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A lemma on extending functions into F-spaces and homomorphisms between Stone–Čech remainders

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Abstract

This papers contains two main results. The first is a theorem about continuous functions from a countably compact Hausdorff space into a compact F-space, which has applications to the algebraic properties of the Stone–Čech compactification βS of a discrete semigroup S. The second main result shows that many continuous homomorphisms from S^* to G^* have to arise from homomorphisms mapping S to G, where S is a discrete semigroup and G is a discrete group and S^* denotes $\beta S \setminus S$. The second result is related to the first because it uses it at a crucial point. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper is devoted to two main results. The first—and the less difficult of the two—says that sometimes extensions of maps which take their values in F-spaces (see [5,12] or [7]) are almost unique. To make this precise, let D be a discrete space, let βD be its Stone–Čech compactification, and write $D^* = \beta D \setminus D$.

Lemma 1.1. *Let D be discrete and let Z be a Hausdorff F-space. Let $\varphi_1, \varphi_2 : \beta D \to Z$ be two continuous functions which coincide on D^*. Then $\varphi_1(d) = \varphi_2(d)$ for all except a finite number of values of $d \in D$.*

A number of applications of this result are given to the theory of compact semigroups βS. If S is any discrete semigroup, βS has a unique semigroup structure in which it becomes a *right topological semigroup* (that is, the maps $p \to pq$ are continuous for every

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$q \in \beta S$) and in which the maps $q \to sq$ are continuous for every $s \in S$. (We refer the reader to [7] for details of this construction and for the theory of βS. General results about compact semigroups can also be found in [7] and also in [2].) We shall, for example, show that if S^* has one right zero then it has infinitely many, and we give a precise cardinal (Theorem 2.5).

The form of the Two-Function Lemma we establish is actually more general than the above special case (see Lemma 2.2). However, it does not tell us when a map $\varphi : D^* \to Z$ does possess an extension to βD. Our second main result, Theorem 5.6, provides a context in which extensions of continuous homomorphisms automatically exist. The Two-Function Lemma plays a small but vital role in the proof: it is used to establish a key algebraic property of extensions, and thence uniqueness.

We shall not state Theorem 5.6 in the introduction, as its statement is rather complicated; but it has several interesting corollaries and we shall cite one or two of these. For example, it implies that, if S is any cancellative discrete semigroup and if G is any countable discrete group, then any continuous injective homomorphism $\varphi : S^* \to G^*$ has the form $\varphi = \overline{\psi}|_{S^*}$, where $\psi : S \to G$ is a homomorphism and $\overline{\psi} : \beta S \to \beta G$ is its continuous extension. It also implies that, if G is any countable discrete group and if $\varphi : \beta\mathbb{N} \to G^*$ is a continuous homomorphism, then $\varphi(\mathbb{N}^*)$ is a finite group. The latter extends a result in [11], which stated that a continuous homomorphism $\varphi : \beta\mathbb{N} \to \mathbb{N}^*$ must have finite range. Further theorems of a type similar to ours can be found in Chapter 10 of [7]; but these are less general. For example, in [7] it is only *injective* homomorphisms from S^* to G^* which are considered, and these are restricted to the case in which S and G are countable and embedable in the circle.

Section 3 collects preliminary results for use in the proof of the main theorem. Two ideas may be worth mentioning here. The set ωG of elements in βG with countable dispersion character (that is, which are represented by ultrafilters which contain a countable set) is a prime subsemigroup of βG (Proposition 3.8). The concept of prime element of βG is introduced; p is *prime* if and only if $p = xq$, with $x \in \beta G$ and $q \in G^*$, implies that $x \in G$. These elements play a key role in the proof.

Section 4 contains a result about homomorphisms $\varphi : S^* \to \kappa G$ where κG is a more general compactification of a group. However, this generality is achieved only at the expense of a very strong hypothesis, that $\varphi(S^*)$ contains an element of G.

Section 5 contains the main theorem, Theorem 5.6, and Section 6 presents some of its corollaries.

We close this introduction with some notes on background knowledge. Information on F-spaces and Stone–Čech compactifications can be found in [5,12,7]; the last reference is the best for the theory of Stone–Čech compactifications of discrete semigroups. We denote the closure of a set A by $\overline{A}$; if A is a subset of a discrete space D then $\overline{A}$ is homeomorphic to βA, and we shall often identify the two.

We shall need some facts about cancellation in semigroups. A semigroup is *left cancellative* if $st = su$ with $s, t, u \in S$ implies $t = u$. It is easy to show that if S is left cancellative and $sq = sr$ with $s \in S$ and $q, r \in \beta S$ then $q = r$. A semigroup is weakly left (respectively right) cancellative if $\{t \in S\colon\ st = u\}$ (respectively $\{t \in S\colon\ ts = u\}$) is finite for each s, $u \in S$. Weak left cancellativity is equivalent to S^* being a left ideal in βS.

Weak left and weak right cancellativity together imply that S^* is a two-sided ideal in βS. In particular, if G is a group then G^* is a two-sided ideal in βG.

Every compact right topological semigroup S has minimal left and minimal right ideals, and the union of all minimal left or minimal right ideals is the unique smallest ideal $K(S)$. Each minimal left ideal L contains idempotents e, and for any such e we have $Se = Le = L$. Moreover $eSe = eLe$ is a group with e as its identity.

Countability assumptions play an essential role in Theorem 5.6. One reason for this is the frequency with which we use a theorem due to Frolik: If A and B are countable subsets of an F-space, then $\overline{A} \cap \overline{B} \neq \emptyset$ implies that $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. (A proof is given in [6, Lemma 1.1].)

2. The Two-Function Lemma and simple applications

Lemma 2.1. *Let (a_n), (b_n) be two sequences in a compact Hausdorff F-space Z such that $a_n \neq b_n$ for all n. Then there is a strictly increasing sequence (n_k) such that*

$$\overline{\{a_{n_k}\colon k = 1, 2, \ldots\}} \cap \overline{\{b_{n_k}\colon k = 1, 2, \ldots\}} = \emptyset.$$

Proof. In a compact Hausdorff F-space, a sequence with an infinite number of distinct terms has $2^{\mathfrak{c}}$ cluster points (see [12, Proposition 1.64] or [5, Exercise 14N5]), and so in particular it does not converge.

If (a_n) has a constant subsequence, say $a_{n_r} = a$ for all r, then since (b_{n_r}) cannot converge to a it is easy to find further subsequences (a_{n_k}), (b_{n_k}) for which the conclusion holds. Using symmetry we may assume that neither (a_n) nor (b_n) has a constant subsequence. Then, using an inductive construction we may replace our original sequences by subsequences (a_n), (b_n) for which $a_m \neq b_n$ for all m, n.

We now construct a subsequence (b_{n_r}) of (b_n) by a diagonal argument. Since $b_n \to a_1$ is false, we may find a subsequence of (b_n) which does not have a_1 as a cluster point. Since this subsequence does not converge to a_2, we may find a further subsequence of the first subsequence which does not have a_2 as a cluster point. And so on. The diagonal sequence (b_{n_r}) does not have any of $a_1, a_2, \ldots$ as a cluster point, and since $a_m \neq b_n$ for all n we have

$$\{a_n\colon n = 1, 2, \ldots\} \cap \overline{\{b_{n_r}\colon r = 1, 2, \ldots\}} = \emptyset,$$

and in particular

$$\{a_{n_r}\colon r = 1, 2, \ldots\} \cap \overline{\{b_{n_r}\colon r = 1, 2, \ldots\}} = \emptyset.$$

We repeat the argument starting with the sequences (a_{n_r}), (b_{n_r}) and interchanging the roles of the a's and b's. We find subsequences $(a_{n_{r_s}})$, $(b_{n_{r_s}})$ with

$$\overline{\{a_{n_{r_s}}\colon s = 1, 2, \ldots\}} \cap \{b_{n_{r_s}}\colon s = 1, 2, \ldots\} = \emptyset.$$

Of course, we still have

$$\{a_{n_{r_s}}\colon s = 1, 2, \ldots\} \cap \overline{\{b_{n_{r_s}}\colon s = 1, 2, \ldots\}} = \emptyset.$$

The conclusion of the lemma follows immediately from Theorem 3.40 in [7]. □

The Two-Function Lemma 2.2. *Let X be a countably compact Hausdorff space and Y a compact Hausdorff F-space. Let $\varphi, \psi : X \to Y$ be continuous mappings. Then*

$$X_0 = \{x \in X\colon\ \varphi(x) \neq \psi(x)\}$$

is countably compact.

Proof. We need to show that any sequence (x_n) in X_0 has a cluster point in X_0. Let (n_k) be the sequence produced by Lemma 2.1 when we take $a_n = \varphi(x_n)$, $b_n = \psi(x_n)$. By countable compactness, (x_{n_k}) has a cluster point x in X. Then $\varphi(x)$ is a cluster point of (a_{n_k}) and so is different from the cluster point $\psi(x)$ of (b_{n_k}). □

Our first corollaries are simple deductions from the lemma (the second is Lemma 1.1).

Corollaries 2.3.
- (i) *Let X be a countably compact Hausdorff space. If $\varphi = \psi$ on a G_δ subset E of X, which is the intersection of a countable family of its closed neighborhoods, then $\varphi = \psi$ on some neighborhood of E.*
- (ii) *If $X = \beta D$ for some discrete space D and $\varphi = \psi$ on D^* then $\varphi(x) = \psi(x)$ for all except a finite number of $x \in X$.*

Proof. (i) Let $E = \bigcap_n W_n$, where W_n is a closed neighborhood of E and $W_{n+1} \subseteq W_n$ for each n. Suppose there exists in each W_n some x_n with $\varphi(x_n) \neq \psi(x_n)$. Then $x_n \in X_0$ for each n but all the cluster points of (x_n) are in E and therefore not in X_0.

(ii) Here $X_0 \subseteq D$, and the only countably compact subsets of a discrete space are finite. □

In the rest of this section S is a discrete semigroup and βS has its usual structure as a compact right topological semigroup.

Definition 2.4. Let $p \in S^*$. The *norm* $\|p\|$ of p is $\min\{\mathrm{card}(U)\colon\ U \in p\}$.

Theorem 2.5. *Suppose that S^* has a right zero z (so that $pz = z$ for all $p \in S^*$, but we are not requiring S^* to be a semigroup here). Then*
- (i) *there is a finite set $F_z \subseteq S$ such that $xz = z$ for all $x \in \beta S \setminus F_z$;*
- (ii) *S^* has $2^{2^{\|z\|}}$ right zeros.*

Proof. The two mappings $x \mapsto z$ (the constant map) and $x \mapsto xz$ (multiplication on the right by z) are both continuous from βS to βS and are equal on S^*. By Corollary 2.3(ii), there is a finite set F_z such that the maps are equal on $\beta S \setminus F_z$. That proves (i).

Choose $U \in z$ with $|U| = \|z\|$. For each $x \in S \setminus F_z$, let $U_x = \{u \in U\colon\ xu = u\}$. (Conceivably U_x is empty.) It follows from the de Bruin–Erdős Lemma (Theorem 9.2 of [4] or Theorem 3.35 of [7]) that $U_x \in z$. The family $\mathcal{F} = \{U_x\colon\ x \in U \cap (S \setminus F_z)\}$ has the property that the intersection of any finite subfamily is both a member of z and a subset of U, and therefore any such intersection has cardinality $\|z\|$. By Theorem 7.7 of [4] (or

Theorem 3.62 of [7]), there are $2^{2^{\|z\|}}$ ultrafilters which refine $\mathcal{F}$. Let q be such an ultrafilter. For each $x \in U \cap (S \setminus F_z)$ and each $u \in U_x$, we have $xu = u$. Allowing u to converge to q and then allowing x to converge to z shows that $zq = q$. Thus, for every $y \in S^*$, we have $yq = yzq = zq = q$. □

Corollary 2.6. *S^* cannot have a zero (for any discrete semigroup S).*

Proof. If S^* did have a (two-sided) zero it would also be the unique right zero and this is impossible. □

Corollary 2.7. *If S^* has a right zero then S must contain a sequence $s_0, s_1, s_2, \ldots$ with the product $s_i s_j = s_j$ whenever $i < j$.*

Proof. With the notation of the proof of Theorem 2.5, take $s_0 \in S \setminus F_z$, $s_1 \in U_{s_0}$, $s_2 \in U_{s_0} \cap U_{s_1}$, and so on. Our requirements are satisfied. □

We cannot improve on Corollary 2.7. To see this, let S be the semigroup generated by a sequence $\{s_1, s_2, \ldots\}$ subject to the relations $s_i s_j = s_j$ when $i < j$. All elements of S are of the form $s_{i_1} s_{i_2} \ldots s_{i_k}$ where $i_1 \geqslant i_2 \geqslant i_k$. If z is any cluster point of (s_n), then when $n \geqslant i_1$ we have $s_{i_1} s_{i_2} \ldots s_{i_k} s_n = s_n$ so that $s_{i_1} s_{i_2} \ldots s_{i_k} z = z$, and therefore z is a right zero for βS (and *a fortiori* for S^*).

The last few results have parallels for right identities. Part (ii) of the following theorem is similar to Theorem 9.28 in [7].

Theorem 2.8. *Suppose that S^* has a right identity e (so that $pe = p$ for all $p \in S^*$). Then*

(i) *there is a finite set $F_e \subseteq S$ such that $xe = x$ for all $x \in \beta S \setminus F_e$;*

(ii) *S^* has $2^{2^{\|e\|}}$ right identities.*

Proof. The argument for (i) is as for Theorem 2.5 starting with the maps $x \mapsto x$ (the identity) and $x \mapsto xe$. The proof for (ii) can be modeled on the proof of Theorem 2.5(ii). □

There are, of course, parallels to the corollaries too. S^* cannot have an identity. If S^* does have a right identity then S contains a subsequence (e_n) with the multiplication $e_i e_j = e_i$ whenever $i < j$. An example similar to the one given above shows that this assertion cannot be improved upon.

It is natural to enquire whether there are results similar to those just obtained for left zeros and left identities. To some extent (but not much) there are. If $z \in S$ is a left zero for S^* (that is, $zq = z$ for all $q \in S^*$) then it is a left zero for all but a finite number of elements of βS, using the above arguments and the continuity of the map $q \mapsto zq$ on βS. However the hypothesis here is unnatural: could we obtain the conclusion if we knew instead that $z \in S^*$ was a left zero for S^*? The answer to this question is, No. The counter-example is a simple one. For the semigroup $(\mathbb{N}, \min)$ each element p of $\mathbb{N}^*$ satisfies $pq = p$ for all $q \in \mathbb{N}^*$, so it is a left zero for $\mathbb{N}^*$, but it is not a left zero for (any element of) $\mathbb{N}$ since $pn = n$ for all $n \in \mathbb{N}$.

In the same way, if $e \in S$ is a left identity for S^* then it is also a left identity for all but a finite number of elements of S, but the example $(\mathbb{N}, \max)$ shows that this need not hold for left identities in S^*.

The Two-Function Lemma also allows us to make a deduction about commuting elements in Stone–Čech compactifications, though again we have to impose a restriction about the elements lying in S.

Theorem 2.9. *Let S be a discrete semigroup and let $A \subseteq S$. Let $s \in S$ have the property that $sp = ps$ for all $p \in A^*$. Then there is a finite subset F of A such that $sx = xs$ for all $x \in A \setminus F$. In particular, if $s \in S$ commutes with every element of S^* then s commutes with almost every element of S.*

Proof. The two continuous maps $x \mapsto xs$, $x \mapsto sx$ of βA to βS coincide on A^*. The Two-Function Lemma immediately gives the conclusion. □

Finally in this section we give a very weak cancellation result.

Proposition 2.10. *Let S be a left cancellative discrete semigroup and let $q, r \in S^*$. Suppose that there is an infinite subset U of S such that $pq = pr$ for all $p \in U^*$. Then $q = r$.*

Proof. The continuous functions $p \mapsto pq$, $p \mapsto pr$ from βU to βS are equal on U^* by hypothesis. From Corollary 2.3(ii) there is $s \in U$ such that $sq = sr$. Since s is left cancellable in βS [7, Lemma 8.1], this gives the result. □

3. Auxiliary results

This section contains results which we shall need to prove our main theorem. The first is an immediate conclusion from a well-known lemma, but is recorded here in the form in which we shall need to use it repeatedly.

Definition 3.1. Let $X_1, \ldots, X_n$ be discrete spaces and let $\theta : \beta X_1 \times \cdots \times \beta X_n \to Z$ be a map to a topological space Z. Then we say θ is *β-separately continuous* if for each k with $1 \leqslant k \leqslant n$, the map $x \mapsto \theta(x_1, \ldots, x_{k-1}, x, p_{k+1}, \ldots, p_n)$ is continuous on βX_k when $x_i \in X_i$ for $1 \leqslant i \leqslant k-1$ and $p_i \in \beta X_i$ for $k+1 \leqslant i \leqslant n$.

A situation in which such maps arise is when $X_1 = \cdots = X_n = S$ are all the same semigroup, and the map θ is the multiplication $(x_1, x_2, \ldots, x_n) \mapsto x_1 x_2 \ldots x_n$ in βS.

Lemma 3.2. *Let $\theta_1 : \beta X_1 \times \cdots \times \beta X_m \to Z$ and $\theta_2 : \beta Y_1 \times \cdots \times \beta Y_n \to Z$ be β-separately continuous maps and let Z be an F-space. Suppose that*

$$\theta_1(x_1, \ldots, x_{h-1}, p_h, p_{h+1}, \ldots, p_m) = \theta_2(y_1, \ldots, y_{k-1}, q_k, q_{k+1}, \ldots, q_n),$$

where $x_i \in X_i$ for $1 \leqslant i \leqslant h-1$ and $p_i \in \beta X_i$ when $h \leqslant i \leqslant m$, and $y_i \in Y_i$ for $1 \leqslant i \leqslant k-1$ and $q_i \in \beta Y_i$ when $k \leqslant i \leqslant n$. Suppose there are countable sets $U_h \subseteq X_h$, $V_k \subseteq Y_k$ with $p_h \in \overline{U}_h$, $q_k \in \overline{V}_k$. Then either there exist $x_h \in U_h$ and $q'_k \in \overline{V}_k$ with

$$\theta_1(x_1, \ldots, x_{h-1}, x_h, p_{h+1}, \ldots, p_m) = \theta_2\big(y_1, \ldots, y_{k-1}, q'_k, q_{k+1}, \ldots, q_n\big), \tag{1}$$

or else there exist $p'_h \in \overline{U}_h$ and $y_k \in V_k$ with

$$\theta_1\big(x_1, \ldots, x_{h-1}, p'_h, p_{h+1}, \ldots, p_m\big) = \theta_2(y_1, \ldots, y_{k-1}, y_k, q_{k+1}, \ldots, q_n). \tag{2}$$

(The point of this lemma is that the number of variables which lie in the sets X or Y rather than in the remainders X^* or Y^* increases by one.)

Proof. The hypotheses ensure that the closures of the two countable sets

$$\theta_1(x_1, \ldots, x_{h-1}, U_h, p_{h+1}, \ldots, p_m), \quad \theta_2(y_1, \ldots, y_{k-1}, V_k, q_{k+1}, \ldots, q_n)$$

intersect. An immediate application of Theorem 3.40 of [7] assures us that either there exist $x_h \in U_h$, $q'_k \in \overline{V}_k$ such that (1) holds, or there exist $p'_h \in \overline{U}_h$, $y_k \in V_k$ such that (2) holds. □

Next we have some lemmas about semigroups.

Lemma 3.3. *Let G be a discrete group. Let $p, q \in \beta G$, and let $G_0 \subseteq G$ be a subgroup.*
(i) *If any two of p, q, pq are in $\overline{G}_0$, so also is the third.*
(ii) *If $pq \in \overline{G}_0$ there is $g \in G$ with pg^{-1}, $gq \in \overline{G}_0$.*

Proof. (i) We do the case in which p, $pq \in \overline{G}_0$. Since $\overline{G}_0$ is a neighbourhood of both pq and p, using right continuity we can find $g \in G_0$ with $gq \in \overline{G}_0$. Thus $q \in \overline{g^{-1}G_0} = \overline{G}_0$.

(ii) As in (i), there is $g \in G$ with $gq \in \overline{G}_0$. Then $(pg^{-1})(gq) = pq \in \overline{G}_0$. From (i), $pg^{-1} \in \overline{G}_0$. □

Lemma 3.4. *Let S, T be compact Hausdorff right topological semigroups with $S \subseteq T$ and let L be a minimal left ideal of S. If $t \in T$, $s \in L$ and $ts \in S$, then $ts \in L$.*

Proof. Take $e^2 = e \in L$ so that $se = s$. Then $ts = tse \in Se = L$. □

Lemma 3.5. *Let S, T be compact Hausdorff right topological semigroups with $S \subseteq T$. If $p \in S$ and $p \notin K(S)$ then $p \notin K(T)$.*

Proof. This follows from Theorem 1.65 in [7]. □

The proof of our Main Theorem 5.6 would be slightly easier if we were considering a two-sided, rather than just a left, ideal. Our next lemma shows that under some circumstances a left ideal can have a very weak right-ideal like property.

Lemma 3.6. *Let S be a compact right topological semigroup and let T be a left ideal in S. Suppose there is $p \in T$ such that $sp \notin K(T)$ for all $s \in S \setminus T$. Then for each $s \in S \setminus T$ there is $q \in T$ with $qs \in T$.*

Proof. Suppose the conclusion false, so that there is $s \in S \setminus T$ such that for all $q \in T$ we have $qs \notin T$, that is, $Ts \subseteq S \setminus T$. Because T is a left ideal in S, Ts is also a left ideal so it contains a minimal left ideal. Therefore there is a minimal idempotent e in Ts, and $Te \subseteq Ts \subseteq S \setminus T$. Since e is minimal, eS is a minimal right ideal in S and eSe is a group with identity e.

Now take any $x \in T$ so that ex is any element of eT. Then $exe \in eSe$, so there is an inverse $(exe)^{-1}$ with $(exe)(exe)^{-1} = e$. Since T is a left ideal, $(exe)^{-1}T \subseteq T$. Thus $eT = (exe)(exe)^{-1}T \subseteq exeT \subseteq eT$. Therefore $(ex)(eT) = eT$, and eT is a minimal right ideal in T. Hence $eT \subseteq K(T)$, that is, $ep \in K(T)$ for all $p \in T$. Thus the hypothesis of the lemma is false for $s = e$. □

Our next few results concern elements in Stone–Čech remainders whose norm is countable. We write

$$\omega S = \{p \in \beta S\colon \ \|p\| \text{ is countable}\}.$$

Since every infinite subset of S contains a countably infinite subset, it is easy to see that

Proposition 3.7. *$\omega S \cap S^*$ is dense in S^*.*

If S is a group G and $p \in \omega G$, then there is a countable subgroup G_p of G with $p \in \overline{G}_p$.

Proposition 3.8. *ωG is a subsemigroup of βG and if $pq \in \omega G$ then both p and q are in ωG.* (*This says that ωG is a prime subsemigroup in normal terminology, but below we shall be using 'prime' in a different sense.*)

Proof. Let G_p, G_q be countable groups in p, q, respectively. Then the group generated by $G_p \cup G_q$ is in pq so that ωG is a semigroup. If $pq \in \omega G$, say $pq \in \overline{G_0}$ where G_0 is countable, then from Lemma 3.3 we see that $p \in \overline{G_0 g}$, $q \in \overline{g^{-1}G_0}$ for some $g \in G$ so that both p and q are in the closures of countable sets. □

Our next lemma looks rather technical. It will be required in this form, but its point is that ωS is well-behaved under homomorphisms.

Lemma 3.9.

(i) *Let S be an infinite discrete semigroup and G a discrete group. Let $S^\dagger$ be a left ideal in βS for which $S^* \subseteq S^\dagger$. Let $\varphi : S^\dagger \to \beta G$ be a continuous homomorphism. Let S_c be a countable subsemigroup of S and write $S_c^\dagger = \overline{S}_c \cap S^\dagger$. Suppose there is $q \in S^\dagger$ with $\varphi(sq) \in \omega G$ for all $s \in S_c$. Then there is a countable subgroup G_c of G such that $\varphi(S_c^\dagger) \subseteq \overline{G}_c$ and $\varphi(q)$, $\varphi(sq) \in \overline{G}_c$ for all $s \in S_c$. Moreover, if $s \in S_c$ and $qs \in S^\dagger$ then $\varphi(qs) \in \overline{G}_c$.*
Furthermore, $\varphi(\overline{S}_c) \subseteq \overline{G}_c$.

(ii) *Let H be a discrete group and G be as in* (i). *If $\varphi : H^* \to \beta G$ is a continuous homomorphism and there is $q \in H^*$ with $\varphi(q) \in \omega G$ then $\varphi(hq) \in \omega G$ for all $h \in H$. Thus, with S taken to be H and $S^\dagger$ taken to be H^*, it follows that, for every countable subsemigroup S_c of H, there is a countable subgroup G_c of G for which the the conclusions of* (i) *hold.*

Proof. (i) Since S_c is countable we can find a countable subgroup G_c of G with $\varphi(sq) \in \overline{G}_c$ for all $s \in S_c$. Given $p \in S_c^\dagger$ we can let $s \in S_c$ converge to p (if $p \in S_c$ then the net tending to p can be constant) to find that $\varphi(pq) \in \overline{G}_c$. But $\varphi(pq) = \varphi(p)\varphi(q)$. So from Lemma 3.8, $\varphi(p)$ and $\varphi(q)$ are in ωG. We may therefore suppose (enlarging G_c if necessary) that $\varphi(q)$ is in $\overline{G}_c$. Then, by Lemma 3.3, we also have $\varphi(p) \in \overline{G}_c$. If in addition $qs \in S^\dagger$, then $\varphi(qs)\varphi(q) = \varphi(q)\varphi(sq) \in \overline{G}_c$ and we see from Lemma 3.3 that $\varphi(qs) \in \overline{G}_c$.

(ii) If H is a group, H^* is an ideal in βH. If $q \in H^*$ and $\varphi(q) \in \omega G$, then for each $h \in H$ we have $\varphi(qh^{-1})\varphi(hq) = \varphi(q^2) \in \omega G$ and so $\varphi(hq) \in \omega G$ by Proposition 3.8. □

We next need the concept of prime element in Stone–Čech compactifications of groups.

Definition 3.10. Let G be a group. We say that $p \in G^*$ is *relatively prime to $q \in G^*$* (*for* G) if $p = xq$ with $x \in \beta G$ implies $x \in G$. If p is relatively prime to all elements of G^* it is called *prime* (*for* G).

Trivially if p is relatively prime to q then it is relatively prime to gq for all $g \in G$. In addition, from Lemma 3.3 we see that if G_0 is a subgroup of G and p is relatively prime to q for G_0 then p is relatively prime to q for G. In fact we can say slightly more.

Proposition 3.11. *Let G be a discrete group and let G_0 be a subgroup of G. Then if $p \in G_0^*$ is prime for G_0, it is prime for G.*

Proof. Let $p = xq$ where $x \in \beta G$, $q \in G^*$. From Lemma 3.3 there is $g \in G$ with $xg^{-1} \in \overline{G}_0$, $gq \in G_0^*$. From $p = xg^{-1}gq$ we get $xg^{-1} \in G_0$, so $x \in G_0 g \subseteq G$. □

The construction of prime elements is very easy.

Proposition 3.12. *Let G be a discrete group and let $A \subseteq G$ be countable and infinite. Then there is an infinite subset A_0 of A such that all elements of A_0^* are prime. The set of prime elements of G^* is dense in G^*.*

Proof. From the remark about subgroups above, we need only consider the case in which G itself is countable, say $G = \{g_1, g_2, \ldots\}$. Choose $a_1 \in A$ arbitrarily and then inductively choose $a_{n+1} \in A$ with $a_{n+1} \notin \{g_1, g_2, \ldots, g_n\}\{a_1, a_2, \ldots, a_n\}$. We put $A_0 = \{a_1, a_2, \ldots\}$. This construction means that for any $g \neq 1$ in G, $gA_0 \cap A_0$ is finite. Take $p \in A_0^*$ and suppose $p = xq$ with $x \in \beta G$, $q \in G^*$. Since $\overline{A_0}$ is open, for every sufficiently small neighbourhood $\overline{U}$ of x in βG we have $\overline{U}q \subseteq \overline{A}_0$. Take any $g_1, g_2 \in U$, so that $g_1 q, g_2 q \in \overline{A}_0$. Then we can find a neighbourhood $\overline{V}$ of q with $g_1\overline{V}, g_2\overline{V} \subseteq \overline{A}_0$, so that

$g_1V, g_2V \subseteq A_0$. Since V is infinite, $g_2g_1^{-1}A_0 \cap A_0 \supseteq g_2g_1^{-1}(g_1V) \cap g_2V = g_2V$ is infinite, whence $g_1 = g_2$. Therefore U is a singleton, and this means that $x \in G$.

Since every subset V of G contains a countable subset, we see that every subset of G^* of the form V^* contains prime elements. Therefore the prime elements are dense. □

Prime elements have a cancellation property.

Proposition 3.13. *Let G be a discrete group. Let $p \in G^*$ be prime, $q_1, q_2 \in \omega G$, and $q_1 p = q_2 p$. Then $q_1 = q_2$. In particular, if G is countable, prime elements are right cancellable.*

Proof. For $i = 1, 2$ let U_i be any countable set with $\overline{U_i}$ a neighbourhood of q_i. By Lemma 3.2, there are two possibilities. The first is that there exist $g \in U_1$ and $q_2' \in \overline{U}_2$ with $gp = q_2'p$. Then $p = g^{-1}q_2'p$ so because p is prime $g^{-1}q_2' \in G$, and then by Veech's Lemma [2, 4.8.9], or by Corollary 8.2 in [7], $g^{-1}q_2' = 1$. Thus $U_1 \cap U_2 \neq \emptyset$. The second alternative is the same with the subscripts 1 and 2 interchanged. Therefore every neighbourhood of q_1 meets every neighbourhood of q_2, so $q_1 = q_2$. □

Finally we make a trivial observation about elements which cannot be prime.

Proposition 3.14. *If p is prime for G and S is any compact subsemigroup of G^*, then $p \notin K(S)$.*

Proof. For each element p of the smallest ideal $K(S)$ there is a left identity $e \in K(S)$, that is $p = ep$. □

4. Homomorphisms: An easy case

The aim in this section is to give a simple (but interesting) result which is not a special case of our main theorem (because the target semigroup is more general). However, the simpler arguments which suffice here follow the same pattern as the main proof.

Theorem 4.1. *Assume that*

(i) *S is a discrete semigroup and $S^\dagger$ is a two-sided ideal in βS,*

(ii) *κG is a compact right topological semigroup which algebraically contains a subgroup G whose identity is the identity of κG and in which G is topologically dense,*

(iii) *$\varphi : S^\dagger \to \kappa G$ is a continuous homomorphism,*

(iv) *$\varphi(S^\dagger) \cap G \neq \emptyset$.*

Then there is a unique homomorphism $\overline{\psi} : \beta S \to \kappa G$ with $\overline{\psi}|_{S^\dagger} = \varphi$. The map $\overline{\psi}$ is continuous on βS.

These hypotheses hold in particular when S is weakly left cancellative and weakly right cancellative, $S^\dagger = S^*$ and φ is a continuous homomorphism from S^* to βG with $\varphi(S^*) \cap G \neq \emptyset$.

Proof. Fix $p \in S^\dagger$ with $\varphi(p) \in G$. For each $s \in S$, $sp \in S^\dagger$. Define $\psi(s) = \varphi(sp)\varphi(p)^{-1}$. Then $\varphi(sp) = \psi(s)\varphi(p)$.

Define $\overline{\psi}$ to be the unique continuous extension of ψ to βS. Notice that, on taking $q \in \beta S$ and letting $s \in S$ converge to q, we get from continuity that $\varphi(qp) = \overline{\psi}(q)\varphi(p)$. Also, since φ is a homomorphism on $S^\dagger$ we have $\varphi(qp) = \varphi(q)\varphi(p)$. Multiplying by $\varphi(p)^{-1}$ on the right gives $\overline{\psi}(q) = \varphi(q)$. Thus $\overline{\psi}$ is a continuous extension of φ to βS.

We must show that $\overline{\psi}$ is a homomorphism on βS. We take any $q \in S^\dagger$. Then for any $t \in \beta S$ we have $qt,\ tp \in S^\dagger$ and so

$$\varphi(qt)\varphi(p) = \varphi(qtp) = \varphi(q)\varphi(tp) = \varphi(q)\overline{\psi}(t)\varphi(p).$$

Since $\varphi(p)$ is invertible we get $\varphi(qt) = \varphi(q)\overline{\psi}(t)$.

Now take $t,\ t' \in \beta S$. Then

$$\varphi(q)\overline{\psi}(tt') = \varphi(qtt') = \varphi\big((qt)t'\big) = \varphi(qt)\overline{\psi}(t') = \varphi(q)\overline{\psi}(t)\overline{\psi}(t').$$

This holds for any $q \in S^\dagger$ so we may in particular take $q = p$ and cancel $\varphi(p)$ to find

$$\overline{\psi}(tt') = \overline{\psi}(t)\overline{\psi}(t'),$$

so that $\overline{\psi}$ is indeed a homomorphism.

Finally notice that if ψ' is any homomorphism on βS which extends φ it must satisfy

$$\varphi(qp) = \psi'(qp) = \psi'(q)\psi'(p) = \psi'(q)\varphi(p)$$

for every $q \in \beta S$. So $\psi'(q) = \overline{\psi}(q)$. $\square$

Examples 4.2.

(i) If G is any compact right topological group we may take $\kappa G = G$ in hypothesis (ii) of Theorem 4.1 and then hypothesis (iv) is automatically satisfied. Thus for a discrete semigroup S, any continuous homomorphism from any ideal $S^\dagger$ in βS to G extends uniquely to a continuous homomorphism from βS to G. In particular, any homomorphism from $S^\dagger$ to a finite group is the Stone–Čech extension of a homomorphism from S to the finite group.

(ii) This example is to show that the hypotheses of the theorem can be satisfied in a non-trivial way.
Put $S = \mathbb{Z}\times\mathbb{Z}$, $G = \mathbb{Z}\times\mathbb{Z}_2$ and $\kappa G = \beta G$. The map $\psi : S \to G$ which is the identity in the first coordinate and the usual quotient in the second is a homomorphism. The extension $\overline{\psi}$ sends elements of $\{0\}\times\mathbb{Z}^*$ to $\{0\}\times\mathbb{Z}_2$. Thus if φ is $\overline{\psi}|_{S^*}$, $\varphi(S^*)$ meets G. But $\varphi(S^*) \nsubseteq G$ and indeed its image contains the whole of G^*.

5. Homomorphisms: The hard case

It will be convenient to use the notation $A^\dagger = S^\dagger \cap \overline{A}$, when $S^\dagger$ is a given subset of βS and $A \subseteq S$. This is meant to parallel the relationship $A^* = S^* \cap \overline{A}$.

The main theorem of the paper is Theorem 5.6 below. We first need to prove a sequence of lemmas in which S and G are restricted to be countable.

In the lemmas which follow we assume that S is a discrete countable semigroup and that G is a discrete countable group. We also assume that $S^\dagger$ is a left ideal in βS such that $S^* \subseteq S^\dagger$, and that $\varphi : S^\dagger \to \beta G$ is a continuous homomorphism. In addition, we assume that there exists $p \in S^\dagger$ such that, for every $s \in S$, $\varphi(sp) \notin K(\varphi(S^\dagger))$.

We note that $\beta S \setminus S^\dagger \subseteq S$ and hence that $S^\dagger$ is a closed G_δ-subset of βS.

We shall prove that there exists a unique homomorphism $\psi : S \to G$ such that $\varphi = \overline{\psi}|_{S^\dagger}$, where $\overline{\psi} : \beta S \to \beta G$ denotes the continuous extension of ψ.

The following lemma is the key to the whole enterprise.

Lemma 5.1. *There exists an infinite $X \subseteq S$ with $\varphi(sp) \notin \beta G\varphi(SX^*p)$ for all $s \in S$.*

Proof. Let e be an idempotent in $K(S^\dagger)$. Then $ep \in K(S^\dagger)$. Since φ is a surjective homomorphism from $S^\dagger$ onto $\varphi(S^\dagger)$ we see that $\varphi(ep) \in K(\varphi(S^\dagger))$ ([8, Surjectivity Lemma 2.3] or [7, Exercise 1.7.3]). We note that this implies that $e \notin S$, because we are assuming that, for each $s \in S$, $\varphi(sp) \notin K(\varphi(S^\dagger))$. Therefore $\beta G\varphi(ep) \cap \varphi(S^\dagger) = \varphi(S^\dagger)\varphi(ep)$ (Lemma 3.4). Hence for each $s \in S$, $\varphi(sp) \notin \beta G\varphi(ep)$.

Now $\beta G\varphi(ep)$ is compact by right continuity. So given $s \in S$ we can find $W(s) \subseteq G$ with $\varphi(sp) \notin \overline{W(s)}$ but $\beta G\varphi(ep) \subseteq \overline{W(s)}$. Then for each $g \in G$, $s' \in S$ we have $s'e \in S^\dagger$ and

$$g\varphi(s'ep) = g\varphi(s'eep) = g\varphi(s'e)\varphi(ep) \in \beta G\varphi(ep) \subseteq \overline{W(s)}.$$

For any given $g \in G$ and $s' \in S$, the map $x \mapsto g\varphi(s'xp)$ is continuous on βS. So there exists $U(g,s,s') \subseteq S$ with $e \in \overline{U(g,s,s')}$ and $g\varphi(s'\overline{U(g,s,s')}p) \subseteq \overline{W(s)}$. Moreover, we can arrange that $s' \notin U(g,s,s')$ since $\overline{U(g,s,s') \setminus \{s'\}}$ is again a neighbourhood of e.

Now $\bigcap_{g,s,s'} \overline{U(g,s,s')}$ is a G_δ set in $\beta S \setminus S = S^*$, which is non-empty as it contains e. So we can find a countably infinite set $X \subseteq S$ with $X^* \subseteq \bigcap_{g,s,s'} \overline{U(g,s,s')}$ [7, Theorem 3.36]).

Thus for any given $s \in S$, for all $g \in G$, $s' \in S$ and $x \in X^*$ we have $g\varphi(s'xp) \in \overline{W(s)}$. Since $\overline{W(s)}$ is closed, we deduce that $\overline{G}\varphi(s'xp) \subseteq \overline{W(s)}$. Consequently $\varphi(sp) \notin \overline{G}\varphi(s'xp)$ as required. □

We must strengthen the properties of X. This is achieved in the next lemma by making it thinner.

Lemma 5.2. *Enumerate the countable set $G \times S \times S$ as $(g_i, s_i, s'_i)_{i\in\omega}$. There is an infinite set $X = \{t_1, t_2, \ldots\} \subseteq S$ such that the conclusion of Lemma* 5.1 *holds and when $m, n > i$ and $m \neq n$,*

$$\varphi(s_i t_m p) \neq g_i \varphi(s'_i t_n p).$$

Proof. We start with X as produced in Lemma 5.1. For a given i and some t_0 fixed in X consider

$$\{t \in X:\ \varphi(s_i t_0 p) = g_i \varphi(s'_i t p)\}.$$

If this set were infinite, then continuity in the t-variable would show that any of its cluster points (which lie in X^*) would violate the conclusion of Lemma 5.1, so in fact the set is finite. The same argument shows that

$$\{t \in X\colon\ \varphi(s_i' t_0 p) = g_i^{-1}\varphi(s_i t p)\}$$

must be finite. Therefore if $t_1, \ldots, t_n$ have been chosen from X, we can find $t_{n+1} \in X$ such that, for $1 \leqslant i, j \leqslant n$,

$$\varphi(s_i t_j p) \neq g_i \varphi(s_i' t_{n+1} p), \quad g_i \varphi(s_i' t_j p) \neq \varphi(s_i t_{n+1} p).$$

We replace X by its subset $\{t_1, t_2, \ldots\}$ to reach our conclusion. □

Lemma 5.3. *Take X as in the conclusion of Lemma* 5.2. *Let $x_1, x_2 \in X^*$, $y_1, y_2 \in \beta G$, $s_1, s_2 \in S$.*

(i) *If $y_1\varphi(s_1 x_1 p) = y_2\varphi(s_2 x_2 p)$ then $x_1 = x_2$.*

(ii) *If in addition $s_1 = s_2$, then $y_1 = y_2$. (This says that for $s \in S$ and $x \in X$, $\varphi(sxp)$ is right cancellable in βG.)*

Proof. (i) Begin by observing that the map $(y, x', x) \mapsto y\varphi(x'xp)$ is β-separately continuous from $\beta G \times \beta S \times \beta S \to \beta G$ (Definition 3.1). Our proof comes from a sequence of applications of Lemma 3.2.

Assume $x_1 \neq x_2$. For $i = 1, 2$ take $U_i \subseteq X$ with $x_i \in U_i^*$ and $U_1 \cap U_2 = \emptyset$. Writing $X = \{t_1, t_2, \ldots\}$ as in Lemma 5.2, we see that if $t_m \in U_1$, $t_n \in U_2$ then $m \neq n$. We also take any $Y_1, Y_2 \subseteq G$ with $y_1 \in \overline{Y}_1$, $y_2 \in \overline{Y}_2$.

Starting from the equation given in (i), Lemma 3.2 and symmetry tell us that we may suppose that there exist $a_1 \in Y_1$, $y_2' \in \overline{Y}_2$ with

$$a_1\varphi(s_1 x_1 p) = y_2'\varphi(s_2 x_2 p).$$

In the same way, continuity in the x variable on the left and the y variable on the right now yields two possibilities. The first, that there exist $u_1 \in U_1$, $y_2'' \in Y_2$ with

$$a_1\varphi(s_1 u_1 p) = y_2''\varphi(s_2 x_2 p),$$

or

$$\varphi\big((s_1 u_1) p\big) = a_1^{-1} y_2''\varphi(s_2 x_2 p),$$

contradicts the conclusion of Lemma 5.1, so is impossible. The second must therefore hold, that is, there exist $x_1' \in \overline{U}_1$, $a_2 \in Y_2$ with

$$a_1\varphi(s_1 x_1' p) = a_2\varphi(s_2 x_2 p). \tag{$\star$}$$

Here $x_1' \in S$ would again contradict the conclusion of Lemma 5.1, so $x_1' \in U_1^*$.

The next step requires a little more care. Let $(a_1^{-1}a_2, s_1, s_2)$ be the rth triple in the enumeration of $G \times S \times S$ in Lemma 5.2. Put $X_r = \{t_n\colon\ n > r\}$ and notice that $X_r^* = X^*$. Then $x_1' \in U_1^* \cap X_r^* = (U_1 \cap X_r)^*$ and $x_2 \in (U_2 \cap X_r)^*$. Applying Lemma 3.2 to ($\star$)

using continuity in the x variables gives two alternatives again. The first is that there exist $t_m \in U_1 \cap X_r$ (which means in particular that $m > r$) and $x_2' \in \overline{U_2 \cap X_r}$ with

$$a_1\varphi(s_1 t_m p) = a_2\varphi(s_2 x_2' p).$$

Again by Lemma 5.1 this is impossible unless $x_2' \in G$, and that means $x_2' \in U_2 \cap X_r$, say $x_2' = t_n$ with $n > r$. But that possibility is ruled out by the properties of X in Lemma 5.2. The second alternative simply interchanges the subscripts 1 and 2, so is equally impossible.

This contradiction tells us that $x_1 = x_2$, and (i) is proved.

(ii) If we start with $s_1 = s_2 = s$ in the equation in (i), then ($\star$) becomes $a_1\varphi(sx_1' p) = a_2\varphi(sx_2 p)$. By (i), this implies that $x_1' = x_2$. Since G is a group we may apply Veech's Lemma ([2, Lemma 4.8.9] or [7, Corollary 8.2]) to find that $a_1^{-1}a_2 = 1$, or $a_1 = a_2$. Thus Y_1 intersects Y_2, which means that every neighbourhood of y_1 meets every neighbourhood of y_2. Thus $y_1 = y_2$. □

The result which we are moving towards is that two elements of the form $\varphi(sxp)$ are relatively prime:

Lemma 5.4. *Let $s_1, s_2 \in S$, $x \in X^*$. Then $\beta G\varphi(s_1 xp) \cap \beta G\varphi(s_2 xp) \neq \emptyset$ implies $\varphi(s_1 xp) \in G\varphi(s_2 xp)$.*

Proof. The hypotheses tell us that there are $y_1, y_2 \in \beta G$ with $y_1\varphi(s_1 xp) = y_2\varphi(s_2 xp)$. Following the proof of Lemma 5.3(i) as far as ($\star$) provides a_1, $a_2 \in G$ such that $a_1\varphi(s_1 x_1' p) = a_2\varphi(s_2 xp)$, for some $x_1' \in \beta S$ (or a corresponding formula with the subscripts 1 and 2 interchanged). Lemma 5.3(i) says that $x_1' = x$, and the result follows. □

We are now in a position to complete the proof for countable semigroups.

Lemma 5.5. *There is a unique homomorphism $\psi : S \to G$ such that $\overline{\psi}|_{S^\dagger} = \varphi$.*

Proof. Take X as in Lemma 5.2. Fix $x \in X^*$ and $s_0 \in S$. For any $s \in S$ write

$$V(s) = \{v \in S^\dagger\colon\ vs \in S^\dagger\}.$$

Our hypotheses imply that $sp \notin K(S^\dagger)$ for all $s \in S$, for otherwise $\varphi(sp) \in \varphi(K(S^\dagger)) = K(\varphi(S^\dagger))$ since a surjective homomorphism $\varphi : S^\dagger \to \varphi(S^\dagger)$ preserves smallest ideals ([8, Surjectivity Lemma 2.3] or [7, Exercise 1.7.3]). Lemma 3.6 therefore tells us that $V(s)$ is non-empty. Moreover, $V(s) = S^\dagger \cap \bigcap_{a \in S \setminus S^\dagger}\{v \in \beta S\colon\ vs \neq a\}$. Now $\{v \in \beta S\colon\ vs \neq a\}$ is clopen in βS for each $a \in S$. Since $S^\dagger$ is a closed G_δ-subset of βS, it follows that $V(s)$ is a closed G_δ-subset of βS.

Now for any $v \in V(s)$ we have $vs \in S^\dagger$ and therefore

$$\varphi(vs)\varphi(s_0 xp) = \varphi(vss_0 xp) = \varphi(v)\varphi(ss_0 xp).$$

Thus $\overline{G}\varphi(s_0 xp) \cap \overline{G}\varphi(ss_0 xp) \neq \emptyset$. From Lemma 5.4 there is an element of G which we denote by $\psi(s)$ for which

$$\varphi(ss_0 xp) = \psi(s)\varphi(s_0 xp);$$

thus $\psi : S \to G$. By Lemma 5.3(ii) the last equation determines $\psi(s)$ uniquely. Let $\overline{\psi}$ be the unique continuous extension of ψ to βS.

Now let q be any element of $S^{\dagger}$. In the last equation we let s in S converge to q. Continuity gives us

$$\varphi(qs_0xp) = \overline{\psi}(q)\varphi(s_0xp).$$

Since φ is a homomorphism on $S^{\dagger}$ we also have $\varphi(qs_0xp) = \varphi(q)\varphi(s_0xp)$, and we may cancel $\varphi(s_0xp)$ (Lemma 5.3) to obtain $\overline{\psi}(q) = \varphi(q)$, so that $\overline{\psi}|_{S^{\dagger}} = \varphi|_{S^{\dagger}}$.

We must show that ψ is a homomorphism. First for any $q \in S^{\dagger}$ and any $s \in S$, either $qs \in S^{\dagger}$ or $qs \in S$. In the former case

$$\begin{aligned}\overline{\psi}(qs)\varphi(s_0xp) &= \varphi(qs)\varphi(s_0xp) = \varphi(qss_0xp) = \varphi(q)\varphi(ss_0xp)\\ &= \overline{\psi}(q)\psi(s)\varphi(s_0xp).\end{aligned}$$

In the second

$$\begin{aligned}\overline{\psi}(qs)\varphi(s_0xp) &= \psi(qs)\varphi(s_0xp) = \varphi(qss_0xp) = \varphi(q)\varphi(ss_0xp)\\ &= \overline{\psi}(q)\psi(s)\varphi(s_0xp).\end{aligned}$$

Again from Lemma 5.3 we deduce that $\overline{\psi}(qs) = \overline{\psi}(q)\psi(s)$. Now given $s,\ s' \in S$, for any $v \in V(s)$ both $v,\ vs \in S^{\dagger}$ (though vs may not be in $V(s)$), so that

$$\overline{\psi}(v)\psi(ss') = \overline{\psi}(vss') = \overline{\psi}(vs)\psi(s') = \overline{\psi}(v)\psi(s)\psi(s').$$

This means that the continuous maps (defined on βS)

$$x \mapsto \overline{\psi}(x)\psi(ss'), \quad x \mapsto \overline{\psi}(x)\psi(s)\psi(s')$$

coincide on the set $V(s)$, which is a closed G_{δ}-subset of βS. By Corollary 2.3(i) to the Two-Function Lemma, these maps are equal on a non-empty open subset of βS. So there exists $x \in S$ such that

$$\psi(x)\psi(ss') = \psi(x)\psi(s)\psi(s')$$

and since $\psi(x) \in G$ it can be cancelled to yield that ψ is a homomorphism.

To see that ψ is unique, let $\psi' : S \to G$ be any homomorphism for which $\overline{\psi'}|_{S^{\dagger}} = \varphi$. Then $\varphi(ss_0xp) = \overline{\psi'}(ss_0xp) = \psi'(s)\overline{\psi'}(s_0xp) = \psi'(s)\varphi(s_0xp)$ and so $\psi' = \psi$. □

We can now prove our second main result.

Theorem 5.6. *Let S be a discrete semigroup and $S^{\dagger}$ be a left ideal in βS with $S^* \subseteq S^{\dagger}$, and let G be a discrete group. Let $\varphi : S^{\dagger} \to \beta G$ be a continuous homomorphism with the properties*

(i) *there is $q \in S^{\dagger}$ such that $\varphi(sq) \in \omega G$ for all $s \in S$,*

(ii) *for each countable subset S_0 of S there exist a countable subsemigroup S_c of S and $p \in S_c^{\dagger} = S^{\dagger} \cap \overline{S}_c$ with $S_0 \subseteq S_c$ and $\varphi(sp) \notin K(\varphi(S_c^{\dagger}))$ for all $s \in S_c$.*

Then there exists a unique homomorphism $\psi : S \to G$ such that $\varphi = \overline{\psi}|_{S^{\dagger}}$.

We note that:

(a) *if G is countable then* (i) *is always satisfied*;

(b) *if S is countable then* (ii) *need only be checked when $S_0 = S_c = S$.*

Proof. Let S_0 be any countable subset of S and let S_c be the countable subset of S guaranteed by hypothesis (ii). By Lemma 3.9, there is a countable subgroup G_c of G such that $\varphi(S_c^\dagger) \subseteq \overline{G_c}$. It follows from Lemma 5.5 that there is a unique homomorphism $\psi_c : S_c \to G$ whose continuous extension $\overline{\psi}_c : \overline{S_c} \to \beta G$ agrees with φ on $S_c^\dagger$. Since the union of any two such semigroups is contained in a third, and the union of all of them is S itself, we easily find a unique ψ defined on S whose extension $\overline{\psi}$ to βS agrees with φ on every $S_c^\dagger$, and so on $\bigcup_c S_c^\dagger = \omega S$. Since ωS is dense in βS, it follows that the two continuous maps $\overline{\psi}$ and φ agree on the whole of βS. Uniqueness is clear, and Theorem 5.6 is proved. □

Remark 5.7. There are two troublesome hypotheses in Theorem 5.6. The first is that $\varphi(S^\dagger)$ should contain elements of ωG. The other is the existence of p with $\varphi(sp) \notin K(\varphi(S_c^\dagger))$ for some semigroup S_c. The first is certainly necessary for φ to arise as the extension of a homomorphism ψ from S to G. To see this, simply observe that if $\psi : S \to G$ then $\psi(S^*)$ must contain cluster points of sets $\psi(A)$ with $A \subseteq S$ countable.

The second is necessary too, but needs a little more argument. For any homomorphism $\psi : S \to G$ for which $\psi(S)$ is infinite, we can find a countable subsemigroup S_0 for which the group G_0 generated by $\psi(S_0)$ is infinite (and countable). Suppose that for every $p \in S_0^\dagger$ there is $s \in S_0$ with $\psi(s)\overline{\psi}(p) = \overline{\psi}(sp) \in K(\overline{\psi}(S_0^\dagger))$. This implies that $\psi(s)\overline{\psi}(p)$ is not prime in βG_0 (Proposition 3.14) and so that $\overline{\psi}(p)$ is not prime. But $\overline{\psi}(S_0^*) = \psi(S_0)^*$ is a clopen subset of G_0^* and so contains prime elements (Proposition 3.12).

6. Corollaries and comments

We now consider some corollaries to the main theorem. First we look at the case in which S is also a group. We show that, if G and H are discrete groups, then every continuous surjective homomorphism from H^* to G^* is the extension of a surjective homomorphism from H to G. This is the only result in this section that requires no countability assumptions about G and which does not use Theorem 5.6.

Lemma 6.1. *Let S be a discrete semigroup. Suppose that $ux = vy$, where $x, y \in \beta S$ and $u, v \in \omega S$. Then there exists $s \in S$ such that $sx \in (\beta S)y$ or there exists $t \in S$ such that $ty \in (\beta S)x$.*

Proof. Let U and V be countable subsets of S such that $U \in u$ and $V \in v$. Since $ux \in \overline{Ux}$ and $vy \in \overline{Vy}$, it follows from Lemma 3.40 in [7] that $sx \in \overline{Vy} \subseteq (\beta S)y$ for some $s \in U$, or else $ty \in \overline{Ux} \subseteq (\beta S)x$ for some $t \in V$. □

Theorem 6.2. *Let H, G be discrete groups, and let $\varphi : H^* \to G^*$ be a continuous homomorphism for which $\varphi(H^*)$ contains an element of ωG which is prime in βG. (This is true in particular when $\varphi(H^*) = G^*$ (by Proposition 3.12).) Then $\varphi = \overline{\psi}|_{H^*}$ for a unique homomorphism $\psi : H \to G$ (and ψ is surjective if φ is surjective).*

Proof. Suppose that $p \in \omega G$ is prime in βG and that $\varphi(q) = p$ for some $q \in H^*$. For any $s \in H$ we have $\varphi(qs^{-1})\varphi(sq) = p^2$. It follows from Proposition 3.9 that $\varphi(qs^{-1})$ and $\varphi(sq)$ are in ωG. It then follows from Lemma 6.1 that $p = x\varphi(sq)$ for some $x \in \beta G$, or $\varphi(sq) = yp$ for some $y \in \beta G$.

Since p is prime, the first possibility implies that $x \in G$. We shall show that the second possibility implies that $y \in G$. To see this, observe that $\varphi(sq) = yp$ implies that $\varphi(qs^{-1})yp = p^2$ and hence that $\varphi(qs^{-1})y = p$ (by Proposition 3.14). Since p is prime, this implies that $y \in G$.

We can thus define $\psi : H \to G$ such that $\varphi(sq) = \psi(s)p$ for every $s \in H$. It follows from continuity that $\varphi(vq) = \varphi(v)p = \overline{\psi}(v)p$ for every $v \in \beta H$. By Proposition 3.14 we then have $\varphi(v) = \overline{\psi}(v)$ for every $v \in H^*$.

To see that ψ is a homomorphism, let $t \in H$. Then $\psi(t)p$ is also a prime element of βG and is also in ωG. By what we have already proved, with $\psi(t)p$ in place of p, there is a function $\psi_t : H \to G$ such that $\varphi(stq) = \psi_t(s)\varphi(tq) = \psi_t(s)\psi(t)p$ for every $s \in H$. We also have $\varphi(stq) = \psi(st)p$ and so $\psi(st) = \psi_t(s)\psi(t)$ [7, Corollary 8.2]. Since $\overline{\psi}_t = \overline{\psi} = \varphi$ on H^*, it follows from Corollaries 2.3 that $\psi_t(s) = \psi(s)$ for all but a finite number of values of s. So $\psi(st) = \psi(s)\psi(t)$ for all but a finite number of values of s. By continuity we have $\overline{\psi}(vt) = \overline{\psi}(v)\psi(t)$ for every $V \in H^*$ and every $t \in H$. If $t' \in H$, we can substitute vt' for v in this equation and deduce from Corollary 8.2 in [7] that $\psi(t't) = \psi(t')\psi(t)$.

To show that ψ is unique, suppose that $\psi' : H \to G$ is also a homomorphism for which $\overline{\psi'} = \varphi$ on H^*. For any $s \in H$, we have $\psi(sp) = \psi'(sp)$ and so $\psi(s)\psi(p) = \psi'(s)\psi(p)$. Thus $\psi(p) = \psi'(p)$, by Corollary 8.2 in [7].

Finally, we suppose that φ is surjective and deduce that ψ must be surjective. For every infinite subset B of G, we must have $\psi(H) \cap B \neq \emptyset$. Otherwise there would be an element in $\overline{B} \cap G^*$, but not in $\overline{\psi(H)} = \overline{\psi}(\beta H)$. So $G \setminus \psi(H)$ is finite. By the pigeon hole principle, for any $u \in G$, there exists $s \in \psi(H)$ such that $us^{-1} \in \psi(H)$. So $u \in \psi(H)$. □

Corollary 6.3. *Let G and H be discrete groups and let $\varphi : H^* \to G^*$ be a continuous homomorphism. If ϕ is not the extension of a homomorphism from H to G, then $\varphi(H^*)$ is nowhere dense in G^*.*

Proof. If $\varphi(H^*)$ contains a non-empty open subset of G^*, it contains a prime element of ωG (by Proposition 3.12). □

Our next theorem shows that, if G and H are discrete groups and G is countable, then any continuous injective homomorphism from H^* to G^* is the extension of a homomorphism from H to G.

Theorem 6.4. *Let G be a countable group and let S be a cancellative semigroup. Then every continuous injective homomorphism from S^* to G^* is the extension of a homomorphism from S to G.*

Proof. Let S_0 be a countable subsemigroup of S. We note that there exists $p \in S_0^* \setminus K(S_0^*)$ (by Corollary 6.33 in [7]). Now, if $s \in S_0$, then $sp \notin K(S_0^*)$. To see this, note that $sp \in K(S_0^*)$ implies that $sp = spq$ for some minimal idempotent q in S_0^* and hence that $p = pq \in K(S_0^*)$ (by Lemma 6.28 in [7]).

If $\varphi(S^*) \to G^*$ is a continuous injective homomorphism, then φ defines an isomorphism from S_0^* to $\varphi(S_0^*)$. Thus, for every $s \in S_0$, $\varphi(sp) \notin K(\varphi(S_0^*))$. It follows from Theorem 5.6, with $S^\dagger = S^*$, that φ is the extension of a homomorphism $\psi : S \to G$. □

Theorem 6.5. *Let G be a countable group and let S be a cancellative discrete semigroup. Then any continuous injective homomorphism from βS to βG is the extension of an injective homomorphism from S to G.*

Proof. Let $\varphi : \beta S \to \beta G$ be a continuous injective homomorphism. Exactly as in the proof of Theorem 6.4, with $S^\dagger = \beta S$, we can show that there is a homomorphism $\psi : S \to G$ such that $\varphi = \overline{\psi}$. It is obvious that ψ is injective. □

Now we consider even more special cases. For simplicity we shall suppose that G is countable—that saves us having to add hypothesis 5.6(i) every time. Our next result is an extension of the main theorem of [11] which says that any homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$ must have finite image. A more general conclusion was given in [7], but this was still restricted to homomorphisms with domains $\mathbb{N}^*$ and ranges in βG, where G was a countable group embedable in the circle.

For the proof we shall need the concept of the *topological center* $\Lambda(T)$ of a compact right topological semigroup T:

$$\Lambda(T) = \{t \in T : s \mapsto ts \text{ is continuous}\}.$$

If $\varphi : T_1 \to T_2$ is a continuous surjective map between compact right topological semigroups then $\varphi(\Lambda(T_1)) \subseteq \Lambda(T_2)$ [8, Surjectivity Lemma 2.3]. A discrete semigroup S is always in the topological center of βS.

Theorem 6.6. *Let G be a countable discrete group and let S be a countable discrete semigroup. Then, for any continuous homomorphism $\varphi : \beta S \to G^*$, $\varphi(S) \cap K(\varphi(\beta S)) \neq \emptyset$ and each maximal group in $K(\varphi(\beta S))$ is finite. Every element in $\varphi(S)$ has finite order.*

Proof. The image $\varphi(\beta S)$ is a compact semigroup with a dense topological center. We apply Lemma 5.5 with $S^\dagger = \beta S$. We take p to be any element of S. Then since φ does not arise from a map $\psi : S \to G$, there must be $s \in S$ with $\varphi(sp) \in K(\varphi(\beta S))$. Since $sp \in S$, $\varphi(sp)$ is in the topological center of $\varphi(\beta S)$. This implies that $\varphi(sp)K(\varphi(\beta S))\varphi(sp)$ is compact. Since it is a maximal group in the minimal ideal of $\varphi(\beta S)$, it is a compact group. But compact groups in F-spaces are finite (see, for example, [11, p. 66]).

Now consider $P = \{p, p^2, p^3, \ldots\}$. We can equally well apply what we have just established to the semigroup P and the homomorphism φ restricted to βP. We conclude that there is an element s of P with $\varphi(sp) \in K(\varphi(\beta P))$; that is, for some integer m we have

$\varphi(p)^m = \varphi(p^m) \in K(\beta P)$. We deduce from this that $\varphi(p)^m$ is in some finite group and so there is n for which $\varphi(p)^{mn}$ is the identity of that group. Thus $\varphi(p)$ has finite order. □

Corollary 6.7. *Let G be a countable group and let $\varphi:\beta\mathbb{N} \to G^*$ be a continuous homomorphism. Then $\varphi(\beta\mathbb{N})$ is finite and $\varphi(\mathbb{N}^*)$ is a finite group.*

Proof. This follows easily from Theorem 6.6. □

Corollary 6.8. *Let G be a countable group and let C be a compact subsemigroup of G^*. Then every element of $\Lambda(C)$ has finite order.*

Proof. Let $p \in \Lambda(C)$. The mapping $\varphi:\mathbb{N} \to G^*$ defined by $\varphi(n) = p^n$ extends to a continuous homomorphism from $\beta\mathbb{N}$ into G^*. It follows from Corollary 6.7 that $\varphi(\mathbb{N})$ is finite. □

Theorem 6.6 also simplifies when S is a group.

Corollary 6.9. *Let G be a countable group and let H be any group. If $\varphi:\beta H \to G^*$is a continuous homomorphism, $\varphi(\beta H)$ is a finite group.*

Proof. We first consider the case in which H is countable. By Theorem 6.6 $\varphi(a) \in K(\varphi(\beta H))$ for some $a \in H$. For every $x \in H$ we have $\varphi(x) = \varphi(a)\varphi(a^{-1}xa^{-1})\varphi(a)$. Our claim then follows from the fact that $\varphi(a)\varphi(\beta(H))\varphi(a)$ is a finite group.

In the general case, we note that $\varphi(H_0)$ is finite for every countable subgroup H_0 of H. It follows that there must be a finite subset F of G such that $\varphi(H_0) \subseteq F$ whenever $H_0 \subseteq H$ is countable. Otherwise, there would be a sequence $(H_n)_{n\in\mathbb{N}}$ of countable subsets of H for which $\varphi(\bigcup_{n\in\mathbb{N}} H_n)$ would be infinite. So $\varphi(H)$ is finite. □

The question of whether there exist elements of finite order (apart from idempotents of course) in any βG appears to be extremely difficult. Zelenuk has shown that, if G is a countable torsion free group, then G^* contains no non-trivial finite subgroups [13]. This was generalized in [9]. In this paper, the finite subgroups of G^* were characterized, where G denoted a countable group. These all have the form $G_0 p$, where G_0 is a finite subgroup of G and p is an idempotent in G^* which commutes with the elements of G_0. There are no known examples of torsion free groups G for which G^* contains elements of finite order which are not idempotent.

If an element $q \in G^*$ does have finite order, then the map $\mathbb{N} \to \beta G$ defined by $\psi(n) = q^n$ naturally extends to a homomorphism from $\beta\mathbb{N}$ to the finite semigroup generated by q.

Here is another corollary to Theorem 6.6.

Corollary 6.10. *Let G be a countable group. Let $(\mathbb{N}, \max)$ be the set $\mathbb{N}$ of integers with the* maximum *multiplication. Then there is no continuous homomorphism $\varphi:\beta\mathbb{N} \to \beta G$ which is injective on $\mathbb{N}$.*

Proof. By Theorem 6.6, $\varphi(n) \in K(\varphi(\beta\mathbb{N}))$ for some $n \in \mathbb{N}$. If $r \in \mathbb{N}$ and $r > n$, then $\varphi(r) = \varphi(n)\varphi(r)\varphi(n)$. Since $\varphi(n)\varphi(\beta\mathbb{N})\varphi(n)$ is finite, $\varphi(\beta\mathbb{N})$ must be finite. □

Definition 6.11. Let p and q be idempotents in a semigroup. We say that $p \leqslant q$ if $pq = qp = p$.

Corollary 6.12. *Let G be a countable group and let C be a compact subsemigroup of G^*. Then $\Lambda(C)$ cannot contain an infinite decreasing sequence of idempotents.*

Proof. If $(p_n)_{n\in\mathbb{N}}$ were an infinite decreasing sequence of idempotents in $\Lambda(C)$, the map $n \mapsto p_n$ would extend to a continuous homomorphism from $(\beta\mathbb{N}, \max)$ into G^*, contradicting Corollary 6.10. □

Corollary 6.13. *Suppose that G and H are countable groups and that $\varphi : H^* \to G^*$ is a continuous homomorphism which is not the extension of a homomorphism from H to G. Then, for every $x, y \in H^*$, $\varphi(xy) \in K(\varphi(H^*))$. In particular, every idempotent in $\varphi(H^*)$ is minimal in $\varphi(H^*)$.*

Proof. By Theorem 5.6, there exists $s \in H$ for which $\varphi(sy) \in K(\varphi(H^*))$. So $\varphi(xy) = \varphi(xs^{-1})\varphi(sy) \in K(\varphi(H^*))$.

Now let q be an idempotent in $\varphi(H^*)$. Then $\varphi^{-1}(\{q\})$ is a compact semigroup and therefore contains an idempotent p (by Theorem 2.5 in [7]). So $q = \varphi(pp) \in K(\varphi(H^*))$. □

Corollaries 6.7 and 6.8 tell us that when G is a discrete group, G^* cannot contain subsemigroups of certain kinds. There is another result of this type in the literature: for a certain restricted class of groups G, there cannot be an infinite compact right-zero subsemigroup in G^* [3, Theorem 8.4]. We have been unable to obtain this conclusion from Theorem 5.6.

Remark 6.14. There can be non-trivial homomorphisms $\varphi : S^* \to G^*$ which do not arise from homomorphisms from S to G, even when S is also a group. Consider the right-zero semigroup $Z = \{z_1, z_2\}$ with two elements. Define $\varphi : \mathbb{Z}^* \to Z$ by $\varphi(\mathbb{Z}^*_+) = \{z_1\}$, $\varphi(\mathbb{Z}^*_-) = \{z_2\}$. Then φ is a continuous homomorphism. It is easy to find copies of Z inside G^* for any infinite group G (for $K(G^*)$ contains infinite right-zero semigroups). However there is no extension of φ to $\beta\mathbb{Z}$ because the image of $\beta\mathbb{Z}$ must be infinite and any closed infinite set must have $2^{\mathfrak{c}}$ elements.

Of course, in this case, $\varphi(S^*) = K(\varphi(S^*))$.

This example also shows that the observation in Example 4.2(i) that any homomorphism from S^* to a finite group is generated by a homomorphism from S to the group, does not extend to finite semigroups.

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