, N$, of the operator A:

(A1) Each $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., $a_i(x, \xi)$ is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and continuous in ξ for almost all $x \in \Omega$. There exist a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$, $1/p + 1/q = 1$, such that

$$\left|a_i(x, \xi)\right| \leqslant k_0(x) + c_0|\xi|^{p-1},$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

(A2) $\sum_{i=1}^N (a_i(x, \xi) - a_i(x, \xi'))(\xi_i - \xi_i') > 0$ for a.e. $x \in \Omega$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) $\sum_{i=1}^N a_i(x, \xi)\xi_i \geqslant \nu|\xi|^p - k_1(x)$ for a.e. $x \in \Omega$, and for all $\xi \in \mathbb{R}^N$ with some constant $\nu > 0$ and some function $k_1 \in L^1(\Omega)$.

As a consequence of (A1), (A2) the semilinear form a associated with the operator A by

$$\langle Au, \varphi\rangle := a(u, \varphi) = \int_\Omega \sum_{i=1}^N a_i(x, \nabla u)\frac{\partial \varphi}{\partial x_i}\, dx, \quad \forall \varphi \in V_0,$$

is well defined for any $u \in V$, and the operator $A : V_0 \to V_0^*$ is continuous, bounded, and strictly monotone. For functions $w, z : \Omega \to \mathbb{R}$ and sets W and Z of functions defined on Ω we use the notations: $w \wedge z = \min\{w, z\}$, $w \vee z = \max\{w, z\}$, $W \wedge Z = \{w \wedge z \mid w \in W,\ z \in Z\}$, $W \vee Z = \{w \vee z \mid w \in W,\ z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$. Next we introduce our basic notion of sub–supersolution.

Definition 2.1. A function $\underline{u} \in V$ is called a *subsolution* of (1.1) if the following conditions are fulfilled:

(i) $\underline{u} \leqslant 0$ on $\partial\Omega$,
(ii) $\underline{u} \vee (\mathrm{dom}(\psi) \cap V_0) \subset \mathrm{dom}(\psi) \cap V_0$,
(iii) there exists a mapping $\hat{\psi} : V \to \mathbb{R} \cup \{+\infty\}$ and a constant $\hat{c} \geqslant 0$ such that the following holds:
  (a) $\underline{u} \in \mathrm{dom}(\hat{\psi})$,
  (b) $\psi(v \vee \underline{u}) + \hat{\psi}(v \wedge \underline{u}) - \psi(v) - \hat{\psi}(\underline{u}) \leqslant \hat{c} \int_\Omega [(\underline{u} - v)^+]^p\, dx$, $\forall v \in \mathrm{dom}(\psi) \cap V_0$,
  (c) $\langle A\underline{u} - f, v - \underline{u}\rangle + \hat{\psi}(v) - \hat{\psi}(\underline{u}) + \int_\Omega j^0(\underline{u}; v - \underline{u})\, dx \geqslant 0$, $\forall v \in \underline{u} \wedge (\mathrm{dom}(\psi) \cap V_0)$.

Similarly we define a supersolution as follows.

Definition 2.2. A function $\bar{u} \in V$ is a *supersolution* of (1.1) if the following conditions are fulfilled:

(i) $\bar{u} \geqslant 0$ on $\partial\Omega$,
(ii) $\bar{u} \wedge (\mathrm{dom}(\psi) \cap V_0) \subset \mathrm{dom}(\psi) \cap V_0$,

(iii) there exists a mapping $\tilde{\psi}: V \to \mathbb{R} \cup \{+\infty\}$ and a constant $\tilde{c} \geqslant 0$ such that the following holds:
(a) $\bar{u} \in \operatorname{dom}(\tilde{\psi})$,
(b) $\psi(v \wedge \bar{u}) + \tilde{\psi}(v \vee \bar{u}) - \psi(v) - \tilde{\psi}(\bar{u}) \leqslant \tilde{c} \int_\Omega [(v - \bar{u})^+]^p \, dx$, $\forall v \in \operatorname{dom}(\psi) \cap V_0$,
(c) $\langle A\bar{u} - f, v - \bar{u} \rangle + \tilde{\psi}(v) - \tilde{\psi}(\bar{u}) + \int_\Omega j^0(\bar{u}; v - \bar{u}) \, dx \geqslant 0$, $\forall v \in \bar{u} \vee (\operatorname{dom}(\psi) \cap V_0)$.

The above definitions of sub–supersolutions require the existence of functionals $\hat{\psi}$ and $\tilde{\psi}$ that satisfy conditions (a)–(c) in Definitions 2.1 and 2.2, respectively, which extend the one for variational inequalities introduced recently in [9]. In fact one can show that the above notions of sub–supersolution extend those for inclusions of hemivariational type introduced in [3,4] and for variational and/or hemivariational inequalities in [5,6,8,9]. Let us consider a few examples.

Example 2.1. Assume $\psi(u) \equiv 0$ and $j: \mathbb{R} \to \mathbb{R}$ smooth, then as already pointed out in the Introduction (1.1) reduces to the Dirichlet problem

$$u \in V_0: \quad Au + j'(u) = f \quad \text{in } V_0^*.$$

We shall see that the above definitions contain the usual notion of sub- and supersolution for the Dirichlet problem. According to Definition 2.1 a function $\underline{u} \in V$ with $\underline{u} \leqslant 0$ on $\partial\Omega$ is a subsolution if (ii) and (iii) of Definition 2.1 can be fulfilled. Since $\operatorname{dom}(\psi) = V$, we see that by choosing $\hat{\psi} = 0$ the conditions (ii) and (iii)(a)–(b) are trivially satisfied. Thus $\underline{u}$ is only required to satisfy condition (iii)(c), i.e.,

$$\langle A\underline{u} - f, v - \underline{u} \rangle + \int_\Omega j'(\underline{u})(v - \underline{u}) \, dx \geqslant 0, \quad \forall v \in \underline{u} \wedge V_0.$$

Let $\varphi \in V_0$; then $v \in \underline{u} \wedge V_0$ is given by $v = \underline{u} \wedge \varphi = \underline{u} - (\underline{u} - \varphi)^+$, which yields

$$\big\langle A\underline{u} - f, -(\underline{u} - \varphi)^+ \big\rangle + \int_\Omega j'(\underline{u})\big(-(\underline{u} - \varphi)^+\big) \, dx \geqslant 0, \quad \forall \varphi \in V_0,$$

and thus we obtain with $w = (\underline{u} - \varphi)^+ \in V_0 \cap L_+^p(\Omega)$ the inequality

$$\langle A\underline{u} - f, w \rangle + \int_\Omega j'(\underline{u}) w \, dx \leqslant 0, \quad \forall w \in W,$$

where $W = \{w = (\underline{u} - \varphi)^+ \mid \varphi \in V_0\}$. Observing that W is dense in $V_0 \cap L_+^p(\Omega)$ (see [2]) we get the usual notion of weak subsolution of the Dirichlet problem. Similarly Definition 2.2 contains the usual notion for a supersolution of the above Dirichlet problem.

Example 2.2. Let $K \subset V_0$ be a closed and convex set, and let $\psi = I_K$, where $I_K : V \to \mathbb{R} \cup \{+\infty\}$ denotes the indicator function related with the given closed convex set $K \neq \emptyset$ and defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K, \end{cases}$$

which is proper, convex, and lower semicontinuous. Problem (1.1) then becomes: Find $u \in K$ such that

$$\langle Au - f, v - u\rangle + I_K(v) - I_K(u) + \int_\Omega j^0(u; v - u)\,dx \geqslant 0, \quad \forall v \in V_0. \tag{2.1}$$

In this case $\underline{u} \in V$ is a subsolution of (2.1) according to Definition 2.1 if the following is satisfied:

(1) $\underline{u} \leqslant 0$ on $\partial\Omega$,
(2) $\underline{u} \vee K \subset K$,
(3) $\langle A\underline{u} - f, v - \underline{u}\rangle + \int_\Omega j^0(\underline{u}; v - \underline{u})\,dx \geqslant 0, \forall v \in \underline{u} \wedge K$.

One readily verifies that with (1)–(3) and taking $\hat{\psi}(v) \equiv 0$ and $\hat{c} = 0$ all the conditions of Definition 2.1 are fulfilled. Analogous conditions can be found for a supersolution $\bar{u}$ of (2.1):

(1′) $\bar{u} \geqslant 0$ on $\partial\Omega$,
(2′) $\bar{u} \wedge K \subset K$,
(3′) $\langle A\bar{u} - f, v - \bar{u}\rangle + \int_\Omega j^0(\bar{u}; v - \bar{u})\,dx \geqslant 0, \forall v \in \bar{u} \vee K$.

Conditions (1)–(3) and (1′)–(3′) which where introduced in [8] to define sub–supersolutions turn out to be special cases of Definitions 2.1 and 2.2, respectively.

Example 2.3. Given a convex lower semicontinuous function $h : \mathbb{R} \to \mathbb{R}$, we introduce $g : V \to \mathbb{R} \cup \{+\infty\}$ by

$$g(v) = \begin{cases} \int_\Omega h(v(x))\,dx & \text{if } h(v) \in L^1(\Omega), \\ +\infty & \text{if } h(v) \notin L^1(\Omega), \end{cases}$$

which is known to be proper, convex and lower semicontinuous. Consider problem (1.1) with $\psi = g$, i.e., find $u \in \operatorname{dom}(g) \cap V_0$ such that

$$\langle Au - f, v - u\rangle + g(v) - g(u) + \int_\Omega j^0(u; v - u)\,dx \geqslant 0, \quad \forall v \in V_0. \tag{2.2}$$

Then, e.g., the following conditions on a function $\underline{u} \in V$ imply that $\underline{u}$ is a subsolution according to Definition 2.1:

(1) $\underline{u} \leqslant 0$ on $\partial\Omega$,
(2) $\underline{u} \vee (\operatorname{dom}(g) \cap V_0) \subset \operatorname{dom}(g) \cap V_0$,
(3) $\underline{u} \in \operatorname{dom}(g)$, and $\langle A\underline{u} - f, v - \underline{u}\rangle + g(v) - g(\underline{u}) + \int_\Omega j^0(\underline{u}; v - \underline{u})\,dx \geqslant 0, \forall v \in \underline{u} \wedge (\operatorname{dom}(g) \cap V_0)$.

Taking $\hat{\psi} = g$ and $\hat{c}$ any nonnegative constant one can see that in view of (1)–(3) all conditions of Definition 2.1 are satisfied. This is because for all $v \in \operatorname{dom}(g) \cap V_0$ the following equation holds for the integral functional g:

$$g(v \vee \underline{u}) + g(v \wedge \underline{u}) - g(v) - g(\underline{u}) = 0. \tag{2.3}$$

The identity (2.3) can easily be proved by splitting up Ω into $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{x \in \Omega \mid v(x) < \underline{u}(x)\}, \qquad \Omega_2 = \{x \in \Omega \mid v(x) \geqslant \underline{u}(x)\},$$

and by considering the resulting integrals. Thus, for example, if $f \in L^{p^{*\prime}}(\Omega)$ (with p^* the critical Sobolev exponent) and $a_i(x,0) = 0$ for $i = 1, \ldots, N$, then $\underline{u} = 0$ is a subsolution if for some $\xi \in \partial h(0)$ the following inequality holds:

$$f(x) \geqslant -j^0(0; -1) + \xi, \quad \text{for a.e. } x \in \Omega.$$

The corresponding conditions for a supersolution $\bar{u}$ are obvious and can be omitted.

Remark 2.1. It should be noted that in specific situations the functionals $\hat{\psi}, \tilde{\psi}$ allow much flexibility for the construction of sub–supersolutions. We provide a construction of sub–supersolutions for more specific problems in the last section.

Let $\partial j : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ denote Clarke's generalized gradient of j defined by

$$\partial j(s) := \{\zeta \in \mathbb{R} \mid j^0(s; r) \geqslant \zeta r, \ \forall r \in \mathbb{R}\}. \tag{2.4}$$

We assume the following hypothesis for j:

(H) The function $j : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and its Clarke's generalized gradient ∂j satisfies the following growth conditions:
- (i) there exists a constant $c_1 \geqslant 0$ such that
$$\xi_1 \leqslant \xi_2 + c_1(s_2 - s_1)^{p-1}$$
for all $\xi_i \in \partial j(s_i)$, $i = 1, 2$, and for all s_1, s_2 with $s_1 < s_2$,
- (ii) there is a constant $c_2 \geqslant 0$ such that
$$\xi \in \partial j(s): \quad |\xi| \leqslant c_2\bigl(1 + |s|^{p-1}\bigr), \quad \forall s \in \mathbb{R}.$$

Let $L^p(\Omega)$ be equipped with the natural partial ordering of functions defined by $u \leqslant w$ if and only if $w - u$ belongs to the positive cone $L^p_+(\Omega)$ of all nonnegative elements of $L^p(\Omega)$. This induces a corresponding partial ordering also in the subspace V of $L^p(\Omega)$, and if $u, w \in V$ with $u \leqslant w$ then

$$[u, w] = \{z \in V \mid u \leqslant z \leqslant w\}$$

denotes the order interval formed by u and w.

In the proofs of our main results we make use of the cut-off function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ related with an ordered pair of functions $\underline{u} \leqslant \bar{u}$, and given by

$$b(x,s) = \begin{cases} (s - \bar{u}(x))^{p-1} & \text{if } s > \bar{u}(x), \\ 0 & \text{if } \underline{u}(x) \leqslant s \leqslant \bar{u}(x), \\ -(\underline{u}(x) - s)^{p-1} & \text{if } s < \underline{u}(x). \end{cases} \tag{2.5}$$

One readily verifies that b is a Carathéodory function satisfying the growth condition

$$\bigl|b(x,s)\bigr| \leqslant k(x) + c_3|s|^{p-1} \tag{2.6}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, with some function $k \in L^q_+(\Omega)$ and a constant $c_3 \geqslant 0$. Moreover, one has the following estimate:

$$\int_\Omega b\big(x, u(x)\big)u(x)\,dx \geqslant c_4\|u\|^p_{L^p(\Omega)} - c_5, \quad \forall u \in L^p(\Omega), \tag{2.7}$$

where c_4 and c_5 are some positive constants. In view of (2.6) the Nemytskij operator $B : L^p(\Omega) \to L^q(\Omega)$ defined by

$$Bu(x) = b\big(x, u(x)\big)$$

is continuous and bounded, and thus due to the compact embedding $V \subset L^p(\Omega)$ it follows that $B : V_0 \to V_0^*$ is compact.

3. Preliminaries

In this section we briefly recall a surjectivity result for multivalued mappings in reflexive Banach spaces (cf., e.g., [10, Theorem 2.12]) which among others will be used in the proof of our main result in this section.

Theorem 3.1. *Let X be a real reflexive Banach space with dual space X^*, $\Phi : X \to 2^{X^*}$ a maximal monotone operator, and $u_0 \in \mathrm{dom}(\Phi)$. Let $A : X \to 2^{X^*}$ be a pseudomonotone operator, and assume that either A_{u_0} is quasi-bounded or Φ_{u_0} is strongly quasi-bounded. Assume further that $A : X \to 2^{X^*}$ is u_0-coercive, i.e., there exists a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ with $c(r) \to +\infty$ as $r \to +\infty$ such that for all $(u, u^*) \in \mathrm{graph}(A)$ one has $\langle u^*, u - u_0\rangle \geqslant c(\|u\|_X)\|u\|_X$. Then $A + \Phi$ is surjective, i.e., $\mathrm{range}(A + \Phi) = X^*$.*

The operators A_{u_0} and Φ_{u_0} that appear in the theorem above are defined by $A_{u_0}(v) := A(u_0 + v)$ and similarly for Φ_{u_0}. As for the notion of *quasi-bounded* and *strongly quasi-bounded* we refer to [10, p. 51]. In particular, one has that any bounded operator is quasi-bounded and strongly quasi-bounded as well. The following proposition provides sufficient conditions for an operator $A : X \to 2^{X^*}$ to be pseudomonotone, which is suitable for our purpose.

Proposition 3.1. *Let X be a real reflexive Banach space, and assume that $A : X \to 2^{X^*}$ satisfies the following conditions*:

(i) *For each $u \in X$ we have that $A(u)$ is a nonempty, closed and convex subset of X^*.*
(ii) *$A : X \to 2^{X^*}$ is bounded.*
(iii) *If $u_n \rightharpoonup u$ in X and $u_n^* \rightharpoonup u^*$ in X^* with $u_n^* \in A(u_n)$ and if $\limsup\langle u_n^*, u_n - u\rangle \leqslant 0$, then $u^* \in A(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.*

Then the operator $A : X \to 2^{X^*}$ is pseudomonotone.

As for the proof of Proposition 3.1 we refer, e.g., to [10, Chapter 2].

4. Existence and comparison result

The main result of this section is given by the following theorem which provides an existence and comparison result for the variational–hemivariational inequality (1.1).

Theorem 4.1. *Let $\bar{u}$ and $\underline{u}$ be super- and subsolutions of* (1.1), *respectively, satisfying* $\underline{u} \leqslant \bar{u}$. *Then under hypotheses* (A1)–(A3) *and* (H), *there exist solutions of* (1.1) *within the order interval* $[\underline{u}, \bar{u}]$.

Proof. Consider the variational–hemivariational inequality (1.1): Find $u \in \mathrm{dom}(\psi) \cap V_0$ such that

$$\langle Au - f, v - u\rangle + \psi(v) - \psi(u) + \int_\Omega j^0(u; v - u)\, dx \geqslant 0, \quad \forall v \in V_0. \tag{4.1}$$

Since we are looking for solutions of (4.1) within $[\underline{u}, \bar{u}]$, we consider the following auxiliary problem: Find $u \in \mathrm{dom}(\psi) \cap V_0$ such that

$$\bigl\langle Au - f + \lambda B(u), v - u\bigr\rangle + \psi(v) - \psi(u) + \int_\Omega j^0(u; v - u)\, dx \geqslant 0, \quad \forall v \in V_0, \tag{4.2}$$

where B is the cut-off operator introduced in Section 2, and $\lambda \geqslant 0$ is some parameter to be specified later.

We proceed in two steps.

Step 1. *Existence for* (4.2). Let us introduce the functional $J : L^p(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \int_\Omega j\bigl(v(x)\bigr)\, dx, \quad \forall v \in L^p(\Omega),$$

which by hypothesis (H) is locally Lipschitz, and moreover, by Aubin–Clarke theorem (see [7, p. 83]) for each $u \in L^p(\Omega)$ we have

$$\xi \in \partial J(u) \quad \Rightarrow \quad \xi \in L^q(\Omega) \quad \text{with } \xi(x) \in \partial j\bigl(u(x)\bigr) \text{ for a.e. } x \in \Omega.$$

Consider now the multivalued operator

$$A + \lambda B + \partial(J|_{V_0}) + \partial(\psi|_{V_0}) : V_0 \to 2^{V_0^*},$$

where $J|_{V_0}$ and $\psi|_{V_0}$ denote the restriction of J and ψ, respectively, to V_0, and $\partial(\psi|_{V_0})$ is the subdifferential of $\psi|_{V_0}$ in the sense of convex analysis. It is well known that $\Phi := \partial(\psi|_{V_0}) : V_0 \to 2^{V_0^*}$ is a maximal monotone operator, cf., e.g., [11]. Since $A : V_0 \to V_0^*$ is strictly monotone, bounded, and continuous, and $\lambda B : V_0 \to V_0^*$ is bounded, continuous and compact, it follows that $A + \lambda B : V_0 \to V_0^*$ is a (singlevalued) pseudomonotone, continuous, and bounded operator. In [5] it has been shown that $\partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is a (multivalued) pseudomonotone operator, which, due to (H), is bounded. Thus $A_0 := A + \lambda B + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is a pseudomonotone and bounded operator. Hence, it follows by Theorem 3.1 that $\mathrm{range}(A_0 + \Phi) = V_0^*$ provided A_0 is u_0-coercive for some

$u_0 \in \mathrm{dom}(\partial(\psi|_{V_0}))$, which can readily be seen as follows: For any $v \in V_0$ and any $w \in \partial(J|_{V_0})(v)$ we obtain by applying (A3), (H)(ii) and (2.7) the estimate

$$\begin{aligned}
&\bigl\langle Av + \lambda B(v) + w, v - u_0\bigr\rangle \\
&\quad = \int_\Omega \sum_{i=1}^N a_i(x, \nabla v)\frac{\partial v}{\partial x_i}\,dx + \lambda\bigl\langle B(v), v\bigr\rangle + \int_\Omega wv\,dx - \bigl\langle Av + \lambda B(v) + w, u_0\bigr\rangle \\
&\quad \geqslant \nu \int_\Omega |\nabla v|^p\,dx - \|k_1\|_{L^1(\Omega)} + c_4\lambda\|v\|^p_{L^p(\Omega)} - c_5\lambda - c_2\int_\Omega \bigl(1 + |v|^{p-1}\bigr)|v|\,dx \\
&\qquad - \bigl|\langle Av + \lambda B(v) + w, u_0\rangle\bigr| \\
&\quad \geqslant \nu\|v\|^p_{V_0} - C\bigl(1 + \|v\|^{p-1}_{V_0}\bigr),
\end{aligned} \tag{4.3}$$

for some constant $C > 0$, by choosing the constant λ in such a way that $c_4\lambda > c_2$. Since $p > 1$, the coercivity of A_0 follows from (4.3). In view of the surjectivity of the operator $A_0 + \Phi$ there exists $u \in \mathrm{dom}(\Phi) \subset \mathrm{dom}(\psi) \cap V_0$ such that $f \in A_0(u) + \Phi(u)$, i.e., there is $\xi \in \partial(J|_{V_0})(u)$ with $\xi \in L^q(\Omega)$ and $\xi(x) \in \partial j(u(x))$ for a.e. $x \in \Omega$, and $\eta \in \Phi(u)$ such that

$$Au - f + \lambda B(u) + \xi + \eta = 0 \quad \text{in } V_0^*, \tag{4.4}$$

where

$$\langle \xi, \varphi\rangle = \int_\Omega \xi(x)\varphi(x)\,dx, \quad \forall \varphi \in V_0, \tag{4.5}$$

and

$$\psi(v) \geqslant \psi(u) + \langle \eta, v - u\rangle, \quad \forall v \in V_0. \tag{4.6}$$

By definition of Clarke's generalized gradient ∂j from (4.5) we get

$$\langle \xi, \varphi\rangle = \int_\Omega \xi(x)\varphi(x)\,dx \leqslant \int_\Omega j^0\bigl(u(x); \varphi(x)\bigr)\,dx, \quad \forall \varphi \in V_0. \tag{4.7}$$

Thus from (4.4)–(4.7) with φ replaced by $v - u$ we obtain (4.2), which proves the existence of solutions of problem (4.2).

Step 2. *$\underline{u} \leqslant u \leqslant \bar{u}$ for any solution u of* (4.2). Let us first show $u \leqslant \bar{u}$. By definition the supersolution $\bar{u}$ satisfies: $\bar{u} \in \mathrm{dom}(\tilde{\psi})$, $\bar{u} \geqslant 0$ on $\partial\Omega$, and

$$\begin{aligned}
&\langle A\bar{u} - f, v - \bar{u}\rangle + \tilde{\psi}(v) - \tilde{\psi}(\bar{u}) + \int_\Omega j^0(\bar{u}; v - \bar{u})\,dx \geqslant 0, \\
&\forall v \in \bar{u} \vee \bigl(\mathrm{dom}(\psi) \cap V_0\bigr).
\end{aligned} \tag{4.8}$$

Let u be any solution of (4.2). We apply the special test function $v = \bar{u} \vee u = \bar{u} + (u - \bar{u})^+$ ($\in \bar{u} \vee (\mathrm{dom}(\psi) \cap V_0)$) in (4.8) and $v = \bar{u} \wedge u = u - (u - \bar{u})^+$ ($\in \mathrm{dom}(\psi) \cap V_0$, due to the hypothesis) in (4.2), and get by adding the resulting inequalities the following one:

$$\langle A\bar{u} - Au, (u-\bar{u})^+\rangle + \lambda\langle B(u), -(u-\bar{u})^+\rangle + \tilde{\psi}(\bar{u}\vee u) - \tilde{\psi}(\bar{u}) + \psi(\bar{u}\wedge u) - \psi(u) + \int_\Omega \big(j^0\big(\bar{u}; (u-\bar{u})^+\big) + j^0\big(u; -(u-\bar{u})^+\big)\big)\,dx \geqslant 0,$$

which yields due to

$$\langle Au - A\bar{u}, (u-\bar{u})^+\rangle \geqslant 0,$$

the inequality

$$\lambda\langle B(u), (u-\bar{u})^+\rangle \leqslant \tilde{\psi}(\bar{u}\vee u) - \tilde{\psi}(\bar{u}) + \psi(\bar{u}\wedge u) - \psi(u) + \int_\Omega \big(j^0\big(\bar{u}; (u-\bar{u})^+\big) + j^0\big(u; -(u-\bar{u})^+\big)\big)\,dx. \tag{4.9}$$

By using (H) and the properties on j^0 and ∂j we get for certain $\bar{\xi}(x) \in \partial j(\bar{u}(x))$ and $\xi(x) \in \partial j(u(x))$ the following estimate of the second term on the right-hand side of (4.9):

$$\begin{aligned}\int_\Omega & \big(j^0\big(\bar{u}; (u-\bar{u})^+\big) + j^0\big(u; -(u-\bar{u})^+\big)\big)\,dx \\ &= \int_{\{u>\bar{u}\}} \big(j^0(\bar{u}; u-\bar{u}) + j^0\big(u; -(u-\bar{u})\big)\big)\,dx \\ &= \int_{\{u>\bar{u}\}} \big(\bar{\xi}(x)\big(u(x)-\bar{u}(x)\big) + \xi(x)\big(-\big(u(x)-\bar{u}(x)\big)\big)\big)\,dx \\ &= \int_{\{u>\bar{u}\}} \big(\bar{\xi}(x)-\xi(x)\big)\big(u(x)-\bar{u}(x)\big)\,dx \leqslant \int_{\{u>\bar{u}\}} c_1\big(u(x)-\bar{u}(x)\big)^p\,dx.\end{aligned} \tag{4.10}$$

Since

$$\langle B(u), (u-\bar{u})^+\rangle = \int_{\{u>\bar{u}\}} (u-\bar{u})^p\,dx,$$

we get from (4.9), (4.10) and due to the definition of the supersolution the estimate

$$(\lambda - c_1 - \tilde{c}) \int_{\{u>\bar{u}\}} (u-\bar{u})^p\,dx \leqslant 0. \tag{4.11}$$

Selecting the parameter λ, in addition, such that $\lambda - c_1 - \tilde{c} > 0$ then (4.11) yields

$$\int_\Omega \big((u-\bar{u})^+\big)^p\,dx \leqslant 0,$$

which implies $(u-\bar{u})^+ = 0$ and thus $u \leqslant \bar{u}$.

The proof for the inequality $\underline{u} \leqslant u$ can be carried out in a similar way. By definition the subsolution $\underline{u}$ satisfies: $\underline{u} \in \operatorname{dom}(\hat{\psi})$, $\underline{u} \leqslant 0$ on $\partial\Omega$, and

$$\langle A\underline{u} - f, v - \underline{u}\rangle + \hat{\psi}(v) - \hat{\psi}(\underline{u}) + \int_\Omega j^0(\underline{u}; v - \underline{u})\,dx \geqslant 0,$$

$$\forall v \in \underline{u} \wedge \big(\mathrm{dom}(\psi) \cap V_0\big). \tag{4.12}$$

Using the test functions $v = \underline{u} \wedge u = \underline{u} - (\underline{u} - u)^+ (\in \underline{u} \wedge (\mathrm{dom}(\psi) \cap V_0))$ in (4.12) and $v = \underline{u} \vee u = u + (\underline{u} - u)^+$ $(\in \mathrm{dom}(\psi) \cap V_0)$ in (4.2), respectively, we get by adding the resulting inequalities the following one:

$$\big\langle Au - A\underline{u}, (\underline{u} - u)^+\big\rangle + \lambda\big\langle B(u), (\underline{u} - u)^+\big\rangle + \hat{\psi}(\underline{u} \wedge u) - \hat{\psi}(\underline{u})$$
$$+ \psi(\underline{u} \vee u) - \psi(u) + \int_\Omega \big(j^0\big(\underline{u}; -(\underline{u} - u)^+\big) + j^0\big(u; (\underline{u} - u)^+\big)\big)\,dx \geqslant 0.$$

Following the same lines as above we arrive at

$$(\lambda - c_1 - \hat{c}) \int_{\{\underline{u} > u\}} (\underline{u} - u)^p\,dx \leqslant 0.$$

Choosing $\lambda - c_1 - \hat{c} > 0$ implies $\underline{u} \leqslant u$. This completes the proof of the theorem. □

5. Compactness and existence of extremal solutions

Let $\mathcal{S}$ denote the set of all solutions of (1.1) within the interval $[\underline{u}, \bar{u}]$ of an ordered pair of sub- and supersolutions. The smallest and greatest elements of $\mathcal{S}$ are called the *extremal solutions* of (1.1) within $[\underline{u}, \bar{u}]$.

Theorem 5.1. *Under the hypotheses of Theorem* 4.1 *the solution set $\mathcal{S}$ is compact in V_0.*

Proof. First we prove that $\mathcal{S}$ is bounded in V_0. Since any $u \in \mathcal{S}$ belongs to the interval $[\underline{u}, \bar{u}]$ it follows that $\mathcal{S}$ is bounded in $L^p(\Omega)$. Moreover, any $u \in \mathcal{S}$ solves (1.1), i.e., u satisfies

$$u \in \mathrm{dom}(\psi) \cap V_0: \quad \langle Au - f, v - u\rangle + \psi(v) - \psi(u) + \int_\Omega j^0(u; v - u)\,dx \geqslant 0,$$
$$\forall v \in V_0.$$

Let u_0 be any (fixed) element of $\mathrm{dom}(\psi) \cap V_0$. By taking $v = u_0$ in the above inequality we get

$$\langle Au, u\rangle \leqslant \langle Au, u_0\rangle + \langle f, u - u_0\rangle + \psi(u_0) - \psi(u) + \int_\Omega j^0(u; u_0 - u)\,dx. \tag{5.1}$$

Since ψ is bounded below by an affine function on V we get the following estimate for some nonnegative constant d:

$$\psi(u) \geqslant -d\big(\|u\|_V + 1\big),$$

which yields by applying Young's inequality and the equivalence of the norm $\|u\|_V \sim \|\nabla u\|_{L^p(\Omega)}$ for $u \in V_0$,

$$\psi(u) \geqslant -\frac{\nu}{2}\|\nabla u\|^p_{L^p(\Omega)} - D,$$

for some constant $D > 0$ not depending on u. By means of the last inequality and by applying (A3), (H)(ii), and Young's inequality we obtain the following estimate:

$$\begin{aligned}\frac{\nu}{2}\|\nabla u\|^p_{L^p(\Omega)} &\leqslant \|k_1\|_{L^1(\Omega)} + c(\varepsilon)\big(\|f\|^q_{V_0^*} + 1\big) + \varepsilon\|u\|^p_{V_0} \\ &\quad + \tilde{\alpha}\big(\|u\|_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} + 1\big),\end{aligned} \tag{5.2}$$

for any $\varepsilon > 0$ and a constant $\tilde{\alpha} > 0$. Hence, the boundedness of $\mathcal{S}$ in V_0 follows by choosing ε sufficiently small and by taking into account that $\mathcal{S}$ is bounded in $L^p(\Omega)$.

Let $(u_n) \subset \mathcal{S}$. From the above boundedness of $\mathcal{S}$ in V_0, we can choose a subsequence (u_k) of (u_n) such that

$$\begin{aligned}&u_k \rightharpoonup u \quad \text{in } V_0, \qquad u_k \to u \quad \text{in } L^p(\Omega), \quad \text{and} \\ &u_k(x) \to u(x) \quad \text{a.e. in } \Omega.\end{aligned} \tag{5.3}$$

Obviously $u \in [\underline{u}, \bar{u}]$. Since u_k solve (1.1), we can put $v = u \in V_0$ in (1.1) (with u_k instead of u) and get

$$\langle Au_k - f, u - u_k\rangle + \psi(u) - \psi(u_k) + \int_\Omega j^0(u_k; u - u_k)\,dx \geqslant 0,$$

and thus

$$\langle Au_k, u_k - u\rangle \leqslant \langle f, u_k - u\rangle + \psi(u) - \psi(u_k) + \int_\Omega j^0(u_k; u - u_k)\,dx. \tag{5.4}$$

Due to (5.3) and due to the fact that $(s, r) \mapsto j^0(s; r)$ is upper semicontinuous we get by applying Fatou's lemma

$$\limsup_k \int_\Omega j^0(u_k; u - u_k)\,dx \leqslant \int_\Omega \limsup_k j^0(u_k; u - u_k)\,dx = 0. \tag{5.5}$$

In view of (5.5) we thus obtain from (5.3), (5.4) and because ψ is weakly lower semicontinuous

$$\limsup_k \langle Au_k, u_k - u\rangle \leqslant 0. \tag{5.6}$$

Since the operator A has the (S_+)-property, the weak convergence of (u_k) in V_0 along with (5.6) imply the strong convergence $u_k \to u$ in V_0, see, e.g., [1, Theorem D.2.1]. Moreover, the limit u belongs to $\mathcal{S}$ as can be seen by passing to the lim sup on the left-hand side of the following inequality:

$$\langle Au_k - f, v - u_k\rangle + \psi(v) - \psi(u_k) + \int_\Omega j^0(u_k; v - u_k)\,dx \geqslant 0, \tag{5.7}$$

where we have used Fatou's lemma, the lower semicontinuity of ψ and the strong convergence of (u_k) in V_0. This completes the proof. □

As for the existence of extremal solutions in $\mathcal{S}$, let us introduce the following notion.

Definition 5.1. Let $(\mathcal{P}, \leqslant)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be *upward directed* if for each pair $x, y \in \mathcal{C}$ there is $z \in \mathcal{C}$ such that $x \leqslant z$ and $y \leqslant z$, and $\mathcal{C}$ is *downward directed* if for each pair $x, y \in \mathcal{C}$ there is $w \in \mathcal{C}$ such that $w \leqslant x$ and $w \leqslant y$. If $\mathcal{C}$ is both upward and downward directed it is called *directed*.

We are now ready to prove our extremality result.

Theorem 5.2. *Let the hypotheses of Theorem* 4.1 *be satisfied, and assume, moreover,*

$$\mathrm{dom}(\psi) \wedge \mathrm{dom}(\psi) \subset \mathrm{dom}(\psi) \quad \textit{and} \quad \mathrm{dom}(\psi) \vee \mathrm{dom}(\psi) \subset \mathrm{dom}(\psi). \tag{5.8}$$

If there is a constant $c \geqslant 0$ such that

$$\psi(w \vee v) - \psi(w) + \psi(w \wedge v) - \psi(v) \leqslant c \int_{\{v>w\}} (v-w)^p \, dx, \tag{5.9}$$

for all $w, v \in \mathrm{dom}(\psi)$, then the solution set $\mathcal{S}$ possesses extremal elements.

Proof. *Step* 1. *$\mathcal{S}$ is a directed set*. As a consequence of Theorem 4.1, we have $\mathcal{S} \neq \emptyset$. Given $u_1, u_2 \in \mathcal{S}$, let us show that there is $u \in \mathcal{S}$ such that $u_k \leqslant u$, $k = 1, 2$, which means $\mathcal{S}$ is upward directed. To this end we consider the following auxiliary variational–hemivariational inequality: Find $u \in \mathrm{dom}(\psi) \cap V_0$ such that

$$\begin{aligned} &\big\langle Au - f + \lambda B(u), v - u \big\rangle + \psi(v) - \psi(u) + \int_\Omega j^0(u; v-u)\, dx \geqslant 0, \\ &\forall v \in V_0, \end{aligned} \tag{5.10}$$

where $\lambda \geqslant 0$ is a free parameter to be chosen later. Unlike in the proof of Theorem 4.1 the operator B is now given by the following cut-off function $b : \Omega \times \mathbb{R} \to \mathbb{R}$:

$$b(x,s) = \begin{cases} (s - \bar{u}(x))^{p-1} & \text{if } s > \bar{u}(x), \\ 0 & \text{if } u_0(x) \leqslant s \leqslant \bar{u}(x), \\ -(u_0(x) - s)^{p-1} & \text{if } s < u_0(x), \end{cases} \tag{5.11}$$

where $u_0 = \max(u_1, u_2)$. By arguments similar to those in the proof of Theorem 4.1 we get the existence of solutions of (5.10) (see Step 1 in the proof of Theorem 4.1). The set $\mathcal{S}$ is shown to be upward directed provided that any solution u of (5.10) satisfies $u_k \leqslant u \leqslant \bar{u}$, $k = 1, 2$, because then $Bu = 0$ and thus $u \in \mathcal{S}$ exceeding u_k. Because $u_k \in \mathcal{S}$, we have $u_k \in \mathrm{dom}(\psi) \cap V_0 \cap [\underline{u}, \bar{u}]$ and

$$\langle Au_k - f, v - u_k \rangle + \psi(v) - \psi(u_k) + \int_\Omega j^0(u_k; v - u_k)\, dx \geqslant 0, \quad \forall v \in V_0. \tag{5.12}$$

Note that (5.8) implies that

$$u + (u_k - u)^+ = u \vee u_k \in \operatorname{dom}(\psi) \cap V_0$$

and

$$u_k - (u_k - u)^+ = u \wedge u_k \in \operatorname{dom}(\psi) \cap V_0.$$

Therefore, one can take as special functions $v = u + (u_k - u)^+$ in (5.10) and $v = u_k - (u_k - u)^+$ in (5.12). Adding the resulting inequalities we obtain

$$\begin{aligned}&\big\langle Au_k - Au, (u_k - u)^+\big\rangle - \lambda\big\langle B(u), (u_k - u)^+\big\rangle \\ &\quad \leqslant \psi(u \vee u_k) - \psi(u) + \psi(u \wedge u_k) - \psi(u_k) \\ &\qquad + \int_\Omega \big(j^0\big(u; (u_k - u)^+\big) + j^0\big(u_k; -(u_k - u)^+\big)\big)\,dx. \end{aligned} \tag{5.13}$$

Arguing as in (4.10), we have for the second term on the right-hand side of (5.13) the estimate

$$\begin{aligned}&\int_\Omega \big(j^0\big(u; (u_k - u)^+\big) + j^0\big(u_k; -(u_k - u)^+\big)\big)\,dx \\ &\quad \leqslant \int_{\{u_k > u\}} c_1\big(u_k(x) - u(x)\big)^p\,dx. \end{aligned} \tag{5.14}$$

For the terms on the left-hand side of (5.13) we have

$$\big\langle Au_k - Au, (u_k - u)^+\big\rangle \geqslant 0 \tag{5.15}$$

and (5.11) yields

$$\begin{aligned}\big\langle B(u), (u_k - u)^+\big\rangle &= -\int_{\{u_k > u\}} \big(u_0(x) - u(x)\big)^{p-1}\big(u_k(x) - u(x)\big)\,dx \\ &\leqslant -\int_{\{u_k > u\}} \big(u_k(x) - u(x)\big)^p\,dx. \end{aligned} \tag{5.16}$$

By means of (5.14)–(5.16) and the assumption we get from (5.13) the inequality

$$(\lambda - c_1 - c)\int_{\{u_k > u\}} \big(u_k(x) - u(x)\big)^p\,dx \leqslant 0. \tag{5.17}$$

Selecting λ such that $\lambda > c_1 + c$ from (5.17) it follows $u_k \leqslant u$.

The proof for $u \leqslant \bar{u}$ follows arguments similar to the ones in Step 2 of the proof of Theorem 4.1, and thus $\mathcal{S}$ is upward directed.

By obvious modifications of the auxiliary problem one can show analogously that $\mathcal{S}$ is also downward directed.

Step 2. *Existence of extremal solutions*. We show the existence of the greatest element of $\mathcal{S}$. Since V_0 is separable we have that $\mathcal{S} \subset V_0$ is separable too, so there exists a countable, dense subset $Z = \{z_n \mid n \in \mathbb{N}\}$ of $\mathcal{S}$. From Step 1, $\mathcal{S}$ is upward directed, so we can construct an increasing sequence $(u_n) \subset \mathcal{S}$ as follows. Let $u_1 = z_1$. Select $u_{n+1} \in \mathcal{S}$ such that

$$\max\{z_n, u_n\} \leqslant u_{n+1} \leqslant \bar{u}.$$

The existence of u_{n+1} is established in Step 1. From the compactness of $\mathcal{S}$ according to Theorem 5.1, we can choose a subsequence of (u_n), denoted again (u_n), and an element $u \in \mathcal{S}$ such that $u_n \to u$ in V_0, and $u_n(x) \to u(x)$ a.e. in Ω. This last property of (u_n) combined with its increasing monotonicity implies that the entire sequence is convergent in V_0 and, moreover, $u = \sup_n u_n$. By construction, we see that

$$\max\{z_1, z_2, \dots, z_n\} \leqslant u_{n+1} \leqslant u, \quad \forall n,$$

thus $Z \subset [\underline{u}, u]$. Since the interval $[\underline{u}, u]$ is closed in V_0, we infer

$$\mathcal{S} \subset \bar{Z} \subset \overline{[\underline{u}, u]} = [\underline{u}, u],$$

which in conjunction with $u \in \mathcal{S}$ ensures that u is the greatest solution of (1.1).

The existence of the least solution of (1.1) can be proved in a similar way. □

Remark 5.1. We note that for the proof of Theorem 5.2 it is enough to assume instead of (5.8) the following condition:

$$\operatorname{dom}(\psi) \wedge \big(\operatorname{dom}(\psi) \cap [\underline{u}, \bar{u}]\big) \subset \operatorname{dom}(\psi)$$

and

$$\operatorname{dom}(\psi) \vee \big(\operatorname{dom}(\psi) \cap [\underline{u}, \bar{u}]\big) \subset \operatorname{dom}(\psi).$$

Remark 5.2. The question may arise whether there are cases of functionals in which condition (5.9) is satisfied with $c > 0$. We illustrate such a case by the following functional. Let $\psi : V_0 \to \mathbb{R}$ be the function $\psi = \psi_1|_{V_0}$ with $\psi_1 : L^p(\Omega) \to \mathbb{R}$ differentiable and convex. The differential at $u \in V_0$ is denoted $\psi'(u) \in V_0^*$ and is equal to $\psi'(u) = i^*\psi_1'(u)$ in V_0^*, with $\psi_1'(u) \in L^q(\Omega)$ and the inclusion map $i : V_0 \to L^p(\Omega)$. We assume that there exists a constant $c > 0$ such that whenever $v, w \in V_0$ one has

$$\psi_1'(v) - \psi_1'(w) \leqslant c(v - w)^{p-1} \quad \text{for a.e. on } \{w < v\}.$$

For all $w, v \in V_0$ we find that

$$\begin{aligned}
&\psi(w \vee v) - \psi(w) + \psi(w \wedge v) - \psi(v) \\
&\quad \leqslant \int_\Omega \psi_1'(w \vee v)(w \vee v - w)\, dx + \int_\Omega \psi_1'(w \wedge v)(w \wedge v - v)\, dx \\
&\quad = \int_\Omega \big(\psi_1'\big(w + (v - w)^+\big) - \psi_1'\big(v - (v - w)^+\big)\big)(v - w)^+\, dx \\
&\quad = \int_{\{w<v\}} \big(\psi_1'(v) - \psi_1'(w)\big)(v - w)\, dx \leqslant c \int_{\{w<v\}} (v - w)^p\, dx.
\end{aligned}$$

6. Applications

Example 6.1. Assume $f \in L^\infty(\Omega) \subset V_0^*$, and let $K \subset V_0$ represent the following obstacle

$$K = \{v \in V_0 \mid v(x) \leqslant \phi(x) \text{ for a.e. } x \in \Omega\}, \tag{6.1}$$

with $\phi : \Omega \to \mathbb{R}$ measurable. Let $g : V \to \mathbb{R} \cup \{+\infty\}$ be the integral functional introduced in Example 2.3 of Section 2 and $I_K : V \to \mathbb{R} \cup \{+\infty\}$ the indicator function related with K given by (6.1) and assume $K \neq \emptyset$. Then the functional $\psi : V \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\psi = I_K + g$$

is proper, convex and lower semicontinuous with $\mathrm{dom}(\psi) = K \cap \mathrm{dom}(g)$. We consider the variational–hemivariational inequality (1.1) with f and ψ as specified above, i.e., we are looking for $u \in K \cap \mathrm{dom}(g)$ such that

$$u \in K \cap \mathrm{dom}(g): \quad \langle Au - f, v - u\rangle + \psi(v) - \psi(u) + \int_\Omega j^0(u; v - u)\,dx \geqslant 0, \quad \forall v \in V_0. \tag{6.2}$$

The following theorem provides conditions that ensure the existence of an ordered pair of constant sub- and supersolutions of (6.2).

Theorem 6.1. *Let $a_i(x, 0) \equiv 0$ for all $1 \leqslant i \leqslant N$, and let the constants $\alpha \leqslant 0$, $\beta \geqslant 0$ satisfy the following conditions*:

(i) $\alpha \leqslant \phi(x)$ *for a.e.* $x \in \Omega$.
(ii) *For some $\xi \in \partial h(\alpha)$, $\eta \in \partial h(\beta)$ the following inequality is satisfied*:

$$-j^0(\alpha; -1) + \xi \leqslant f(x) \leqslant j^0(\beta; 1) + \eta \quad \textit{for a.e. } x \in \Omega. \tag{6.3}$$

Then the constant functions $\underline{u} = \alpha$ and $\bar{u} = \beta$ form an ordered pair of sub- and supersolutions of (6.2).

Proof. First let us verify that $\underline{u}(x) \equiv \alpha$ is a subsolution according to Definition 2.1. As already noted above we have $\mathrm{dom}(\psi) = K \cap \mathrm{dom}(g)$. Since $\alpha \in \mathrm{dom}(g)$ and $\alpha \leqslant 0$ and due to $\alpha \leqslant \phi$ (see (i)) we get $\alpha \vee (\mathrm{dom}(\psi) \cap V_0) \subset \mathrm{dom}(\psi) \cap V_0$, and thus (i) and (ii) of Definition 2.1 are satisfied. To verify (iii) of Definition 2.1 we need to construct an appropriate functional $\hat{\psi}$ that satisfies (a)–(c) of Definition 2.1. To this end we set $\hat{\psi} = g$. Then (a) is satisfied, because $\alpha \in \mathrm{dom}(g)$. For $v \in \mathrm{dom}(\psi) \cap V_0 = K \cap \mathrm{dom}(g)$ we obtain

$$\begin{aligned}&\psi(v \vee \underline{u}) + \hat{\psi}(v \wedge \underline{u}) - \psi(v) - \hat{\psi}(\underline{u}) \\ &\quad = g(v \vee \alpha) + g(v \wedge \alpha) - g(v) - g(\alpha) = 0,\end{aligned} \tag{6.4}$$

which shows that (b) of Definition 2.1 is satisfied with $\hat{c} = 0$. The second equality of (6.4) can easily be shown to be true by splitting up the domain Ω into $\Omega = \Omega_1 \cup \Omega_2 = \{x \in \Omega \mid v(x) \geqslant \alpha\} \cup \{x \in \Omega \mid v(x) < \alpha\}$, and by evaluating the individual integrals. To see that also

(c) of Definition 2.1 is valid let $v \in \alpha \wedge (K \cap \operatorname{dom}(g))$. Then $v - \alpha \leqslant 0$ in Ω and by (6.3) we get (note that $h : \mathbb{R} \to \mathbb{R}$ is the integrand of the functional g given in Example 2.3)

$$\langle A\alpha - f, v - \alpha \rangle + g(v) - g(\alpha) + \int_\Omega j^0\big(\alpha; v(x) - \alpha\big)\,dx$$

$$\geqslant \int_\Omega \big(j^0(\alpha; -1) + f(x) - \xi\big)\big(\alpha - v(x)\big)\,dx \geqslant 0, \tag{6.5}$$

which proves that α is a subsolution.

Let us show that β is a supersolution of (6.2). One readily sees that $\beta \wedge K \subset K$ and $\beta \wedge \operatorname{dom}(g) \subset \operatorname{dom}(g)$ holds, and thus (i) and (ii) of Definition 2.2 are satisfied. It remains to verify (iii) of Definition 2.2. To this end we show that with $\tilde{\psi} = g$ and applying (6.3) the conditions (a)–(c) of Definition 2.2 can be fulfilled. We have $\beta \in \operatorname{dom}(g)$ and for $v \in K \cap \operatorname{dom}(g)$ the following equalities are satisfied:

$$\psi(v \wedge \bar{u}) + \tilde{\psi}(v \vee \bar{u}) - \psi(v) - \tilde{\psi}(\bar{u})$$

$$= g(v \wedge \beta) + g(v \vee \beta) - g(v) - g(\beta) = 0, \tag{6.6}$$

which shows that (b) of Definition 2.2 holds with $\tilde{c} = 0$. Finally, to verify (c) let $v \in \beta \vee (K \cap \operatorname{dom}(g))$; then $v \geqslant \beta$ and we obtain by means of (6.3),

$$\langle A\beta - f, v - \beta \rangle + g(v) - g(\beta) + \int_\Omega j^0\big(\beta; v(x) - \beta\big)\,dx$$

$$\geqslant \int_\Omega \big(j^0(\beta; 1) - f(x) + \eta\big)(v(x) - \beta)\,dx \geqslant 0,$$

which proves that the constant $\beta \geqslant 0$ is a supersolution. □

Corollary 6.1. *Let the hypotheses of Theorem* 6.1, (A1)–(A3) *and* (H) *be satisfied. Then the variational–hemivariational inequality* (6.2) *possesses extremal solutions within the order interval* $[\alpha, \beta]$ *and the solution set* $\mathcal{S}$ *of all solutions of* (6.2) *within* $[\alpha, \beta]$ *is compact.*

Proof. By Theorem 6.1 the constants α and β form an ordered pair of sub- and supersolutions, respectively, and thus Theorems 4.1 and 5.1 can be applied which provide the existence of solutions within $[\alpha, \beta]$ and the compactness of $\mathcal{S}$. For the existence of extremal solutions we apply Theorem 5.2. To this end we only need to verify conditions (5.8) and (5.9) for the specific functional $\psi = I_K + g$ considered here. It can easily be seen that the following is true: $K \vee K \subset K$, $K \wedge K \subset K$, $\operatorname{dom}(g) \vee \operatorname{dom}(g) \subset \operatorname{dom}(g)$, and $\operatorname{dom}(g) \wedge \operatorname{dom}(g) \subset \operatorname{dom}(g)$, and hence condition (5.8) holds (note $\operatorname{dom}(\psi) = K \cap \operatorname{dom}(g)$). For $w, v \in K \cap \operatorname{dom}(g)$ we have

$$\psi(w \vee v) - \psi(w) + \psi(w \wedge v) - \psi(v)$$

$$= g(w \vee v) - g(w) + g(w \wedge v) - g(v) = 0,$$

and thus also (5.9) is satisfied with $c = 0$. This completes the proof. □

Example 6.2. Assume as in Example 6.1 that the operator A satisfies $a_i(x,0) \equiv 0$ for all $1 \leqslant i \leqslant N$. Let $\psi : V_0 \to \mathbb{R}$ be given by

$$\psi(v) = \frac{\lambda}{p} \int_\Omega |v|^p \, dx, \quad \forall v \in V_0,$$

then the following corollary provides a sufficient condition for zero to be a subsolution of problem (1.1). In the proof we will demonstrate the flexibility in the choice of the auxiliary functional $\hat{\psi}$.

Corollary 6.2. *Let* $f \in L^{p^{*\prime}}(\Omega)$ (p^* *the critical Sobolev exponent*) *such that* $f(x) \geqslant -j^0(0;-1)$ *for a.e.* $x \in \Omega$*, where* $j : \mathbb{R} \to \mathbb{R}$ *verifies assumption* (H)*. Then* $\underline{u} = 0$ *is a subsolution of problem* (1.1) *with* ψ *as specified above.*

Proof. We need to verify the conditions of Definition 2.1. Since $\operatorname{dom}(\psi) = V_0$, (i) and (ii) of Definition 2.1 are trivially satisfied. To check condition (iii) we may choose the function $\hat{\psi} : V \to \mathbb{R}$ in the form

$$\hat{\psi}(v) = \frac{m\lambda}{p} \int_\Omega |v|^p \, dx, \quad \forall v \in V,$$

where $m \in [0, \infty)$. Condition (iii)(a) is evident. Condition (iii)(b) is verified, because we have

$$\begin{aligned}
&\psi(v^+) + \hat{\psi}(-v^-) - \psi(v) - \hat{\psi}(0) \\
&\quad = \frac{\lambda}{p}\left[\int_\Omega |v^+|^p \, dx + m \int_\Omega |v^-|^p \, dx - \left(\int_\Omega |v^+|^p \, dx + \int_\Omega |v^-|^p \, dx\right)\right] \\
&\quad = \frac{(m-1)\lambda}{p} \int_\Omega |v^-|^p \, dx = \frac{(m-1)\lambda}{p} \int_\Omega \left|(-v)^+\right|^p \, dx, \quad \forall v \in V_0,
\end{aligned}$$

and thus condition (iii)(b) is satisfied with $\hat{c} = 0$ for $m \in [0,1]$, and a positive $\hat{c} = (m-1)\lambda/p$ for $m > 1$. It remains to verify condition (iii)(c), that is

$$\langle A0 - f, v\rangle + \frac{m\lambda}{p} \int_\Omega |v|^p \, dx + \int_\Omega j^0(0; v) \, dx \geqslant 0, \quad \forall v \in 0 \wedge V_0.$$

Writing $v = -w^-$ with $w \in V_0$, this reads

$$\int_\Omega \left(f + \frac{m\lambda}{p}(w^-)^{p-1} + j^0(0;-1)\right) w^- \, dx \geqslant 0,$$

which in view of our assumptions is true for any $m \in [0, \infty)$. □

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