Weak Convergence Theorems for Nonexpansive Mappings in Banach Spaces

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Let C be a closed convex subset of a Banach space E, and let T: C → C be nonexpansive (that is, | Tx - Ty | ≤ | x - y | for all x and y in C). J.-B. Baillon [1] has recently shown that if E = Lp, 1 < p < ∞, and T has a fixed point, then for each x in C the Cesàro means of the iterates {Tnx} converge weakly to a fixed point of T. The purpose of this note is to point out that his ideas also lead to the following results. Recall that a sequence {xn} C E is weakly almost convergent (cf. [9]) to y E E if (∑k=0 n-1 xk)/n → y uniformly in k, and that an operator A C E → E is said to be m-accretive if R(A+E) = E and | x1 - x2 | ≤ | x1 - x2 + r(y1 - y2) | for all y1, y2 ∈ Ax, i = 1, 2, and r > 0.

**THEOREM 1.** Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. If T: C → C is a nonexpansive mapping with a fixed point, then {Tnx} is weakly almost convergent to a fixed point of T.

**THEOREM 2.** Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, T: C → C a nonexpansive mapping with a fixed point, and {cn} a real sequence such that 0 < cn < 1 and ∑n=1 ∞ cn(1 - cn) = ∞. If x1 ∈ C and xn+1 = cnTxn + (1 - cn) x for n ≥ 1, then {xn} converges weakly to a fixed point of T.

**THEOREM 3.** Let E be a uniformly convex Banach space with a Fréchet differentiable norm, Jr (r > 0) the resolvent of an m-accretive operator A C E × E with 0 ∈ R(A), and {rn} a positive sequence. Suppose that either

(a) {rn} is bounded away from zero, or

(b) the modulus of convexity of E satisfies δ(ε) ≥ Keε for some K > 0 and p ≥ 2, and ∑n=1 ∞ rn/p = ∞.

If x1 ∈ E and xn+1 = Jrnxn for n ≥ 1, then {xn} converges weakly to a zero of A.

Theorem 1 has been known so far only in Hilbert space (cf. [5, 12]) while Theorems 2 and 3 have been known for those uniformly convex Banach spaces.
that satisfy Opial’s condition (cf. [8, 10, 7]). Theorem 1 implies (cf. [9]) that \( \{T^n x\} \) is weakly summed by every strongly regular matrix to a fixed point of \( T \), and that \( T^n x - T^{n+1} x \to 0 \) if and only if \( \{T^n x\} \) converges weakly to a fixed point of \( T \) (cf. [3, 5, 12, 2, 1] for previous results in this direction). It was originally obtained (in collaboration with R. E. Bruck) by modifying Baillon’s arguments. We omit the details because Bruck [6] has since found a much simpler proof. Analogous results hold for semigroups of nonexpansive mappings (cf. [2, 11]). In order to prove Theorems 2 and 3 we first establish a Proposition which also has other applications. For \( x \neq 0 \) and \( y \) in a smooth Banach space we denote \( \lim_{t \to 0} (|x + ty| - |x|)/t \) by \((x, y)\).

**PROPOSITION.** Let \( C \) be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm, and let \( \{T_n : 1 \leq n < \infty\} \) be a family of nonexpansive self-mappings of \( C \) with a nonempty common fixed point set \( F \). If \( x_1 \in C \) and \( v_{n+1} = T_n x_n \) for \( n \geq 1 \), then \( \lim_{n \to \infty} (f_1 - f_2, x_n) \) exists for all \( f_1, f_2 \) in \( F \).

**Proof.** Let \( a_n = a_n(t) = |t x_n - (1 - t) f_1 - f_2| \) \((0 \leq t \leq 1)\), \( \delta \) the modulus of convexity of the space, \( M = |x_1 - f_1| \), \( \gamma(r) = (M/2) \delta(4r/M) \), \( S_{n,m} = T_{n+m} \cdots T_{n+1} T_{n} \), and \( b_{n,m} = |S_{n,m}(t x_n + (1 - t) f_1) - (t x_{n+m} + (1 - t) f_1)| \). Note that \( a_{n,m} \leq b_{n,m} + a_n \). After some manipulation we see that \( \gamma(|T(cx + (1 - c)y) - c T x - (1 - c) T y|) \leq |x - y - (1 - x) T x - T y| \) for all \( 0 < c < 1 \), \( |x - y - M \) and nonexpansive \( T: C \to C \). Hence \( \gamma(b_{n,m}) \leq \gamma(x_n - f_1) \leq |x_{n+1} - f_1| \to 0 \) as \( n \to \infty \). Consequently, \( \limsup_{n \to \infty} a_n \leq \liminf_{n \to \infty} a_n \) and \( \lim_{n \to \infty} a_n(t) = a(t) \) exists. Let \( d_n = (f_1 - f_2, x_n - f_1) \). Given \( \varepsilon > 0 \) there is \( 0 < t < 1 \) such that \( 0 \leq a_n(t)/t - d_n < \varepsilon \) for all \( n \geq 1 \). Therefore \( \limsup_{n \to \infty} d_n \leq a(t)/t \) and the result follows.

**Proof of Theorem 2.** Since \( \sum_{n=1}^\infty e_n(1 - e_n) = \infty \), \( \{x_n - T x_n\} \) converges strongly to zero. Therefore every weak subsequential limit of \{\( x_n \)\} is a fixed point of \( T \) [4]. Let \( f_1 \) and \( f_2 \) be two such limits. By the Proposition (with \( T_n = e_n T_n \cdot (1 - e_n) I ), \( f_1 - f_2, f_1 = (f_1 - f_2, f_2) \), so that \( f_1 = f_2 \).

**Proof of Theorem 3.** Let \( y_{n+1} = (x_n - x_{n+1})/r_n \). In both cases \( y_n \to 0 \). Since \( x_n - f_1 x_n = y_n \), every subsequential weak limit of \{\( x_n \)\} is a zero of \( A \). Again the result now follows from the Proposition (with \( T_n = f_{x_n} \).

**Remark 1.** In the setting of the Proposition, let the space be uniformly convex and \( P \) the nearest point projection onto \( F \). Then the strong \( \lim_{n \to \infty} P x_n \) exists (cf. [11]).

**Remark 2.** In the setting of Theorem 3, let \( E \) be uniformly convex with a uniformly Gâteaux differentiable norm, and suppose that \( 0 \notin R(A) \). Then \( \lim_{n \to \infty} x_n = \infty \). The same conclusion can be reached in the setting of Theorem 2 if \( T \) is fixed point free and we assume, for example, that \( C = E \).
and the sequence \{c,\} is bounded away from 0 and 1. We do not know however if 
\[ |(\sum_{i=0}^{n-1} T^i x)/n| \to_{n \to \infty} \infty \]
when \( T \) is fixed point free. This is known to be true in Hilbert space.

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\section*{Note added in proof}
1. In the setting of Theorem 2, assume that 0 \( \leq c, \leq 1 \) and that \( c, \to 1 \). Then \( \{x,\} \) is weakly almost convergent to a fixed point of \( T \). Proofs of this result and of Theorem 1 can be found in my ANL Report entitled "Nonlinear ergodic theory in Banach spaces." 2. In Theorem 3, (a) can be replaced by (a') \( |r,| \) does not converge to zero. 3. In the setting of Remark 2, the condition \( \sum_{n=1}^{\infty} c,n(1 - c,n) = \infty \) also implies that \( |x,n| \to \infty \) if and only if \( T \) is fixed point free. This follows from the ideas of my note entitled "On infinite products of resolvents," \textit{Atti Accad. Naz. Lincei} 63 (1977), 338–340.

\section*{References}