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Weak Convergence Theorems for Nonexpansive Mappings in Banach Spaces

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Let C be a closed convex subset of a Banach space E, and let $T: C \to C$ be nonexpansive (that is, $| Tx - Ty | \leqslant | x - y |$ for all x and y in C). J.-B. Baillon [1] has recently shown that if $E = L^p$, $1 < p < \infty$, and T has a fixed point, then for each x in C the Cesàro means of the iterates $\{T^n x\}$ convegre weakly to a fixed point of T. The purpose of this note is to point out that his ideas also lead to the following results. Recall that a sequence $\{x_n\} \subset E$ is weakly almost convergent (cf. [9]) to $y \in E$ if $(\sum_{i=0}^{n-1} x_{i+k})/n \rightharpoonup y$ uniformly in k, and that an operator $A \subset E \times E$ is said to be m-accretive if $R(I + A) = E$ and $| x_1 - x_2 | \leqslant | x_1 - x_2 + r(y_1 - y_2)|$ for all $y_i \in Ax_i$, $i = 1, 2$, and $r > 0$.

THEOREM 1. *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. If $T: C \to C$ is a nonexpansive mapping with a fixed point, then $\{T^n x\}$ is weakly almost convergent to a fixed point of T.*

THEOREM 2. *Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, $T: C \to C$ a nonexpansive mapping with a fixed point, and $\{c_n\}$ a real sequence such that $0 \leqslant c_n \leqslant 1$ and $\sum_{n=1}^{\infty} c_n(1 - c_n) = \infty$. If $x_1 \in C$ and $x_{n+1} = c_n T x_n + (1 - c_n) x_n$ for $n \geqslant 1$, then $\{x_n\}$ converges weakly to a fixed point of T.*

THEOREM 3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm, J_r $(r > 0)$ the resolvent of an m-accretive operator $A \subset E \times E$ with $0 \in R(A)$, and $\{r_n\}$ a positive sequence. Suppose that either*

(a) *$\{r_n\}$ is bounded away from zero, or*

(b) *the modulus of convexity of E satisfies $\delta(\epsilon) \geqslant K\epsilon^p$ for some $K > 0$ and $p \geqslant 2$, and $\sum_{n=1}^{\infty} r_n{}^p = \infty$.*

If $x_1 \in E$ and $x_{n+1} = J_{r_n} x_n$ for $n \geqslant 1$, then $\{x_n\}$ converges weakly to a zero of A.

Theorem 1 has been known so far only in Hilbert space (cf. [5, 12]) while Theorems 2 and 3 have been known for those uniformly convex Banach spaces

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that satisfy Opial's condition (cf. [8, 10, 7]). Theorem 1 implies (cf. [9]) that $\{T^nx\}$ is weakly summed by every strongly regular matrix to a fixed point of T, and that $T^nx - T^{n+1}x \to 0$ if and only if $\{T^nx\}$ converges weakly to a fixed point of T (cf. [3, 5, 12, 2, 1] for previous results in this direction). It was originally obtained (in collaboration with R. E. Bruck) by modifying Baillon's arguments. We omit the details because Bruck [6] has since found a much simpler proof. Analogous results hold for semigroups of nonexpansive mappings (cf. [2, 11]). In order to prove Theorems 2 and 3 we first establish a Proposition which also has other applications. For $x \neq 0$ and y in a smooth Banach space we denote $\lim_{t\to 0}(|x + ty| - |x|)/t$ by (x, y).

PROPOSITION. *Let C be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm, and let $\{T_n : 1 \leqslant n < \infty\}$ be a family of nonexpansive self-mappings of C with a nonempty common fixed point set F. If $x_1 \in C$ and $x_{n+1} = T_nx_n$ for $n \geqslant 1$, then* $\lim_{n\to\infty}(f_1 - f_2, x_n)$ *exists for all $f_1 \neq f_2$ in F.*

Proof. Let $a_n = a_n(t) = |tx_n + (1 - t)f_1 - f_2|$ $(0 \leqslant t \leqslant 1)$, δ the modulus of convexity of the space, $M = |x_1 - f_1|$, $\gamma(r) = (M/2)\,\delta(4r/M)$, $S_{n,m} = T_{n+m-1}T_{n+m-2} \cdots T_n$, and $b_{n,m} = |S_{n,m}(tx_n + (1 - t)f_1) - (tx_{n+m} + (1 - t)f_1)|$. Note that $a_{n+m} \leqslant b_{n,m} + a_n$. After some manipulation we see that $\gamma(|T(cx + (1 - c)y) - cTx - (1 - c)Ty|) \leqslant |x - y| - |Tx - Ty|$ for all $0 \leqslant c \leqslant 1$, $|x - y| \leqslant M$, and nonexpansive $T: C \to C$. Hence $\gamma(b_{n,m}) \leqslant |x_n - f_1| - |x_{n+m} - f_1| \leqslant \epsilon_n \to_{n\to\infty} 0$. Consequently, $\limsup_{n\to\infty} a_n \leqslant \liminf_{n\to\infty} a_n$ and $\lim_{n\to\infty} a_n(t) = a(t)$ exists. Let $d_n = (f_1 - f_2, x_n - f_1)$. Given $\epsilon > 0$ there is $0 < t < 1$ such that $0 \leqslant a_n(t)/t - d_n < \epsilon$ for all $n \geqslant 1$. Therefore $\limsup_{n\to\infty} d_n \leqslant a(t)/t$, $\liminf_{n\to\infty} d_n \geqslant a(t)/t - \epsilon$, and the result follows.

Proof of Theorem 2. Since $\sum_{n=1}^{\infty} c_n(1 - c_n) = \infty$, $\{x_n - Tx_n\}$ converges strongly to zero. Therefore every weak subsequential limit of $\{x_n\}$ is a fixed point of T [4]. Let f_1 and f_2 be two such limits. By the Proposition (with $T_n = c_nT + (1 - c_n)I$), $(f_1 - f_2, f_1) = (f_1 - f_2, f_2)$, so that $f_1 = f_2$.

Proof of Theorem 3. Let $y_{n+1} = (x_n - x_{n+1})/r_n$. In both cases $y_n \to 0$. Since $|x_n - J_1x_n| \leqslant |y_n|$, every subsequential weak limit of $\{x_n\}$ is a zero of A. Again the result now follows from the Proposition (with $T_n = J_{r_n}$).

Remark 1. In the setting of the Proposition, let the space be uniformly convex and P the nearest point projection onto F. Then the strong $\lim_{n\to\infty} Px_n$ exists (cf. [11]).

Remark 2. In the setting of Theorem 3, let E be uniformly convex with a uniformly Gâteaux differentiable norm, and suppose that $0 \notin R(A)$. Then $\lim_{n\to\infty} |x_n| = \infty$. The same conclusion can be reached in the setting of Theorem 2 if T is fixed point free and we assume, for example, that $C = E$

and the sequence $\{c_n\}$ is bounded away from 0 and 1. We do not know however if $|(\sum_{i=0}^{n-1} T^i x)/n| \to_{n\to\infty} \infty$ when T is fixed point free. This is known to be true in Hilbert space.

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Note added in proof. 1. In the setting of Theorem 2, assume that $0 < c_n < 1$ and that $c_n \to 1$. Then $\{x_n\}$ is weakly almost convergent to a fixed point of T. Proofs of this result and of Theorem 1 can be found in my ANL Report entitled "Nonlinear ergodic theory in Banach spaces." 2. In Theorem 3, (a) can be replaced by (a′) $\{r_n\}$ does not converge to zero. 3. In the setting of Remark 2, the condition $\sum_{n=1}^{\infty} c_n(1 - c_n) = \infty$ also implies that $|x_n| \to \infty$ if and only if T is fixed point free. This follows from the ideas of my note entitled "On infinite products of resolvents," *Atti Accad. Naz. Lincei* **63** (1977), 338–340.

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