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On homological coherence of discrete groups ☆

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Abstract

We explore a weakening of the coherence property of discrete groups studied by F. Waldhausen. The new notion is defined in terms of the coarse geometry of groups and should be as useful for computing their K-theory. We prove that a group Γ of finite asymptotic dimension is weakly coherent. In particular, there is a large collection of $R[\Gamma]$-modules of finite homological dimension when R is a finite-dimensional regular ring. This class contains word-hyperbolic groups, Coxeter groups and, as we show, the cocompact discrete subgroups of connected Lie groups.

1. Introduction

Let A be a ring with a unit. A left A-module is *coherent* if it has a resolution by finitely generated projective A-modules. It is *regular coherent* or said to have *finite homological dimension* if such resolution can be chosen to be finite. This notion is particularly useful when A is a group ring $R[\Gamma]$; alas, homologically finite-dimensional modules over generic group rings are very rare. We will describe a weaker notion of coherence and a new method for constructing finite-dimensional modules using coarse geometric properties of the group Γ. Throughout the paper the ring R is assumed to be noetherian.

We should recall that F. Waldhausen [11] discovered a remarkable collection of discrete groups Γ such that all finitely presented modules over the group ring $R[\Gamma]$ are regular coherent. It includes free groups, free abelian groups, torsion-free one-relator groups, their

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various amalgamated products and HNN extensions and so, in particular, the fundamental groups of submanifolds of the three-dimensional sphere. Waldhausen called this property of the group *regular coherence* and used it to compute the algebraic K-theory of these groups. He also wondered if a weaker property of the group ring would suffice in his argument (see, for example, the paragraph after the proof of Theorem 11.2 in [11]). When we compute the K-theory of geometrically finite groups of finite asymptotic dimension in [4–6], by proving surjectivity of the integral assembly map, we indeed require a weaker coherence property than that of Waldhausen; however, it is not directly related to his argument.

Using the coarse combinatorial geometry of the group, we will define a class of finite presentations of $R[\Gamma]$-modules which we call *admissible*. We will also use the geometry to introduce a large collection of finite dimensional $R[\Gamma]$-modules which we call *lean* and which includes all modules with admissible presentations.

1.1. Example. To illustrate the geometric nature of our method, we give a new proof of coherence of the group of integers $\mathbb{Z}$. In this case, we consider an $R[\mathbb{Z}]$-homomorphism of two free modules $f : R[\mathbb{Z}]^m \to R[\mathbb{Z}]^n$ and show that the kernel of f is finitely generated when R is noetherian.

A geometric viewpoint on f is introduced by filtering each of the free modules by the R-submodules associated to the subsets $[a,b] = \{a, a+1, \dots, b-1, b\}$ of $\mathbb{Z}$. Let $R[a,b]^k$ stand for the k-tuples of group ring elements where all group elements in the formal sum expressions come from $[a,b]$. Notice that for each homomorphism f there is a number d such that $f(R[a,b]^m) \subset R[a-d, b+d]^n$ for all choices of $a \leqslant b$.

Let k be an element of the kernel $\ker(f)$ and let k be written as a sum $k = \sum k_i$, where $k_i \in R[5di, 5d(i+1)]^m$, and only finitely many k_i are nonzero. Observe that because of the property of the number d and the fact that $k_i + \sum_{j \neq i} k_j \in \ker(f)$, we have $f(k_i) = s_{i,l} + s_{i,r}$, where

$$s_{i,l} = -f\left(\sum_{j<i} k_j\right) \in R[5di - d, 5di + d]^n \quad \text{and}$$

$$s_{i,r} = -f\left(\sum_{j>i} k_j\right) \in R\big[5d(i+1) - d, 5d(i+1) + d\big]^n.$$

In fact, $s_{i,r} = -s_{i+1,l}$ for all i. Since R is a noetherian ring, $\operatorname{im}(f) \cap R[-d,d]^n$ is finitely generated, so there is a number e such that

$$\operatorname{im}(f) \cap R[-d,d]^n = f\big(R[-d-e, d+e]^m\big) \cap R[-d,d]^n.$$

Now choose $t_i \in R[5di - d - e, 5di + d + e]^m$ so that $f(t_i) = s_{i,r} = -s_{i+1,l}$ and thus all $k_i - t_i + t_{i+1}$ are in the kernel:

$$\begin{aligned} f(k_i - t_i + t_{i+1}) &= f(k_i) - f(t_i) + f(t_{i+1}) = (s_{i,l} + s_{i,r}) - s_{i,l} + s_{i+1,l} \\ &= (s_{i,l} + s_{i,r}) - s_{i,l} - s_{i,r} = 0. \end{aligned}$$

Since all elements $k_i - t_i + t_{i+1} \in R[5di - d - e, 5d(i+1) + d + e]^m$, we conclude that the $R[\mathbb{Z}]$-module $\ker(f)$ is generated by the R-submodule $\ker(f) \cap R[-d - e, 5d + d + e]^m$ which itself is finitely generated as R is noetherian.

For a general discrete group Γ, given an $R[\Gamma]$-module F with finite generating set Σ, it is also an R-module with the generating set $B = \Sigma \times \Gamma$. There is a locally finite set function $s : B \to \Gamma$ which maps (σ, γ) to γ. On the other hand, one can associate to every subset S of Γ the R-submodule generated by $\Sigma \times S$.

Recall that a finitely presented group Γ can be given a *word metric* specific to the presentation. This makes Γ a proper metric space. It is known that all word metrics on the group are quasi-isometric.

1.2. Definition. Consider general functors $f : \mathcal{P}(\Gamma) \to \mathrm{Mod}_R(F)$ from the power set of Γ ordered by inclusion to the R-submodules of F such that $f(\Gamma) = F$ and $f(T)$ is a finitely generated R-module for each bounded subset $T \subset \Gamma$. We will refer to F as an *Γ-filtered* R-module. If f is Γ-equivariant in the sense that $f(\gamma S) = \gamma f(S)$ for all $\gamma \in \Gamma$ and $S \subset \Gamma$ then F as an *equivariant Γ-filtered* R-module.

A homomorphism $\phi : F_1 \to F_2$ between finitely generated $R[\Gamma]$-modules with fixed choices of filtrations f_i, $i = 1, 2$, is *boundedly controlled* with respect to the bound $D > 0$ if $\phi f_1(S) \subset f_2(B_D(S))$ for all subsets $S \subset \Gamma$. Here $B_D(S)$ stands for the D-enlargement of a subset S in a metric space X that is the subset $\{x \in X \mid d(x, S) \leqslant D\}$. Let I be the image of ϕ and let $i(S) = \mathrm{im}(\phi) \cap f_2(S)$. If ϕ in addition satisfies $\phi F_1 \cap f_2(S) \subset \phi f_1(B_D(S))$ then it is called *boundedly bicontrolled* of filtration D. When Γ is infinite, neither of the properties is satisfied by all $R[\Gamma]$-homomorphisms.

1.3. Example. A boundedly controlled idempotent homomorphism of an equivariant filtered module is always boundedly bicontrolled. Indeed, if $\phi : F \to F$ is an idempotent so that $\phi^2 = \phi$ then $\phi|I = \mathrm{id}$, so $\phi F \cap f(S) \subset \phi f(S)$.

1.4. Definition. A pair of subsets S, T of a metric space X is (*coarsely*) *antithetic* if for each number $D > 0$ there is $D_1 > 0$ so that $B_D(S) \cap B_D(T) \subset B_{D_1}(S \cap T)$.

Examples of such pairs include any two subsets of a simplicial tree as well as complementary half-spaces in a Euclidean space.

1.5. Definition. A Γ-filtration f of an R-module F is *lean* if it satisfies the following two properties for some fixed number $d = d_f > 0$:

(1) for any subset S of Γ and $y \in f(S)$,

$$y \in \sum_{\gamma \in S} f\big(B_d(\gamma)\big);$$

(2) for any antithetic pair of subsets S and T, if $y \in f(S)$ and $y \in f(T)$ then $y \in f(B_d(S \cap T))$.

An $R[\Gamma]$-module is called *lean* if it has a lean equivariant Γ-filtration by R-submodules.

Notice that a lean $R[\Gamma]$-module is finitely generated. The class of lean $R[\Gamma]$-modules certainly contains all free finitely generated $R[\Gamma]$-modules.

1.6. Definition. An $R[\Gamma]$-module is *finitely presented* if it is the cokernel of a homomorphism, called *presentation*, between free finitely generated $R[\Gamma]$-modules. If the homomorphism is boundedly bicontrolled, we call the presentation *admissible*.

1.7. Definition. The group ring $R[\Gamma]$ is *weakly coherent* if every $R[\Gamma]$-module with an admissible presentation has a projective resolution of finite type. We say the ring $R[\Gamma]$ is *weakly regular coherent* if every $R[\Gamma]$-module with an admissible presentation has finite homological dimension.

Groups of finite asymptotic dimension were introduced by M. Gromov [10]. Examples from this apparently very large class are the Gromov hyperbolic groups [10], Coxeter groups [9], various generalized products of these, including the groups acting on trees with vertex stabilizers of finite asymptotic dimension [2], and, more generally, fundamental groups of developable complexes of finite-dimensional groups [1]. We show in Section 3 that cocompact lattices in connected Lie groups also have finite asymptotic dimension.

The following is the main result of the paper.

1.8. Theorem. *Let R be a noetherian ring and Γ be a discrete group of finite asymptotic dimension. Then*

(1) *lean $R[\Gamma]$-modules have projective resolutions of finite type,*
(2) *all $R[\Gamma]$-modules with admissible presentations are lean.*

If, in addition, R has finite homological dimension then

(3) *lean $R[\Gamma]$-modules also have finite homological dimension.*

1.9. Corollary. *Let R be a finite-dimensional noetherian ring and Γ be a discrete group of finite asymptotic dimension. Then the group ring $R[\Gamma]$ is weakly regular coherent.*

1.10. Example. To illustrate the construction of interesting lean finite-dimensional modules, recall that idempotents between $R[\Gamma]$-modules are boundedly bicontrolled. We will see that images and cokernels of boundedly bicontrolled maps between lean modules are lean. Existence of idempotents over group rings is well-known. Now given any idempotent between free finitely-generated $\mathbb{Z}[\Gamma]$-modules, reduction modulo a composite integer m gives another idempotent whose image and cokernel are nonprojective modules over $\mathbb{Z}[\Gamma]$.

We will prove weak coherence properties for discrete groups of finite asymptotic dimension in Section 2. Section 3 shows that cocompact lattices in connected Lie groups have finite asymptotic dimension.

2. Weak coherence and finite asymptotic dimension

2.1. Definition. A family of subsets in a general metric space X is *d-disjoint* if $\mathrm{dist}(V, V') = \inf\{\mathrm{dist}(x, x') \mid x \in V, x' \in V'\} > d$ for all V, V'. The *asymptotic dimension* of X is defined by M. Gromov [10] as the smallest number n such that for any $d > 0$ there is a uniformly bounded cover U of X by $n + 1$ d-disjoint families of subsets $U = U^0 \cup \cdots \cup U^n$.

It is known that asymptotic dimension is a quasi-isometry invariant and so is an invariant of a finitely generated group viewed as a metric space with the word metric associated to a given presentation.

The proof of Theorem 1.8 is based on the following characterization of metric spaces of finite asymptotic dimension and a sequence of lemmas.

2.2. Definition. A map between metric spaces $\phi : (M_1, d_1) \to (M_2, d_2)$ is an *asymptotic* or *uniform embedding* if there are two real functions f and g with $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ such that

$$f\big(d_1(x, y)\big) \leqslant d_2\big(\phi(x), \phi(y)\big) \leqslant g\big(d_1(x, y)\big)$$

for all pairs of points x, y in M_1.

2.3. Theorem (Dranishnikov [7,8]). *A group Γ has finite asymptotic dimension if and only if there is a uniform embedding of Γ in a finite product of locally finite simplicial trees.*

We can use the notions of lean filtered R-modules and boundedly controlled and bicontrolled homomorphisms of such modules associated to any proper metric space X, with or without a group action. Thus an X-filtration of an R-module F is a functor $f : \mathcal{P}(X) \to \mathrm{Mod}_R(F)$ from the power set of X to the R-submodules of F such that $f(X) = F$ and $f(T)$ is a finitely generated R-module for each bounded subset $T \subset X$. Now conditions (1) and (2) in Definition 1.5 define the class of *lean X-filtered modules.*

2.4. Lemma. *Let P be a finite product of locally finite simplicial trees, with the product word metric. Then the kernel of a surjective boundedly bicontrolled homomorphism between lean P-filtered R-modules is lean.*

Proof. Suppose $P = \prod_{1\leqslant i\leqslant m} T_i$ and $\pi : P \to T = T_m$ is the mth coordinate projection. Given a surjective boundedly bicontrolled homomorphism $\phi : F \to G$ between two lean P-filtered R-modules, let $D \geqslant 0$ be a number such that $\mathrm{fil}(\phi) < D$, and let f and g be lean filtrations of F and G respectively, both of filtration D.

We will show that the kernel $K = \ker(\phi)$ equipped with the restriction of the P-filtration f is lean. Fix a vertex t_0 in T. Given another vertex $t \in T$, we define its *shadow* as the subset $\mathrm{Sh}(t) = \{t' \in T \mid t \in [t_0, t']\}$. For every $t \in \partial B_{6kD}(t_0)$, $0 \leqslant k$, let

$$S(t) = \mathrm{Sh}(t) \cap \big(B_{6(k+2)D}(t_0) - B_{6(k+1)D}(t_0)\big).$$

Since D is a filtration of f, if k is in the kernel K then k can be written as the sum $\sum l_t$, t as above, where $l_t \in f(\pi^{-1}(S_t))$. This is certainly a finite sum. More generally, let $S(t, l, u)$, for $t \in T$ with $\mathrm{dist}(t_0, t) \leqslant l \leqslant u$, be the subset $\mathrm{Sh}(t) \cap (B_l(t_0) - B_u(t_0))$. Then

$$\phi(l_t) \in g\big(\pi^{-1} S\big(t, 6(k+1)D - D, 6(k+2)D + D\big)\big).$$

Using that $\phi(l_t) = -\phi(\sum_{t' \neq t} l_{t'})$,

$$\phi\Bigg(\sum_{t' \neq t} l_t\Bigg) \in g\big(\pi^{-1} S\big(t, 6(k+1)D + D, 6(k+2)D - D\big)\big),$$

and that D is a filtration of g, we see that $\phi(l_t) = y_t^1 + y_t^2$ with

$$y_t^1 \in g\big(\pi^{-1} S\big(t, 6(k+1)D - 2D, 6(k+1)D + 2D\big)\big) \quad \text{and}$$
$$y_t^2 \in g\big(\pi^{-1} S\big(t, 6(k+2)D - 2D, 6(k+2)D + 2D\big)\big).$$

Notice that

$$\operatorname{diam} S\big(t, 6(k+1)D - 2D, 6(k+1)D + 2D\big) \leqslant 16D \quad \text{and}$$
$$\operatorname{diam} S\big(t, 6(k+2)D - 2D, 6(k+2)D + 2D\big) \leqslant 20D.$$

It is clear that the subsets $S_{t,*}$ so obtained are pairwise disjoint. Since $\mathrm{fil}(\phi) < D$, there are elements

$$z_t^1 \in f\big(\pi^{-1} S\big(t, 6(k+1)D - 3D, 6(k+1)D + 3D\big)\big) \quad \text{and}$$
$$z_t^2 \in f\big(\pi^{-1} S\big(t, 6(k+2)D - 3D, 6(k+2)D + 3D\big)\big)$$

with $\phi(z_t^i) = y_t^i$. It is easy to see that $\sum_t (z_t^2 - z_t^1) = 0$. Now $k_t = -z_t^1 + l_t - z_t^2$ are elements in the kernel K, each contained in

$$F_t = f\big(\pi^{-1} S\big(t, 6(k+1)D - 4D, 6(k+2)D + 4D\big)\big),$$

so k can be written as a finite sum

$$k = \sum k_t. \tag{$*$}$$

It follows that K is generated as an R-module by the submodules $K_t = K \cap F_t$ for all t as above. For each t, the diameter of the set $S(t, 6(k+1)D - 4D, 6(k+2)D + 4D)$ is bounded above by $28D$ which is independent of t. In particular, this proves the statement when $P = T$. In this case K_t are finitely generated as submodules of finitely generated modules over the noetherian ring R.

In general, one can use induction on the number m of tree factors in P. Let P_i be the product $\prod_{j \geqslant i} T_i$. Let $\pi_{i-1} : P_{i-1} \to P_i$ be the obvious projection. Now given an element k in the kernel K such that there is $S \subset T$ with $k \in f(S)$ and $\operatorname{diam}(\pi_{i-1}(S)) < C$, we would like to see that k can be written as a sum $\sum k_t$ so that $k_t \in f(S_t)$ and $\operatorname{diam}(\pi_i(S_t)) < B$ where B is a number which depends on C and D but not on n. This is easily achieved exactly as in the construction of the sum $(*)$ above with $B = 2C + 15D$. Applying this construction inductively, one obtains a decomposition of the original $k \in K$ as the sum $\sum k_t$ with $k_t \in f(S_t)$ and $\operatorname{diam}(S_t) < (C + 15D)2^m$.

Property (2) of the lean modules for $\ker(\phi)$ is inherited from F. □

2.5. Lemma. *Every $R[\Gamma]$-homomorphism $\phi : F \to G$ between a lean $R[\Gamma]$-module F and an equivariant Γ-filtered module G is boundedly controlled as a homomorphism between filtered R-modules.*

Proof. Let f be a lean equivariant Γ-filtration of F. Consider $z \in f(S)$, then $z = \sum r_i z_i$ where $z_i \in f(B_d(x_i))$ for some $x_i \in S$. Since ϕ is an $R[\Gamma]$-homomorphism, there is a number $D \geqslant 0$ such that $\phi(z)$ is in $g(B_{d+D}(x))$ for all $z \in f(B_d(x))$ and all $x \in \Gamma$. Then $\phi(z) = \sum r_i \phi(z_i) \in \sum g(B_{d+D}(x_i)) \subset g(B_{d+D}(S))$. □

2.6. Lemma. *Every surjective boundedly controlled homomorphism of lean filtered modules is boundedly bicontrolled. Therefore every surjective $R[\Gamma]$-homomorphism of lean $R[\Gamma]$-modules is boundedly bicontrolled.*

Proof. If $y \in g(S)$ then $y = \sum r_i y_i$ with $y_i \in g(B_{d_G}(x_i))$, $x_i \in g(S)$. Each $g(B_{d_G}(x))$ is a finitely generated R-module, so there is a constant $C \geqslant 0$ and $z_i \in f(B_{d_G+C})(x)$ so that $\phi(z_i) = y_i$. Now $z = \sum r_i z_i$ is in $f(B_{d_G+C}(S))$. □

2.7. Lemma. *Let $\phi : M_1 \to M_2$ is an injective asymptotic embedding of proper metric spaces. If S and T are coarsely antithetic in M_1 then $\phi(S)$ and $\phi(T)$ are antithetic in M_2. Conversely, if U and V are antithetic in M_2 then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are antithetic in M_1.*

Proof. We will show the first statement, the proof of the second is similar. Assume ϕ has the properties listed in Definition 2.2. Now for any choice of $d \geqslant 0$ with $f(d) > D$

$$B_D\phi(S) \cap B_D\phi(T) \subset \phi\big(B_d(S)\big) \cap \phi\big(B_d(T)\big) = \phi\big(B_d(S) \cap B_d(T)\big) \subset \phi\big(B_{d_1}(S \cap T)\big)$$
$$\subset B_{g(d_1)}\phi(S \cap T) \subset B_{g(d_1)}\big(\phi(S) \cap \phi(T)\big).$$

Here the equality follows from the injectivity assumption. So we can take $D_1 = g(d_1)$. □

2.8. Proposition. *If $\phi : M_1 \to M_2$ is an injective asymptotic embedding between proper metric spaces then the M_2-filtration $f_*(S) = f(\phi^{-1}(S))$ induced from an M_1-filtration f is lean if and only if f is lean.*

Proof. We show the necessity half of the argument. Notice that the fact that $d_2(\phi(x), \phi(y)) \leqslant g(d_1(x,y))$ implies $B_d(x) \subset \phi^{-1}(B_{g(d)}(\phi(x))$ for all $d \geqslant 0$. Suppose f is lean, then given $y \in f_*(S) = f(\phi^{-1}(S))$ and

$$y \in \sum_{x \in \phi^{-1}(S)} f\big(B_d(x)\big),$$

we have

$$y \in \sum_{x \in \phi^{-1}(S)} f\big(\phi^{-1}\big(B_{g(d)}\big(\phi(x)\big)\big)\big) = \sum_{x \in \phi^{-1}(S)} f_*\big(B_{g(d)}\big(\phi(x)\big)\big) \subset \sum_{z \in S} f_*\big(B_{g(d)}(z)\big).$$

For the second property, if $y \in f_*(S) \cap f_*(T) = f(\phi^{-1}(S)) \cap f(\phi^{-1}(T))$ then

$$\begin{aligned} y &\in f\big(B_d\big(\phi^{-1}(S)\big) \cap B_d\big(\phi^{-1}(T)\big)\big) \subset f\big(\phi^{-1}\big(B_{g(d)}(S)\big) \cap \phi^{-1}\big(B_{g(d)}(T)\big)\big) \\ &= f\big(\phi^{-1}\big(B_{g(d)}(S) \cap B_{g(d)}(T)\big)\big) \subset f_*\big(B_{d_1}(S \cap T)\big) \end{aligned}$$

for some d_1. So f_* is lean with characteristic constant d_1. □

2.9. Corollary. *Let Γ be a finitely generated group viewed as a metric space with the word metric induced by a fixed presentation. If Γ has a uniform embedding $i_0 : \Gamma \to P$ in a finite product P of locally finite simplicial trees then the kernel of a surjective $R[\Gamma]$-homomorphism of lean $R[\Gamma]$-modules is lean. In particular, it is finitely generated.*

Proof. The given homomorphism $\phi : F_1 \to F_2$ between two lean $R[\Gamma]$-modules can be thought of as a boundedly controlled homomorphism between lean R-modules with the P-filtrations f_0 defined by $f_0(S) = f(i_0^{-1}(S))$. From Proposition 2.8, we see that f_0 is lean if and only if f is lean. When ϕ is surjective, it is boundedly bicontrolled by Lemma 2.6. The rest follows from Lemma 2.4. □

2.10. Lemma. *The image of a boundedly bicontrolled homomorphism of lean filtered modules is lean.*

Proof. Let D be a filtration degree of the homomorphism $\phi : F \to G$. If I is the image of f, it has the natural Γ-equivariant filtration given by $i(S) = I \cap g(S)$. If $y \in g(S)$ then there is $z \in f(B_D(S))$ with $\phi(z) = y$ written as $z = \sum r_i z_i$ for some $z_i \in f(B_{d_G}(x_i))$ and $x_i \in B_D(S)$. So $y = \sum r_i \phi(z_i)$ and $\phi(z_i) \in g(B_{d_G+D}(x_i))$. In other words,

$$y \in \sum_{x \in S} i\big(B_{d_G+2D}(x)\big).$$

To see that the second characteristic property of lean modules is inherited by the image from G, we show that generally the image of a boundedly bicontrolled homomorphism with the kernel satisfying property (1) also satisfies property (2) in Definition 1.5. Let $y \in g(S) \cap g(T)$, then there are $z_S \in f(B_D(S))$ and $z_T \in f(B_D(T))$ such that $\phi(z_S) = \phi(z_T) = y$. Thus $k = z_S - z_T$ is in the kernel $K = \ker(\phi)$. Using property (1) of the kernel K, write $k = k_S + k_T$ where $k_S \in f(B_{d_f+D}(S))$ and $k_T \in f(B_{d_f+D}(T))$ so that $z_S - k_S = z_T + k_T$ and again $\phi(z_S - k_S) = \phi(z_T + k_T) = y$. Now since F has property (2) and $z_S - k_S = z_T + k_T$ is in $f(B_{d_f+D}(S)) \cap f(B_{d_f+D}(T))$, it is also in $f(B_{2d_f+D}(S \cap T))$. So $y \in g(B_{2d_f+2D}(S \cap T))$. □

2.11. Corollary. *The cokernel of a boundedly bicontrolled homomorphism of lean P-filtered R-modules is lean.*

Proof of Theorem 1.8. Given a lean $R[\Gamma]$-module F, let F_1 be the free $R[\Gamma]$-module on the finite generating set Σ of F. We view it as a lean R-module with the canonical filtration induced from the product generating set $\Sigma \times \Gamma$. Then the surjection $\pi : F_1 \to F$ is boundedly bicontrolled. The kernel $K_1 = \ker(\pi)$ is lean by Lemma 2.4. Construct a free finitely generated $R[\Gamma]$-module F_2 with a projection $\pi_1 : F_2 \to K_1$. By Lemma 2.5, π_1 is boundedly controlled, hence by Lemma 2.6 it is boundedly bicontrolled. This shows that F is finitely presented as the quotient of the composition $d_1 = i_1\pi_1$ which is boundedly bicontrolled. This construction also inductively gives a resolution by free finitely generated $R[\Gamma]$-modules.

Part (2) of Theorem 1.8 follows directly from Corollary 2.11.

For part (3), consider the nth syzygy module $K_n = \ker(d_n)$ where n is the homological dimension of the ring R. It is known from the syzygy theorem that G is a projective R-module if it fits into a resolution

$$0 \longrightarrow G \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow F \longrightarrow 0$$

of an R-module F over a regular ring R of homological dimension $\mathrm{hd}(R) \leqslant n$ and all modules $P_1, \ldots, P_n$ are projective, cf. [12, Lemma 4.1.6]. This certainly applies to K_n. Since $R[\Gamma]$-modules which are free as R-modules are also free $R[\Gamma]$-modules, it follows easily that $R[\Gamma]$-modules projective as R-modules are projective as $R[\Gamma]$-modules. Since K_n is lean, it is finitely generated over $R[\Gamma]$. This shows that F has a finite projective resolution of length at most n. □

3. The asymptotic dimension of uniform lattices

This section proves that the asymptotic dimension of cocompact discrete subgroups of a connected Lie group G is the dimension of the homogeneous space of maximal compact subgroups in G.

3.1. Definition. A map between metric spaces $\phi : (M_1, d_1) \to (M_2, d_2)$ is *eventually continuous* if there is a real function g such that $d_2(\phi(x), \phi(y)) \leqslant g(d_1(x, y))$ for all pairs of points x, y in M_1.

3.2. Proposition. *If $M_1 = M_2$, the identity map* $\mathrm{id}: M_1 \to M_2$ *is a uniform embedding if and only if the identity map is eventually continuous in both ways, that is, there are real functions g and $\bar{g}$ such that $d_2(x, y) \leqslant g(d_1(x, y))$ and $d_1(x, y) \leqslant \bar{g}(d_2(x, y))$ for all pairs of points x, y in M.*

Proof. If the identity is an asymptotic embedding, we may choose g for g and define

$$\bar{g}(z) = \sup\{z' \mid f(z') \leqslant z\}.$$

Then $d_1(x, y) \leqslant \bar{g}(d_2(x, y))$ since $f(d_1(x, y)) \leqslant d_2(x, y)$.

To see that the identity is an asymptotic embedding, we may again choose g for one of the bounding functions and define

$$f(z) = \inf\{z' \mid \bar{g}(z') \leqslant z\}.$$

Then $f(d_1(x, y)) \leqslant d_2(x, y)$ since $d_1(x, y)) \leqslant \bar{g}(d_2(x, y))$. $\lim_{z\to\infty} f(z) = \infty$ because X is not compact. □

3.3. Definition. Given a space M, two metrics d_1 and d_2 on M form a *uniform pair* if the identity map $\mathrm{id}: (M_1, d_1) \to (M_2, d_2)$ is an asymptotic embedding.

When two metrics are a uniform pair, metric balls of uniformly bounded diameter in one metric are uniformly bounded in the other metric.

The following result is from [3, Chapter V].

3.4. Proposition. *Let G be a connected Lie group and K be its maximal compact subgroup. Then there is a simply connected nilpotent Lie group N and a simply transitive action of N on the homogeneous space G/K by isometries with respect to the N-invariant metric d_1. If d_2 is the G-invariant metric on G/K then the identity map of G/K with these two metrics is eventually continuous. In other words, the two metrics d_1 and d_2 form a uniform pair.*

Let Γ be a cocompact lattice in a connected Lie group. A uniform embedding of Γ in N can be obtained by uniformly embedding Γ in G/K as the pullback of the orbit Γ_0 of x_0 via the simply transitive action of N on G/K with either metric d_1 or d_2 and then lifting the embedding to N . There is no natural action of Γ on N but notice that the embedding of Γ is commensurable.

3.5. Theorem. *Let N be a simply connected nilpotent Lie group with the left-invariant Riemannian metric. Then*

$$\mathrm{asdim}(N) = \dim(N).$$

Proof. A simply connected solvable group N of dimension n is isomorphic to the semidirect product $T \rtimes N_0$, where N_0 is a normal simply connected solvable Lie group and T is isomorphic to the group of real numbers which act on N_0. There is a corresponding vector space splitting of the Lie algebra $n = t \oplus n_0$ which is orthogonal with respect to a positive definite bilinear form β on n. If the metric d in N is the Riemannian metric associated to β and T has the metric associated to the restriction of β to t then the projection $\pi : N \to T$ is a distance nonincreasing map. In fact, if $y = y_t + y_0$ then the length $l(y) = l(y_t) + l(y_0)$. One can show that

$$B_r\big(\pi^{-1}[a,b]\big) = \pi^{-1}\big([a-r, b+r]\big).$$

For details, see [3, Section V]. For any point $x \in [a,b]$, the function

$$\rho(a,b,x) : \pi^{-1}\big([a,b]\big) \to \pi^{-1}(x)$$

given by $\rho(a,b,x)(g) = g(x - \pi(g))$ is bounded by $b - a$, that is,

$$d\big(g, \rho(a,b,x)(g)\big) \leqslant b - a$$

for all a, b, x, and $g \in \pi^{-1}([a,b])$. Also, $\rho(a,b,x)$ is equivariant with respect to the left multiplication action by N_0.

There is a useful equivalent characterization of asymptotic dimension [7,10]. For a metric space X, $\operatorname{asdim}(X) \leqslant n$ if for arbitrarily large number D there is a uniformly bounded cover $\mathcal{U}$ of X such that every metric ball of radius D has nonempty intersection with at most $n+1$ sets in $\mathcal{U}$.

We will use induction on the dimension of N. Starting with dimension one, let the covering of $N = \mathbb{R}$ be by the closed segments

$$\mathcal{U}^1 = \big\{U_i^1 = \big[4Di, 4D(i+1)\big] \bigm| i \in \mathbb{Z}\big\}.$$

It is clear that $\operatorname{asdim}(\mathbb{R}) = 1$. Notice also that each set U_i^1 in $\mathcal{U}^1$ has the property that there is the point $x_i = 4Di + 2D \in U_i^1$ such that the metric ball centered at x_i with radius D is contained entirely in U_i, and another covering $\mathcal{U}^2$ can be obtained by translating $\mathcal{U}^1$ (that is left-multiplying) by $2D$. Because of the first property, each metric ball with radius D intersects at most 3 subsets from the new covering $\mathcal{U}^1 \cup \mathcal{U}^2$.

Now suppose that $\dim(N) = n$, then $\dim(N_0) = n - 1$ in the semidirect product decomposition above. We assume that

(1) N_0 is given the N_0-invariant Riemannian metric,
(2) N_0 has a covering consisting of two subcoverings $\mathcal{U}_{n-1}^1$ and $\mathcal{U}_{n-1}^2$ by uniformly bounded subsets with the property that each ball of radius D intersects at most n subsets in each covering $\mathcal{U}_{n-1}^1$ and $\mathcal{U}_{n-1}^2$ and at most $n+1$ subsets in the union $\mathcal{U}_{n-1}^1 \cup \mathcal{U}_{n-1}^2$.

In order to construct two similar coverings $\mathcal{U}_n^1$ and $\mathcal{U}_n^2$ of N, consider the translates $t_i N_0$ of N_0 for $t_i = 4Di$, $i \in \mathbb{Z}$, and the corresponding coverings $\mathcal{U}_{n-1,i}^1$ and $\mathcal{U}_{n-1,i}^2$ of $t_i N_0$. We will use the notation

$$S_i^l(U) = \rho^{-1}(t_i - 2D, t_i, t_i)(U), \qquad S_i^r(U) = \rho^{-1}(t_i, t_i + 2D, t_i)(U)$$

for any subset U of $t_i N_0$. Now define four collections of subsets of N as

$$\mathcal{U}_n^{1,l} = \{S_i^l(U) \mid U \in \mathcal{U}_{n-1,i}^1,\ i \in \mathbb{Z}\}, \qquad \mathcal{U}_n^{1,r} = \{S_i^r(U) \mid U \in \mathcal{U}_{n-1,i}^1,\ i \in \mathbb{Z}\},$$
$$\mathcal{U}_n^{2,l} = \{S_i^2(U) \mid U \in \mathcal{U}_{n-1,i}^2,\ i \in \mathbb{Z}\}, \qquad \mathcal{U}_n^{2,r} = \{S_i^r(U) \mid U \in \mathcal{U}_{n-1,i}^2,\ i \in \mathbb{Z}\}.$$

Let $\mathcal{U}_n^1 = \mathcal{U}_n^{1,l} \cup \mathcal{U}_n^{2,r}$ and $\mathcal{U}_n^2 = \mathcal{U}_n^{1,r} \cup \mathcal{U}_n^{2,l}$. It is clear that either of the two coverings $\mathcal{U}_n^1$ and $\mathcal{U}_n^2$ has the property that a metric ball with radius D in N intersects at most $n+1$ sets from the covering. It is also clear that a metric ball with radius D intersects at most $n+2$ sets from the covering $\mathcal{U}_n^1 \cup \mathcal{U}_n^2$, as required in the induction step. So by induction $\operatorname{asdim}(N) \leqslant n$.

To see the reverse inequality, recall that Gromov [10] defines another notion of asymptotic dimension which he denotes simply asdim. This notion is different from the asymptotic dimension conventionally used in this and other papers in the literature. In order to avoid confusion in this proof, we will use the notation asdim_* for this possibly different number. Now Gromov shows that for a compact acyclic manifold M, $\operatorname{asdim}_*(\widetilde{M}) = \dim(M)$. The general inequality $\operatorname{asdim}_* \leqslant \operatorname{asdim}$ gives $\dim(M) \leqslant \operatorname{asdim}(\widetilde{M})$. Applying this inequality in the case of $M = \Gamma\backslash N$, for any cocompact lattice Γ in N, we see that $\dim(N) = \dim(\Gamma\backslash N) \leqslant \operatorname{asdim}(N)$. □

A map between metric spaces $\phi : (X_1, d_1) \to (X_2, d_2)$ is a *uniform embedding* if there are two real functions f and g with $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ such that

$$f(d_1(x, y)) \leqslant d_2(\phi(x), \phi(y)) \leqslant g(d_1(x, y))$$

for all pairs of points x, y in X_1. It is known from [10] that asymptotic dimension does not decrease under uniform embeddings.

3.6. Corollary. *Let Γ be a cocompact lattice in a connected Lie group G. Then*

$$\operatorname{asdim}(\Gamma) = \dim(G/K).$$

Proof. Clearly, $\operatorname{asdim}(\Gamma) = \operatorname{asdim}(G/K)$ since Γ embeds uniformly and commensurably in the homogeneous space G/K. Now there are mutual uniform embeddings of G/K in a simply connected nilpotent Lie group N with the N-invariant Riemannian metric, and vice versa, according to [3, Section IV]. Thus the three metric spaces have the same asymptotic dimension. □

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