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On the stability properties of linear dynamic time-varying unforced systems involving switches between parameterizations from topologic considerations via graph theory

M. De la Sen, A. Ibeas

Department of Electricity and Electronics, Faculty of Science and Technology, Instituto de Investigacion y Desarrollo de Procesos IIDP, University of the Basque Country Aptdo. 644-Bilbao, 48080-Bilbao, Spain

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Abstract

This paper deals with the stability of linear time-varying systems involving switches through time between different parameterizations of a dynamic linear time-varying system. Graph theory is used to describe the combinations of possible switches of the various sets of parameterizations which ensure the stability of the configurations. Each graph vertex is associated with a particular parameterization while edges (arcs) are associated with switches between the graphs (directed graphs or digraphs). An axiomatic framework is first established concerned with previously known stability results from systems theory related to the achievement of stability when switches between several parameterizations of a dynamic system take place. The axiomatic context is then used to obtain stability results mainly based on the topology of the links between the various configurations associated with a state-trajectory as well as on the nature of the vertices related to the stability of the various isolated parameterizations.

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1. Introduction

The stabilization of dynamic systems is a crucial research topic from several decades. The stabilization may be achieved or improved through different alternative conventional techniques like observer-based controller design, sliding-mode controller synthesis, optimal controllers, model-matching, pole-placement or adaptive stabilization, [9,3,27,2,15,12,1,29,14,33,23] which can also be combined into mixed techniques. On the other hand, there are in nature time-varying systems defined via switches through time in-between several configurations which may be time-variant or time-invariant. A serious inconvenience to analyze those systems by conventional techniques is that their properties depend on the basis chosen to represent the system in state–space, [9,15]. Examples are the use of descriptions consisting of linearized models around different equilibrium points which describe certain chemical processes or the configurations including switches between several estimation algorithms in parallel multiestimation schemes which set more efficiently the identification performances through time than conventional single estimation schemes,

E-mail address: wepdepam@lg.ehu.es (M. De la Sen).

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[9,3,27,2,15]. Those schemes are sometimes used in tandem configurations with associated adaptive controller parameterizations to improve the tracking performances in adaptive control compared to the use of single adaptive controller parameterizations, [3,27,2]. This manuscript is motivated by the well-known fact from Lyapunov stability theory [9,15,12,1], that switches between stable parameterizations of a dynamic linear system ensure the stability of the whole configuration if a minimum residence time at each parameterization is kept while very fast switches may lead to instability, [9,3,27,2,15]. In this paper, the stability of configurations consisting of a set of parameterizations with switching in-between them are discussed from topological considerations using standard graph theory and an "ad hoc" axiomatic framework constructed from logic considerations and a set of previous background of analytical results. On the other hand, it is well-known the fact that when dealing with very general classes of systems, expert systems formulated ad hoc for certain applications in a expert rule-based context of artificial intelligence, the clear formulation of an axiomatic context is crucial to then derive new results which should agree with the empirical observation or knowledge as well as with results obtained from conventional mathematical techniques, [24,22,30,7,11]. The paper is organized as follows. Section 2 is devoted to describe a simple background on graph theory (see, for instance, [21,20,13,8,6,31,26,32,16]) to be then used in Section 3 which describes the axiomatic context and basic definitions and assumptions about the configurations of switched parameterizations of a dynamic system built from elementary well-known results. In that formulation, auto-loops associated with graph vertices are only allowed if the vertices are terminal; i.e. if they are end vertices of walks with no later associated switches between the various system parameterizations. Section 4 is devoted to enounce and derive the main results about stability based on graph theory. Section 4 proposes some alternative formulations in formulating the graphs associated with the state-trajectories mainly concerned with the presence or not of auto-loops in the vertices associated with the parameterizations. Section 6 is devoted to discuss a case study concerned with parallel multiestimation-based adaptive control where switches in-between the various estimators and associate controller parameterizations are decided by a supervisor on the basis of the identification performance, [3,27,2,18]. Finally, conclusions end the paper. The formulation is, in particular, appropriate for describing multimodel and multiestimation schemes, [3,27,2,5,19,17] from a topological point of view playing a main attention to the interconnection topology among the various subsystems and the residence time at each node. On the other hand, it seems to be also promising for certain classes of hybrid subsystems whose topology is easily described by automata, [10], as well as for reliable control designs under a multiple controller, [4]. Furthermore, it is also interesting to promote certain nodes describing estimator–parameterized controller pairs under a certain residence time, especially for the case when a steady-state regime is reached then being subject to an infinite residence time and for finding particular patterns from data under certain different initial conditions and classes of external disturbances, [28,25].

2. Background on graph theory

A *graph* (*digraph* or *directed graph if oriented*) is a pair $G := (V, E)(G := (V, A))$ consisting of a finite set $V \neq \emptyset$ of *vertices* and a set of (two-element ordered pairs) subsets $e(a) := \{v_1, v_2\}$ called *edges* (*arcs* for a digraph), of vertices $v_1 \in V$ and $v_2 \in V$, for all $e \in E$ $(a \in A)$. Since V is a carrier set and E (or A) is a signature of operations with elements of V then G is also an algebra, [13,8,6,31,26,32]. It is said that v_1 and v_2 are incident with $e(a)$ and that v_1 and v_2 are adjacent (or neighbors) of each other if $e(a) := \{v_1, v_2\}$. If G is a digraph then the arcs are directed so that the associated pair of vertices are strictly ordered pairs. $G_1 := (V, E_1)$, with $E \neq E_1 \subset E$, is called a partial graph of G while $G_1' := (V_1, E_1')$, $E_1' \subset E//V_1$, with$E//V_1$ denoting the set of all edges in E which have both their vertices in V_1, is called a subgraph of G (similar definitions stand for a digraph as the subsequent definitions stand as well). The number of vertices is called the cardinal ϑ of V or the *degree* or *power* of the graph G. A *walk* in G is defined by

$$W := (e_1, e_2, \ldots, e_n) = (\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}),$$

with $e_i \in E$, $v_j \in V$ $(i = 1, 2, \ldots, n;\ j = 0, 1, \ldots, n)$ being a sequence of edges in the graph G whose *length* $L(W) := n$, with *start vertex* v_0 and *end or terminal vertex* v_n, is the number of edges. If e_i are pair-wise distinct then W is called a *trail*. If, in addition, the involved vertices are pair-wise distinct then the trail is a *path*. A closed trail with $n \geqslant 3$ for which v_j are pair-wise distinct (except $v_0 = v_n$) is called a *cycle*. *Composed trails* -or *paths* are (trail-path) walks where at least one edge or vertex is repeated. A *Hamiltonian cycle* is a cycle which contains each vertex exactly once. A *Hamiltonian graph* is that which contains (at least) a Hamiltonian cycle, [13,26,32]. If G is a digraph then the respective above definitions are specified as *directed walk*, *trail*, *path*, *cycle* etc. If the edges have an associate positive weight, we can also define weighted lengths $L_w(W) := \sum_{i=1}^{n} \omega_i$ of walks W where ω_i is the weight of the edge

e_i $(i = 1, 2, \ldots, n)$. Note that although the length $L(W)$ is a well-posed metric, the weighted length $L_w(W)$ is not a metric for all admissible weights and any walk W. An important concept stated "ad hoc" for the subsequent formulation proposed in this paper is that of a proper subwalk W'. A proper subwalk $W \neq W' \subset W$ is defined by convention as a strip containing an ordered proper subset of the edges (arcs) of W subject to the constraints:

- W' has no terminal vertex.
- $L(W') = \infty$ if and only if W has no terminal vertex.

Note that

- If W has no terminal vertex then $L(W') \leqslant \infty$ (since $L(W) < \infty \Rightarrow L(W') < \infty$ and $L(W) = \infty \Rightarrow L(W') \leqslant \infty$ with $L(W') = \infty$ if it both W' and W contain infinitely many edges/arcs).
- If W has a terminal vertex then $L(W') < \infty$ (since W' has no terminal vertex by definition).
- If $L(W') = \infty$ then W has no terminal vertex (since if W has a terminal vertex then $L(W) < \infty \Rightarrow L(W') < \infty$ since $W \neq W' \subset W$).

3. Axiomatic formalism and main assumptions and definitions

The problem considered through this paper is that of set of switched parameterizations of a dynamic system with those switches between the various parameterizations taking place through time. This problem occurs in nature as, for instance, when a set of parallel estimators with switching is used in estimation and adaptive control [3,27,2] or when the amount of unmodeled dynamics of a certain nominal model is changing through time because of the degree of plant filtering effects to the various input frequencies. For instance, the parasitic capacitor coupling effects with earth in amplifiers, which imply an effective increase in the system order, only take place for high frequencies, [29,14,33,23]. Analytic previous results on global stability (GS) of dynamic systems; i.e. stability in the sense that all the state-trajectories are bounded for any bounded initial conditions, and global asymptotic stability (GAS); i.e. GS with the state-trajectory converging to zero as time tends to infinity, are given below. These results are then used to settle the axiomatic framework to obtain topologically based stability results when a finite or infinite number of switches in-between several parameterizations of a system are involved, [9,3,27,2].

3.1. Background results on stability of switched parameterizations from conventional analysis techniques

Let $\mathbf{R}$ and $\mathbf{Z}$ be the sets of real and integer numbers, respectively. Assume an unforced dynamic system $\mathbf{D_s}$ of n-dimensional state $x(t)$, which takes through time parameterizations $P : [t_i, t_{i+1}) \cap \mathbf{R}_\mathbf{0}^+ \to \mathbf{P}$ with $\mathbf{R}_\mathbf{0}^+ := \mathbf{R}^+ \cup \{0\}$ (the set of nonnegative real numbers being $\mathbf{R}^+ := \{z \in \mathbf{R} : z > 0\}$) and $t_i \in \mathbf{S} := \{t_i \in \mathbf{R}_\mathbf{0}^+\}_0^\infty$ (for all $i \in \mathbf{Z}_\mathbf{0}^+$ with $\mathbf{Z}_\mathbf{0}^+ := \mathbf{Z}^+ \cup \{0\} = \{z \in \mathbf{Z} : z \geqslant 0\}$) being the strictly ordered sequence of switches such that $t_0 = 0 \in \mathbf{S}$. The set $\mathbf{P}$ consists of a finite or infinite number of distinct parameterizations of $\mathbf{D_s}$. Three major known features presented as rigorous results in the literature (see, for instance, [9,3,27,2,15,12]) applicable to configurations consisting of switches through time in-between parameterizations of (in general) linear time-varying systems are described below. The problem is restricted to a linear $\mathbf{D_s}$ in order to formulate it by guaranteeing the absence of finite escape times. Those results will be then used in settling partly the axioms to be then used to derive the (topologically based) stability results for the switches between parameterizations.

Fact 1. *A finite number of switches between stable parameterizations leads to a stable configuration. An infinite number of switches between stable parameterizations leads to a stable configuration if a minimum residence time constraint $T_{\text{res}} > 0$ at each one (which is proved to exist) is respected in-between each pair ofconsecutive switches or, in a more general context, for some given integer N_r $(1 \leqslant N_r < \bar{N}_r < \infty)$, it exists a minimum time-varying residence time $T_{\text{res}}(P(t_k t), t_k) \geqslant N_r T$, T being the sampling period, such that the subsequent stability constraint holds for a prescribed real constant $0 < \rho \leqslant 1$:*

$$\infty \geqslant t_{k+1} \geqslant \text{Min}(t = t_k + T_{\text{res}}[P(t_k, t), t_k) : \|x(t)\| \leqslant \rho \|x(t_k)\|, \rho \in (0, 1] \cap \mathbf{R}) \tag{1}$$

over a subset $\mathbf{S} \supseteq \mathbf{S_T} := \{t_k \in \mathbf{S} : \infty > \delta_r \geqslant |t_i - t_j|;\ |t_{i+1} - t_i| \geqslant T_{\mathrm{res}};\ \forall t_i, t_j \in \mathbf{S} \cap \mathbf{S_T},\ t_{i+1} \in \mathbf{S}\}$; *i.e. the sequence* $\mathbf{S_T}$ *is defined for switches from each current parameterization to a (stable) one and by taking into account all the previous residence times for switches in* $\mathbf{S}$ *and associate parameterizations, from the last switch at the subset* $\mathbf{S_T}$. *The residence time compatible with the stability constraint is only checked for time instants in* $\mathbf{S_T}$ *so that if* $\mathbf{S_T} \neq \mathbf{S}$ *then not all the switches belonging to* $\mathbf{S}$ *have to accomplish with the residence time constraint but only those belonging to* $\mathbf{S_T}$. $\mathbf{S} := \{t_i \in \mathbf{R_0^+}\}_1^k$ *might consist of a finite number of k switches and if* $\mathbf{S_T} \neq \mathbf{S}$ *is finite then its last element is a switching time instant to a stable parameterization. If* $\rho < 1$, *and each individual parameterizations is GAS (GS), then the system is GAS in the Lyapunov's sense since there are no finite escape times, since the system is linear,* $x(t)$ *is bounded and converges asymptotically to zero for a finite, since all parameterizations are GAS (GS), or infinite number of switches. If* $\rho = 1$ *then the system is (at least) GS since* $\|x(t_k)\| \leqslant \|x(0)\|$ *and* $\|x(t_k)\| \leqslant \|x(0)\| + K$ *for all* $t_k \in \mathbf{S}$ *and all* $t \in \mathbf{R_0^+}$, *some* $K \in \mathbf{R}^+$. *Since* T_{res} *exists some upper-bound* $T \geqslant T_{\mathrm{res}}$ *may be computed either from 'a priori' knowledge on the system or experimentally starting with an arbitrary small tentative value to be then increased with small positive increments until stability is guaranteed by keeping the state-trajectory norm below a prescribed upper-bound.*

Fact 2. *Stable configurations might be found provided that*

(a) *There is a non-empty subset* P_s *of (GAS or GS) stable parameterizations of cardinal at least unity of the set of parameterizations* P.
(b) *The residence time* $T_{\mathrm{res}}[P(t_k, t), t_k)$ *satisfies the constraint* (1) *for the dynamic system* $\mathbf{D_s}$ *parameterized at some member or at some non-empty subset of* P_s *for all* $t \in [t_k, t_{k+1})$, $\forall t_k \in \mathbf{S_T}$.

In other words, the so-called stability constraint $\|x(t)\| \leqslant \rho \|x(t_k)\|;\ 0 < \rho \leqslant 1$ should be tested on a subsequence $\mathbf{S_T}$ of the sequence of switches (such that the dynamic system is parameterized by a non-empty subset of possible stable parameterization) by respecting at such a parameterization a minimum residence time which depends, in general, of all the previous parameterizations and respective residence times since the above test was performed at the previous member of $\mathbf{S_T}$. At an intuitive level, it is required to have at least one stable parameterization available in the configuration of the dynamic system. Furthermore, if the number of switches is finite then the last one is to a stable parameterization and the stability constraint should be tested at GAS parameterizations in-between finite time intervals and a minimum residence time should be respected when the constraint is checked to the subsequent parameterization switching. If either ρ is unity or the stability constraint is guaranteed with (non-asymptotically or critically) GS parameterizations (so that ρ cannot be strictly less than unity) then the whole configuration of parameterizations and associated switches is GS.

Fact 3. *If all the parameterizations are unstable then the whole configuration might be still (critically) GS if the stability constraint holds for certain patterns in the topology of switches, equilibrium points and initial conditions if the stability constraint is respected,* [27,15].

Remark. (1) Facts 1–2 are established in terms of sufficiency-type conditions so that they are compatible with Fact 3.

(2) Fact 2 is a direct extension of Fact 1 and might be applied under the existence of (at least one) stable parameterizations of the dynamic system. It is only required that the stability constraint to be tested and accomplished at certain stable parameterizations after a finite number of switches have taken place involving either unstable or stable parameterizations where such a constraint has not been guaranteed. Since not all stable parameterizations are involved in testing the stability constraint, those involved in guaranteeing the associate test will be called marked parameterizations in the graph-based formulation context and axiomatic foundations used to describe the topological context of the configuration obtained from switches between parameterizations of the dynamic system.

The subsequent simple example illustrates the above Facts 1–3.

Example. Assume a linear (piecewise invariant) time-variant first-order system $\dot{x}(t) = \alpha(t)x(t);\ x(0) = x_0$ with $|x_0| < \infty$. Assume a sequence $\{t_i\}_0^\chi$, $0 \leqslant \chi \leqslant \infty$ of switches between configurations defined by $\alpha(t) = \alpha(t_i)$, $\forall t \in [t_i, t_{i+1})$ with

$$x(t_{i+1}) = \prod_{j=0}^{i} [\mathrm{e}^{\alpha(t_j)(t_{j+1}-t_j)}]x(t_0) = [\mathrm{e}^{\sum_{j=0}^{i}(\alpha(t_j)(t_{j+1}-t_j))}]x(t_0).$$

If $\chi < \infty$, then the system is GS (GAS) if $\alpha(t_x) \leqslant 0$ ($\alpha(t_\chi) < 0$) since $|x(t)| \leqslant K_1 < \infty$, $\forall t \geqslant 0$ (and, furthermore, $x(t) \to 0$ as $t \to \infty$ if $\alpha(t_x) < 0$). If $\chi = \infty$ (i.e. infinitely many switches take place) then the system is GS (GAS) if $\sum_{j=0}^{i} \alpha(t_j)(t_{j+1} - t_j) \leqslant K_2 < \infty$ for any integer $i \geqslant 0$ so that it suffices that $\alpha(t_{\ell_j}) \leqslant 0$ ($\alpha(t_{\ell_j}) < 0$) during a sufficiently large residence time interval $t_{\ell_{j+1}} - t_{\ell_j} \geqslant T_{\mathrm{res}}(\alpha(t_i); i = t_{\ell_{j-1}}, t_{\ell_{j-1}+1}, \ldots, t_{\ell_j})$, $|t_{\ell_{j+1}} - t_{\ell_j}| \leqslant \delta_r < \infty$ for $t_{\ell_{j+1}} = t_{\ell_j} + \kappa(\ell_j)$, some $\ell_j, \kappa(\ell_j) \in \mathbf{Z}^+$, with the configuration $\alpha(t) = \alpha(t_{\ell_j})$, $\forall t \in [t_{\ell_j}, t_{\ell_{j+1}})$ for all $t_i \in \{t_j\}_0^\infty$ and some (in the most general case proper) subsequence of switching time instants $t_{\ell_i} \in \{t_{\ell_j}\}_0^\infty \subset \{t_j\}_0^\infty$ such that ℓ_i takes values at a subset of $\mathbf{Z}_0^+$ as $i \in \mathbf{Z}_0^+$, where the stability constraint is guaranteed so that $|x(t_{\ell_{j+1}})| \leqslant \rho |x(t_{\ell_j})| \leqslant K_3 < \infty$ with $\rho \leqslant 1$ and $|x(t)| \leqslant K_3 < \infty$ for all $t \in \mathbf{R}_0^+$ and some positive finite real constant K_3. If $\rho < 1$ then $x(t)$ converges asymptotically to zero at the subsequence of switching instants where the stability constraint holds so that as a result $x(t)$ converges asymptotically to zero as time tends to infinity since no finite escape times can occur in-between such switching instants (since the system is linear). Note that the above stability properties agree with the general Facts 1–2. Now, assume that $\alpha(t) = \alpha_0 \mathrm{e}^{-\beta t}$ ($\alpha_0 > 0$, $\beta > 0$), $\forall t \in \mathbf{R}_0^+/\mathbf{R_1} := \{z \in \mathbf{R}_0^+ : z \notin \mathbf{R_1}\} \subset \mathbf{R}_0^+$ with $\mathbf{R_1}$ being a (perhaps) non-connected interval) of finite measure L_1 where $\alpha(t) = \alpha_1 > 0$, $\forall t \in \mathbf{R_1}$. All the time instants where the value of $\alpha(t)$ switches from α_1 or to α_1 are switching instants between parameterizations of the system. Then, the system is GS with $\alpha(t) > 0$, for all $t \in \mathbf{R}_0^+$ since $|x(t)| = |\mathrm{e}^{\int_0^t \alpha(\tau)\,\mathrm{d}\tau} x_0| \leqslant (\mathrm{e}^{\alpha_1 L_1} + \mathrm{e}^{\alpha_0/\beta})|x_0| < \infty$ for all $t \in \mathbf{R}_0^+$. This is a particular situation included in the general background Fact 3.

3.2. Graph theory problem setting up

Graph theory is now used to investigate the stability of a dynamic system possessing a set of parameterizations with mutual switches between them. For that purpose, the following topological formalism is first introduced. Consider a linear (in general time-varying) dynamic system $\mathbf{D_s}$ with ϑ (finite or infinitely many) distinct parameterizations $\mathbf{P} := \{p_0, p_1, \ldots, p_{\vartheta-1}\}$, [20,21], to which the graph $G := (V, E)$ of so-called power, degree or cardinal $\mathrm{Card}(V) = \vartheta$ being the number of vertices in $V := \{v_0, v_1, \ldots, v_{\vartheta-1}\}$, with $E := \{e_1, e_2, \ldots, e_\vartheta\}$ being its associate set of edges/(or arcs if G is a digraph). Each one of the ϑ vertices v_i in V represents a possible parameterization p_i of $\mathbf{D_s}$ while the switches from a parameterization to the next one is represented by an edge $e_{j_i}\{v_{j_{i-1}}, v_{j_i}\}$ for $v_{j_i} \in V$ with $j_i \in \{1, 2, \ldots, \vartheta\}$ (or by an arc if G is a digraph what means that a transition to the last previous parameterization is forbidden). A walk $W = (e_{j_1}, e_{j_2}, \ldots,) = (\{v_{j_0}, v_{j_1}\}, \{v_{j_1}, v_{j_2}\}, \ldots,)$ in G (in particular a trail, path, cycle etc.) with start vertex v_{j_0} in V is associated via a bijective mapping to a state-trajectory of $\mathbf{D_s}$ involving (possibly) parameterizations switches whose initial condition x_0 is in the parameterization $p_{j_0} \in \mathbf{P}$. A walk having repeated vertices is associate with repeated parameterizations which define a certain state-trajectory of the dynamic system. If the walk involves $(n + 1)$ vertices then the number of switches is n (finite) and the system remains at the last parameterization (which is reached in finite time) during a time interval of infinite duration which allows describing the concept of stability as a limit property with time tending to infinity. If there is no terminal vertex then the state-trajectory associated with W possess infinitely many switches. *The residence time at each parameterization is the weighted length of the edge/arc to the next parameterization with the only vertices possessing reflexive arcs* (*then of weighted length infinity*) *being the terminal ones*, if any, for some walk describing a state-trajectory of $\mathbf{D_s}$. Since there is a natural bijection between state-trajectories of the dynamic system $\mathbf{D_s}$ and walks in the graph, we might refer the stability properties of a state-trajectory of the dynamic system to related properties in the associated graph walks or individual vertices. Thus, the subsequent definitions are introduced.

Definition. (1) A vertex in the set V of vertices of G is asymptotically stable, critically stable (both being stable) or unstable if its associate parameterization in the dynamic system $\mathbf{D_s}$ is GAS, GS or unstable.

(2) A (so-called) reduced vertex describes a whole configuration satisfying Fact 3 being considered as a vertex for all purposes.

(3) A walk W_s in G is GS (GAS) if the associate state-trajectory $x_{\mathbf{D_s}}(x_0; 0, \infty)$ of $\mathbf{D_s}$, whose state $x(t) \in \mathbf{R}^n$ remains uniformly bounded for all $t \in \mathbf{R}^+$ provided that it is bounded at $t = 0$ (in addition, if it converges asymptotically to zero as time tends to infinity).

(4) The graph G is GS (GAS) if any walk with edges/arcs of G is GS (GAS).

Remark. (1) Note that any walk being a reflexive edge/arc around a critically stable or asymptotically stable vertex corresponds to a globally stable or globally asymptotically stable state-trajectory with a single parameterization with no switches for any bounded initial state. If $\mathbf{D_s}$ is linear time-invariant, so that with a single constant parameterization, then G has a single vertex which is asymptotically stable (critically stable) if all the eigenvalues of the dynamics matrix of $\mathbf{D_s}$ have strictly negative real parts (at least one of them has zero real part).

(2) The graph G has no walk constraints if any vertex is linked to all the remaining ones with an edge/arc so that the input and output degrees (indeg and outdeg) of each vertex are both equal to $\vartheta - 1$, which is infinity if the graph power is infinity. As a result, switches are allowed from a parameterization to any other existing parameterization in $\mathbf{D_s}$. A walk W in G describes a state-trajectory of $\mathbf{D_s}$ during a certain time (for stability analysis such a time should be infinity). Each vertex in the walk describes a parameterization of $\mathbf{D_s}$ which is reached at a switching time. The residence time at this parameterization and the switch to the next vertex (representing a new parameterization) is described by an incident output edge/arc whose accumulated weight equalizes such a time, the arc being reflexive if and only if the vertex is terminal. The state-trajectory solution at time instants, which are not switching times in $\mathbf{D_s}$, is always mapped into some edge/arc adjacent with two vertices of G describing the previous and next (if any) switching between parameterizations.

(3) Since the overall time associated with any state-trajectory is infinity, any walk $W := (e_{j_0}, e_{j_1}, \ldots, e_{j_m})$ of length $L(W) := m \leqslant \infty$, the associated graph G has an accumulated weighted length of residence times at each vertex $L_w(W) := \sum_{j_i \in Z_m}^{m} T_{j_i} = \infty$; $Z_m = \{0, 1, \ldots, m\}$ such that $T_i := t_{i+1} - t_i$; $t_i \in S := \{t_j\}_0^m$ $(0 \leqslant m \leqslant \infty)$. Any proper subwalk $W' \subset W$, associated with a subtrajectory of $\mathbf{D_s}$, has a finite accumulated weight (i.e. duration) $L_w(W') := \sum_{j_i \in Z'_m}^{m'} T_{j_i} < \infty$ if W possess a terminal vertex and $L_w(W') := \sum_{j_i \in Z'_m}^{m'} T_{j_i} \leqslant \infty$, otherwise with $m' < m \leqslant \infty$ and all residence times in the walk fulfilling $T_i > 0$. These considerations for the evolution through time of the state-trajectory or corresponding walk W in the associated graph are motivated by the fact that the residence time at any single parameterization is the weighted length of the edge/arc to the next parameterization.

(4) To investigate the stability, note that all the state-trajectories of $\mathbf{D_s}$ are, by nature, of infinite duration so that the associated walks W in G satisfy some of the subsequent properties:

- $L(W) := m < \infty$ and $L_w(W) = \infty$. Thus, W has $(m + 1)$ vertices, m edges/arcs, a finite number m of switches in-between parameterizations and a terminal vertex whose associate parameterization has an infinite time duration. Any proper subwalk $W' \subset W$ satisfies $L(W') < m < \infty$, $L_w(W') < \infty$ and always involves a finite number of switches in-between parameterizations with all the residence times being finite.
- $L(W) = \infty$ and $L_w(W) = \infty$. Thus, W has infinitely many vertices and edges/arcs, infinitely many switches in-between parameterizations and no terminal vertex. All the residence times are finite. Any proper subwalk $W' \subset W$ satisfies $L(W') \leqslant \infty$, $L_w(W') \leqslant \infty$ with $(L_w(W') < \infty \Leftrightarrow L(W') < \infty) \Leftrightarrow (L_w(W') = \infty \Leftrightarrow L(W') = \infty)$ so that it can involve a finite or infinite number of switches in-between parameterizations. Since $L(W) = \infty$ either one or more vertices are reached infinitely many times if $\mathrm{Card}(V) < \infty$ or, alternatively,$\deg(G) = \mathrm{Card}(V) = \infty$ (i.e. the dynamic system $\mathbf{D_s}$ has infinitely many parameterizations associated with the terminal vertex-free walk W).

(5) The finite (or infinite) set of switching times between all the set of adjacent vertices $v_{j_i}, v_{j_{i+1}} \in V$ of G defining a walk W is $\mathbf{S} := \{t_i\}_0^{\chi \leqslant \infty} = \{t_i \in \mathbf{R}_0^+ : t_{i+1} > t_i\}$ with the residence time at v_{j_i} being $T_i = t_{i+1} - t_i$.

Since only one vertex at each time is active at each walk, not all vertices should be tested for the stability constraint and (in general) not all the vertices of the graph are in any walk describing a state-trajectory, the following variables and functions are defined, $\forall v \in V$:

- s_v is the *stability notch* of the vertex v defined as follows: $s_v = 1, 0$ and -1 if and only if v is asymptotically stable, critically stable and unstable, respectively.
- $a_v : \mathbf{R}_0^+ \to \{0, 1\}$ is the (binary) *activity function* of the vertex v defined as $a_v(t) = 1$ if v is active in some walk W describing a state-trajectory of $\mathbf{D_s}$ and $a_v(t) = 0$, otherwise.
- $p_v : \mathbf{R}_0^+ \to \mathbf{P} \subset \mathbf{R}^{\mathbf{q}_v}$ is the (real $\mathbf{Z}^+ \ni q_v$-vector) *parameterization function* of $\mathbf{D_s}$ represented by the vertex v (q_v may depend on v in the general case). If $\mathbf{D_s}$ is piecewise time-invariant then $p_v \in \mathbf{R}^{\mathbf{q}_v}$ is a constant real vector.
- $m_v : \mathbf{R}_0^+ \to \{0, 1\}$ is the (binary) *marks function* of the (so-called *marked*) *vertex* v in the walk W at time t defined as $m_v(t) = 1$ if v is active and its associate parameterization in $\mathbf{D_s}$ fulfills the stability constraint tested from the last previous marked vertex, otherwise $m_v(t) = 0$ if v is not a marked vertex at time t.

- $h_v : \mathbf{R}_0^+ \to \{0,1\}^{\chi_{hv}(t)}$ is the (size dependent on time) *marks history* $\chi_{hv}(t)$*-tuple* ($\chi_{hv}(t) \leqslant \infty$) of the *marked vertex* v at time t defined as

$$h_v(t) := [h_v(t^-) \vdots m_v(t_{\delta v})] = [h(t_{(\delta-1)v}) \vdots m_v(t_{\delta v})] = [m_v(t_{0v}), m_v(t_{1v}), \ldots, m_v(t_{\delta v})],$$

  with $h_v(0^-) = \emptyset$, as then stated in an axiomatic context, for all time t verifying $\mathbf{S} \supset (\mathbf{S}^* \cap \mathbf{S_v}) \supset \mathbf{S_v^*} \ni t_{\delta v} \geqslant t (\in \mathbf{R}_0^+) > t_{(\delta-1)v}$ where $\mathbf{S_v}$ ($\mathbf{S_v^*}$) are the sets of switching instants towards parameterizations of $\mathbf{D_s}$ represented by the active (active and marked) vertex v of G; i.e. $t_i \in \mathbf{S_v} \Leftrightarrow a_v(t) = 1, \forall t \in [t_i, t_{i+1}), \forall t_{i+1} \in \mathbf{S}$; and

$$t_i \in \mathbf{S_v^*} \Leftrightarrow m_v(t) = 1, \;\; \forall t \in [t_i, t_{i+1}), \;\; \forall t_{i+1} \in \mathbf{S}.$$

  Note that $\mathbf{S_T}$ is introduced via imbedding from the background Facts 1–3. However, it is not required that $\mathbf{S}^* \equiv \mathbf{S_T}$; i.e. it is not requested to perform the stability constraint test at all stable vertices taking part of a walk associate with a state-trajectory of the dynamic system, although it is required that $\mathbf{S}^* \subset \mathbf{S_T}$.
- The state-trajectory strip of $\mathbf{D_s}$ on $[a,b) \cap \mathbf{R}_0^+$ is denoted as $x_{\mathbf{D_s}}(x_0; a, b)$ with $x_{\mathbf{D_s}}(x_0) := x_{\mathbf{D_s}}(x_0; 0, \infty)$ so that its state is $x(t) \in \mathbf{R^n}, \forall t \in \mathbf{R}_0^+$, with initial state $x(0) = x_0$.
- The bijective mapping from each state-trajectory of $\mathbf{D_s}$ with initial state x_0 at the parameterization $p_0 \in \mathbf{P}$ into its associated walk $W = (\{v_{j_0}, v_{j_1}\}, \{v_{j_1}, v_{j_2}\}, \ldots : v_{j_i} \in V;\, j_i \in \mathbf{Z}_0^+)$ is denoted by $x_W : x_{\mathbf{D_s}}(x_0) \to W$. The notation also extends to sub-trajectories by specifying the time interval as $x_W(0,a) : x_{\mathbf{D_s}}(x_0) \to W' \subset W$ or $x_W(a,b) : x_{\mathbf{D_s}}(x(a)) \to W' \subset W$.

Then, each vertex v in the graph G is characterized at time $t \in [t_v, t_i)$, with $t_v \in \mathbf{S_v}$, $t_i \in \mathbf{S}$ and $[t_v, t_i) \cap \mathbf{S} = \emptyset$, by an information six-tuple:

$$z_v(t) := (s_v, a_v(t), p_v(t), m_v(t), h_v(t) \vdots I(\mathbf{D_s})),$$

with $I(\mathbf{D_s})$ being either empty if the vertex v is inactive at time t or a pair with the vertex information associated with the dynamic system defined by

$I(\mathbf{D_s}) = (x(x_0, t), t_v)$ if $a_v(t) = 1$ (i.e. if the vertex is active at time since $t_v \in \mathbf{S_v}$ to the next switch (if any) $t_i \in \mathbf{S}$ or until time tends to infinity with $[t_v, t_i) \cap \mathbf{S} = \emptyset$); and

$$I(\mathbf{D_s}) = \emptyset \quad \text{if } a_v(t), =0.$$

If the information $z_v(t)$ is the partial one referred to a particular walk W then it is denoted by $z_{vW}(t)$. The marked history as well as the history and the stability functions contain information about if the active vertex is asymptotically or critically stable or unstable and about its history and marked history. If the vertex is active during a time interval then it takes part of the walk W. If it is active in the same W more than once (i.e. it is active during at least two non-connected time intervals) then the above six-tuple is updated as many times as necessary. The whole state-trajectory is reconstructed from the information part of W which contains the strictly ordered once active vertices with possible repetitions. We have in mind that the information about the residence time at an active vertex at each connected residence time interval in a walk W is the weighted length of the edge/arc to the next vertex in W, namely, $L_{wW}(\{v, v_i\}) = t_i - t_v \leqslant L_w(W)$ where v_i is the next vertex after the switching time t_v such that the edge/arc $\{v, v_i\}$ is a subwalk of W. Let $\mathrm{Suc}(v, W) = \lfloor v' \in V : \{v, v'\} \subset W \rfloor$ be the set of successors of v in V. Then, the accumulated weight of all edges/arcs in W being output-incident with vertex v is

$$L_{avwW}\left(\left\{v, \underset{v_i \in \mathrm{Suc}(v,W)}{v_i}\right\}\right) = \sum_{v_i \in \mathrm{Suc}(v,W)} L_{vwW}(\{v, v_i\}) = \sum_{t_{vi} \in \mathbf{S_{vW}};\, t_{j_i} \in \mathbf{S_{v'W}}} (t_{j_i} - t_{vi}) \leqslant L_w(W),$$

with $\mathbf{S_{vW}} \subset \mathbf{S_v} \subset \mathbf{S}$ being the strictly ordered set of switches to parameterizations associated with the vertex v in the walk W. If v is visited only once in W then the above expression reduces to $L_{avwW}(\{v, v_{i\, v_i \in \mathrm{Suc}(v,W)}\}) = L_{avw}(\{v, v_i\})$. To map appropriately the state-trajectory of the dynamic system into the walk W at transition times in-between vertices, define the accumulated time-scheduled weight of the walk W at time t as

$$L_{avwW(t)} := \sum_{t_{i+1}, t_i \in \mathbf{S_{W(t)}}} (t_{i+1} - t_i) + t - t_{\chi_W(t)} \leqslant L_w(W),$$

where W consists of χ_W $(\leqslant\infty)$ edges/arcs visiting (χ_W+1) vertices with $\mathbf{S_W} := \{t_i\}_0^{\chi_W}$ being a sequence of strictly ordered switching times, $\mathbf{S_{W(\mathit{t})}} := \{t_i\}_0^{\chi_W(t)}$ and $\chi_W \geqslant \chi_W(t) := \mathrm{Max}(i \in \mathbf{Z_0^+} : t \geqslant t_i \in \mathbf{S_W})$. Then, the accumulated time-scheduled weight of all edges/arcs in W being output-incident with vertex v is

$$L_{av\,wW(t)} := \sum_{t_i, t_{i+1} \in \mathbf{S_{vW(\mathit{t})}}}^{\chi_{vW}(t)} (t_{i+1} - t_i) + t - t_{\chi_v(t)},$$

with $\mathbf{S_{vW(\mathit{t})}} := \{t_i\}_0^{\chi_v(t)}$ being a strictly ordered sequence of switching times, $\mathbf{S_{vW(\mathit{t})}} := \{t_i\}_0^{\chi_{vW}(t)}$ towards vertex v in walk W, and $\chi \geqslant \chi_{vW} \geqslant \chi_{vW}(t) := \mathrm{Max}(i \in \mathbf{Z_0^+} : t \geqslant t_i \in \mathbf{S_{vW}})$ and χ_{vW} is the number of times that the vertex V is visited in walk W.

Each edge (“e”) or arc (“a”) output-incident with a vertex $v \in V$ has an associated history of weights (i.e. times of residence) for all time which can be obtained from the information six-tuple $z_v(t)$ which can be directly obtained from the set of vertices with non-empty history which is the strip $g_e(t)$, $g_a(t)$ defined by

$$g_{e(a)}(t) := (g_{e(a)}^{(1)}(t_{i_1}), g_{e(a)}^{(2)}(t_{i_2}) \ldots, g_{e(a)}^{(\ell(t))}(t_{i_\ell}))$$

(the subscript “e” or “a” is specified “ad hoc” for edge or arc) for all $t \geqslant t_i \in \mathbf{S_v}$ where $g_{e(a)}(t_i) := t_{i+1} - t_i > 0$ for $t_i \in \mathbf{S_v}$ and $t_{i+1} \in \mathbf{S}(\notin \mathbf{S_v})$ with $(t_i, t_{i+1}) \cap S = \emptyset$ and $g_{e(a)}(t_i) := t - t_i$ for $t_i \in \mathbf{S_v}$ if $[t_i, \infty) \cap \mathbf{S} = \emptyset$; i.e. there is no switching to a vertex input-incident with “e” (“a”) after t_i. If v is not output-incident with e/a or its first time it is output-incident is at $t' > t$ then $g_{e(a)}(t) = \emptyset$. If the definition refers to a particular walk W then an extra subscript W is added; i.e. $g_{eW(aW)}(t) = \emptyset$. As a result, note that for all walk W and $\forall t \in \mathbf{R_0^+}$ such that $a_v(t) = 1$ and $(t_i, t] \cap \mathbf{S_v} = \emptyset$, the constraints $L_w(W) \geqslant L_{av\omega W(t)} \geqslant \sum_{t \geqslant t_i \in \mathbf{S_v}} g_{e(a)}(t_i) + t - t_i$ hold.

3.3. Axiomatic framework

A set of axioms is now stated in order to then formulate the main results. The decomposition of V as the disjoint union of stable, critically stable and unstable vertices $V := V^s \cup V^{cs} \cup V^u$. In general, we will identify the overall set of distinct vertices as $V_0 := V^s \cup V^{cs} \cup V^u$. However, the cardinal of that set may be reduced by grouping together sets of walk including vertices and associate edges/arcs which satisfy Background Result 3, i.e. the associate subwalks are (critically) stable. These configurations are improper or reduced vertices and by convention the set V includes these configurations, if any, as improper vertices so that $V := V^s \cup V^{cs} \cup V^u \cup V^r$ is disjoint union where vertices in the set $V^s \cup V^{cs} \cup V^u$ taking part of V have been deleted from the original set of vertices V_0 and incorporated to V^r as (improper) reduced vertices. If there exists a non-empty set of (so-called) reduced vertices V^r leading to a stable configuration including the edges/arcs incident with them, include them, together with all their incident edges/arcs in the set V^{cs} of critically stable vertices; i.e. $s_{v_r} = 0, \forall v_r \in V^r$. All vertex $V \ni v \notin V^r$ is called a *proper vertex* (as opposite to *reduced vertex* in V^r). Let us denote by $T(e^{*-}, e^{*+})$ (respectively, $T(a^{*-}, a^{*+})$ the time interval in-between the output edge e^* (arc a^*) of the current stable marked vertex v^* and the stable previous marked vertex v^{*-} in a walk W which has to be at least equal to a minimum residence time T_{res} in order to guarantee the closed-loop stability of the state-trajectory $x_W(x_0, \mathbf{D_s})$.

Axioms. (1) A walk W is a unique strictly ordered sequence of edges/arcs associated with active vertices during consecutive time intervals. Also, $0 \in \mathbf{S}$ for any walk W, i.e. $t = 0$ is a switching time irrespective of the initial vertex of W.

(2) $t = 0 \Rightarrow h_v(0^-) = \emptyset, \forall v \in V$.

(3) A vertex (reduced vertex) v has a reflexive edge (or arc) for a walk W if and only if v is the terminal vertex (terminal reduced vertex) of W.

(4) $\forall t \in \mathbf{R_0^+}$, $\exists$ (a unique) $v^* \in V : a_{v^*}(t) = 1$ for each W.

(5) $\forall t \in \mathbf{R}^+$, $\exists$ (a unique) edge $e^* = \{v^*, v\}$ (or arc $a^* = \{v^*, v\}$) $g_{e^*(a^*)}(t) > 0$, $v, v^* \in V$. (From Axiom 3, for terminal vertices or terminal reduced vertices $v = v^*$ so that $e^* = \{v^*, v^*\}$ (or $a^* = \{v^*, v^*\}$)).

(6) For any walk W, $L_w(W) = \infty$ and $L_w(W_1) \leqslant \infty$ for any proper subwalk $W \neq W_1 \subset W$. If W has a terminal (proper or reduced) vertex then $L_w(W_1) < \infty$ for any proper subwalk $W \neq W_1 \subset W$.

(7) If $s_v = 0(1)$ then any walk $W = (W_1, v)$ with proper terminal vertex $v \in V$ is critically stable (globally asymptotically stable). If $s_{v_r} = 0$ then any walk $W = (W_1, v_r)$ with $v_r \in V^r$ is critically stable.

(8) If W is a walk with (non-necessarily distinct) non-terminal stable marked vertices $v^*, v'^* \in V^s$ after some finite time $t^* \in \mathbf{R}^+$ (i.e. $h_{v^*}(t) \neq \emptyset$ and $h_{v'^*}(t) \neq \emptyset$ for all $t \geqslant t^*$) then W, and thus $x_W(x_0, \mathbf{D_s})$, is GAS for all bounded x_0 provided that $f(h_v(t_i^*))$ satisfies the stability constraint (1) with $h_v(t_i^*) := [h_v(t_{i-1}^*) \dot{:} h_{ve(a)}(t_i^*)]$ for $t_{i+1}^* \geqslant t_i^* + T_{\mathrm{res}}$, $\forall t_i^*, t_{i+1}^* \in \mathbf{S}^* \cap [t^*, \infty)$ with some real $\rho \in (0, 1)$. W, and thus $x_W(x_0, \mathbf{D_s})$, is (at least) GS for all bounded x_0 provided that the stability constraint holds with $\rho = 1$. W, and thus $x_W(x_0, \mathbf{D_s})$, is GS for all bounded x_0 provided that the stability constraint holds with $\rho \in (0, 1)$ provided that $v^*, v'^* \in V^{cs} \cup V^r$.

(9) If either $h_v(t) = \emptyset, \forall v \in V^s \cup V^{cs} \cup V^r, \forall t \in \mathbf{R}^+$ or $V \neq V^s \cup V^{cs} \cup V^r = \emptyset$ then W, and thus $x_W(x_0, \mathbf{D_s})$, is unstable for all bounded x_0.

(10) Only vertices in $V^s \cup V^{cs} \cup V^r$ can possess a non-empty marked history as time tends to infinity.

Comments to the axioms. (1) Axiom 1 implies that for any state-trajectory of $\mathbf{D_s}$, the initial time is a switching time of certain (initial) parameterization and state represented by a start proper vertex (or trail associated with a reduced vertex) of the graph.

(2) Axiom 2 establishes that no vertex has a non-empty history on $(-\infty, 0)$.

(3) Axiom 3 states that reflexive edges/arcs are only incident with proper or reduced terminal vertices to reflect the two following facts:

- Any state-trajectory of $\mathbf{D_s}$ has an infinite time duration so that only (sub)trajectories of finite duration are possible prior to reach terminal vertices.
- The residence time intervals at ancestor vertices for the same walk are finite which are the weights of (non-reflexive) edges/arcs of all those ancestor vertices.

Note that this axiom might be replaced by using reflexive edges/arcs at all vertices whose weights are the respective residence times at the involved vertices. Under such a description, all the vertices in a walk have reflexive edges/arcs whole the remaining output (unweighted) edges/arcs are only used to construct the walk of the graph. This alternative description is not used since the problem information is not increased while the difficulties, for instance, derived from schematics graphics descriptions increase because of the (one per vertex) proliferation of reflexive edges/arcs.

(4) Axioms 4 and 5 mean that each walk W in G, associated via a bijective mapping with a state-trajectory of $\mathbf{D_s}$, might be constructed by successive aggregation of subwalks involving active vertices and edges/arcs and incident edges/arcs. In the case of walks with terminal vertices, a reflexive edge/arc finishes the walk with an infinity time duration.

(5) Axiom 6 establishes that the accumulated weight of any walk associated with a state-trajectory is infinite, because of its infinite associated residence time, and that any partial subtrajectory has a finite duration so that its associate subwalk has a finite accumulated weight provided that the whole walk ends at a terminal proper or reduced vertex. If W contains infinitely many switches then the proper subwalks can have either finite or infinite time lasting.

(6) Axiom 7 reflects the fact that state-trajectories with either asymptotically stable or critically stable final configurations are described by terminal critically stable or asymptotically stable vertices in the graph what may be extended to stable reduced vertices and associate configurations of $\mathbf{D_s}$.

(7) Axioms 8 and 10 establish that any stable/critically stable proper vertex or critically stable reduced vertex might possess a non-empty marked history as time tends to infinity. Note that although the start vertex is marked (Axiom 1), it has no marked history for $t \leqslant 0$ (Axiom 2) so that Axiom 10 is compatible and complementary with Axioms 1 and 2.

(8) Axiom 9 states that if there are unstable vertices and there is no marked history in a walk at any time then the trajectory is unstable.

(9) Axiom 10 also reflects the feature that reduced stable terminal vertices are critically stable (by nature) describing oscillatory behaviors in the dynamic system $\mathbf{D_s}$ either associated with configurations of combined unstable, stable and critically stable equilibrium points or switching processes in-between unstable configurations to the light of background fact 3.

4. Stability results

A set of stability results based on the topology of links between parameterizations and mutual switches based on the parallel graph problem description are now obtained. Basically, these results rely on the background results Facts 1–3 of Section 3, which are known from analytical tools for stability analysis in the presence of parameterization switches the literature, together with the axioms of Section 3 which were introduced in a natural way by using intuition as well as Facts 1–3. Basically, if switching stops in finite time, the whole configuration is globally stable (asymptotically stable) if the last parameterization is stable (asymptotically stable). If there are infinitely many switches in-between parameterizations, it is requested to have infinitely many stable (asymptotically stable) parameterizations (including possible repetitions in the same walk of parameterizations through time) where the system remains parameterized at least a minimum residence time. The associated (asymptotically stable or critically stable) vertices are "marked" since they satisfy the stability constraint compatible with the requested residence time imposed by the time interval from the last occurred mark. The critical parameterization and the sets of switches leading to stability according to Fact 3, even under individual instability conditions in the absence of switches, are addressed in a similar way. For that last situation, the concept of (critically stable) reduced (i.e. non-proper) vertices is used to include all the local configuration (parameterizations plus switches). Those vertices are added to the set of proper ones for stability discussion. Some of the results are, in particular, devoted to mixed configurations that include Eulerian, Hamiltonian and rotational cycles generated by the fact that the sequences of parameterizations are constrained according to those cyclic behaviors.

Claim 1. *Assume that the start active vertex of a walk W is in* $V/V^u = V^s \cup V^{cs} \cup V^r$. *Then,*

$$\mathbf{S}^* := \{t^*_{\ell_k}\}_{\ell_0}^{\ell_\chi} \subset \mathbf{S} := \{t_k\}_0^{\chi \leqslant \infty} \quad \textit{with } \mathbf{Z}_0^+ \ni \ell_k \leqslant k \in \mathbf{Z}_0^+$$
$$\Rightarrow (\mathrm{Card}(\mathbf{S})) = \chi + 1 \geqslant (\mathrm{Card}(\mathbf{S}^*)) = \ell_\chi + 1 - \ell_0 \geqslant 1$$

so that $\mathrm{Card}(\mathbf{S}^*) = 1$ *if and only if* $\mathbf{S}^* = \{0\}$ *and* $\mathrm{Card}(\mathbf{S}) < \infty \Rightarrow \mathrm{Card}(\mathbf{S}^*) < \infty$.

If the start active vertex is in V^u, so that $0 \notin \mathbf{S}^*$, then $\mathbf{S}^* = \emptyset$ and $\mathrm{Card}(\mathbf{S}^*) = 0$ for all walk W with no marked vertex.

Proof. W is unique and built with a sequence of edges/arcs of consecutive active vertices with $h_v(0^-) = \emptyset$ from Axioms 2–5. Since all the marked vertices are in $\mathbf{S}$ then $\mathbf{S}^* \subseteq \mathbf{S}$ and then $\infty \geqslant \chi + 1 = \mathrm{Card}(\mathbf{S}) \geqslant \mathrm{Card}(\mathbf{S}^*) \geqslant 1$ if $0 \in \mathbf{S} \cap \mathbf{S}^*$ (Axiom 1) $\Rightarrow$ $\mathrm{Card}(\mathbf{S}) \geqslant \mathrm{Card}(\mathbf{S}^*) \geqslant 1$ and the proof is complete. On the other hand, if $0 \notin \mathbf{S}^*$ then $\mathrm{Card}(\mathbf{S}^*) = 0$ for any walk W with no marked vertex from Axioms 1 and 2 since $h_v(0^-) = \emptyset$, $\forall t \in \mathbf{R}_0^+$. □

Claim 2. *If* $V = V^s = \{v\}$ *then* $\mathbf{D_s}$ *is GAS for any bounded initial condition and the only associated walk W is GAS. If either* $V = V^{cs} = \{v\}$ *or* $V = V^r = \{v\}$ *consists of a reduced (critically stable) vertex then* $\mathbf{D_s}$ *is GS for any bounded initial condition and the only associated walk W is GS.*

Proof. The only (trivial) switch occurs at $0 \in \mathbf{S} \cap \mathbf{S}^* = \{0\}$, from Axiom 1, since no edge/arc exists in G apart from the reflexive edge incident with v which is trivially terminal. Then the result follows directly. □

Claim 3. *Assume that* $V^s \cup V^{cs} = \emptyset$ *and that there is no reduced vertex. Then,* $\mathbf{D_s}$ *has no stable-state trajectory so that there is no stable walk in the associated graph.*

Proof. Since $V^s \cup V^{cs} \cup V^r = \emptyset$ then $h_v(t) = \emptyset$, $\forall t \in \mathbf{R}_0^+$ since $0 \notin \mathbf{S}^*$ (from Axioms 1 and 8 since there is no stable vertex). Then, any walk W has all vertices, the (unstable) start marked vertex included, in a set of proper unstable vertices since $V^r = \emptyset$. As a result W as well as the associated state-trajectory $x(x_0, \mathbf{D_s})$ of $\mathbf{D_s}$ are unstable from Axiom 9. □

Claim 4. *Assume that* $V^s \cup V^{cs} \neq \emptyset$ *and* $V^r = \emptyset$. *Then, a necessary and sufficient condition for a state-trajectory* $x(x_0, \mathbf{D_s})$ of $\mathbf{D_s}$ *to be stable that its associate walk* $W(x(x_0, \mathbf{D_s}))$ *satisfies any of the two conditions below*:

C1. $W(x(x_0, \mathbf{D_s}))$ *has a terminal vertex in* $V^s \cup V^{cs}$.
C2. $W(x(x_0, \mathbf{D_s}))$ *has (at least) one vertex in* $V^s \cup V^{cs}$ *with non-empty marked history* $h_v(t) \neq h_v(t - t_{s^*}) \Rightarrow (h_v(t') \neq \emptyset, \forall t' \geqslant t)$, $\forall t \geqslant \mathrm{Max}(t^*, t_s^*)$, *some finite* $t^*, t_{s^*} \in \mathbf{R}_0^+$.

A necessary and sufficient condition for a state-trajectory $x(x_0, \mathbf{D_s})$ of $\mathbf{D_s}$ to be asymptotically stable is that its associate walk $W(x(x_0, \mathbf{D_s}))$ has a terminal vertex in V^s provided that $L_w(W_1) < \infty$ for all $W \neq W_1 \subset W$, or infinitely many marks in elements of V^s as time tends to infinity if $L_w(W_1) = \infty$ for some proper subwalk W_1 with $W \neq W_1 \subset W$.

Proof. *Necessity*. Proceed by contradiction. Assume that $V^s \cup V^{cs} = \emptyset$, $V^r = \emptyset$ with a walk $W(x(x_0, \mathbf{D_s}))$ with a terminal vertex in V^u so that $L_w(W) = \infty$ and $L_w(W_1) < \infty$ for all $W \neq W_1 \subset W$ (Axiom 6) so that $L_w(W_1) \leqslant L_w(W) = \infty$, $\forall W \neq W_1 \subset W$ (Axiom 6). Then, the system is unstable from Axiom 9. Equivalently, if there is a terminal vertex in $V^s \cup V^{cs}$ (Condition C1) then the walk $W(x(x_0, \mathbf{D_s}))$ and its associate state-trajectory $x(x_0, \mathbf{D_s}))$ of $\mathbf{D_s}$ are stable. Now, assume that there is no terminal vertex. Then, $x(x_0, \mathbf{D_s}))$ might be stable only if $V^r \neq \emptyset$ (what is excluded by assumption) since $V^s \cup V^{cs} = \emptyset$ or if there is at least one vertex with non-empty marked history as time tends to infinity. This is impossible from Axioms 8 to 10 for any walk of infinite weighted length if $V^s \cup V^{cs} \cup V^r = \emptyset$ or if there is no vertex in $V^s \cup V^{cs}$ with non-empty marked history (Condition C2).

Sufficiency. It follows directly from Axiom 7.

The asymptotic stability referred to in the second part of the claim follows directly using Condition C2 with the restriction that the terminal vertex is in V^s provided that any proper subwalk is of finite weighted length or the walk W (not necessarily possessing an asymptotically stable terminal vertex) possess infinitely many marks in the marked history provided that some proper subwalk is of infinite weighted length from Axioms 6 and 7 since $\|x(t_k^*)\| \leqslant \rho \|x(t_{k-1}^*)\| \leqslant \rho^k \|x(0)\| \rightarrow 0$ as $k \rightarrow \infty$ with all the sequence $\{\|x(t_k^*)\|\}_{k=0}^{\infty}$ being bounded if $x(0)$ is bounded with $t_i^* = t_{i+k_i} \in \mathbf{S}^*$ ($t_0^* = t_0 \in \mathbf{S}^*$) for some sequence of integers $k_i \in \mathbf{Z}_\mathbf{0}^+$, $\infty \geqslant i \in \mathbf{Z}_\mathbf{0}^+$. Since $0 < T_{\text{res}} \leqslant |t_{i+1}^* - t_i| \leqslant \delta_r < \infty$ and $\mathbf{D_s}$ is linear no finite escape times can occur so that the state-trajectory is bounded for all time and it tends asymptotically to zero as time tends to infinity. The sufficiency part follows directly from Axiom 7. The proof has been completed. □

Claim 4 extends in a natural fashion for a G with all vertices being stable as follows:

Claim 5. *Assume that G is a graph with $V = V^s \cup V^{cs}$ and $V^r = \emptyset$. Then, a necessary and sufficient condition for a state-trajectory $x(x_0, \mathbf{D_s})$ of $\mathbf{D_s}$ to be stable is that its associate walk $W(x(x_0, \mathbf{D_s}))$ satisfies any of the two conditions below*:

C1. $W(x(x_0, \mathbf{D_s}))$ *has a terminal vertex.*

C2. $W(x(x_0, \mathbf{D_s}))$ *has (at least) one vertex with non-empty marked history* $h_v(t) \neq h_v(t - t_{s^*}) \Rightarrow (h_v(t') \neq \emptyset, \forall t' \geqslant t), \forall t \geqslant \text{Max}(t^*, t_s^*)$, *some finite* $t^*, t_{s^*} \in \mathbf{R}_\mathbf{0}^+$.

A necessary and sufficient condition for a state-trajectory $x(x_0, \mathbf{D_s})$ of $\mathbf{D_s}$ to be asymptotically stable is that its associate walk $W(x(x_0, \mathbf{D_s}))$ has a terminal vertex in V^s provided that $L_w(W_1) < \infty$ for all $W \neq W_1 \subset W$ or infinitely many marks in elements of V^s as time tends to infinity if $L_w(W_1) = \infty$ for some proper subwalk W_1 with $W \neq W_1 \subset W$. If either C1 or C2 is satisfied by any walk $W(x(x_0, \mathbf{D_s}))$ then all trajectories of $\mathbf{D_s}$ and their associate graphs are stable. If, furthermore, $V^{cs} = \emptyset$ then all trajectories and associate graphs are asymptotically stable under the same conditions.

Proof. It follows directly as that of Claim 4 using $V = V^s \cup V^{cs}$ and $V^r = \emptyset$ since the graph has no unstable vertex so that the required properties apply for all walk. □

Claims 4 and 5 extend now in a natural way to the presence of graphs with reduced (stable) vertices, but only (non-asymptotic) stability is guaranteed as follows since reduced vertices behave as critically stable proper ones. They also extend allowing switches if the graph is connected since it is always possible to reach (terminal or non-terminal with marked history) vertices in the non-empty set V/V^u even for initial conditions in V^u.

Claim 6. *Assume that $V = V^s \cup V^{cs} \cup V^r$. Then, a necessary and sufficient condition for any state-trajectory $x(x_0, \mathbf{D_s})$ of $\mathbf{D_s}$ to be stable is that its associate walk $W(x(x_0, \mathbf{D_s}))$ satisfies any of the two conditions below*:

C1. $W(x(x_0, \mathbf{D_s}))$ *has a terminal vertex.*

C2. *$W(x(x_0, \mathbf{D_s}))$ has (at least) one vertex with non-empty marked history $h_v(t) \neq h_v(t-t_{s*}) \Rightarrow (h_v(t') \neq \emptyset, \forall t' \geqslant t)$, $\forall t \geqslant \mathrm{Max}(t^*, t_s^*)$, some finite $t^*, t_{s*} \in \mathbf{R}_0^+$.*

Assume that G is a connected graph with $V \supset V/V^u = V^s \cup V^{cs} \cup V^r$ being a proper inclusion (i.e. $V^u \neq \emptyset$) so that $V^s \cup V^{cs} \cup V^r = \emptyset$ (i.e. the overall set of stable proper and reduced vertices is non-empty). Then, for any bounded initial condition of $\mathbf{D_s}$, it is possible to construct stable state-trajectories $(x(x_0, \mathbf{D_s}))$, involving at least one parameterization switch, with associate stable walk $W(x(x_0, \mathbf{D_s}))$ if any of the two conditions below:

C1. $W(x(x_0, \mathbf{D_s}))$ has a terminal vertex in V/V^u.
C2. $W(x(x_0, \mathbf{D_s}))$ has (at least) one vertex in V/V^u with non-empty marked history $h_v(t) \neq h_v(t-t_{s*}) \Rightarrow (h_v(t') \neq \emptyset, \forall t' \geqslant t)$, $\forall t \geqslant \mathrm{Max}(t^*, t_s^*)$, some finite $t^*, t_{s*} \in \mathbf{R}_0^+$.

Proof. It follows by directly extending that of Claim 5 and using the properties of connected graphs for the second part. In that second part, a switch at least is allowed for a trajectory starting in V^u since the graph G is connected and $\mathrm{Min}(\mathrm{Card}(V^u), \mathrm{Card}(V/V^u)) \geqslant 1$ since both V/V^u and V^u are non-empty. For any trajectory with start vertex in V/V^u, at least two switches are allowed to build a stable state-trajectory of $\mathbf{D_s}$ and associate stable walk $W(x(x_0, \mathbf{D_s}))$ of G if $\mathrm{Min}(\mathrm{Card}(V^u), \mathrm{Card}(V/V^u)) = 1$. □

The connection structure of the graph as well as the properties of its reachability matrix lead to the following necessary and sufficient condition of existence of stable state-trajectories and associated walks:

Claim 7. *Assume that the set of vertices V satisfies the constraint of proper inclusion $V \supset V^s \cup V^{cs} \cup V^r \neq \emptyset$ (i.e. $V^u \neq \emptyset$) and that the reachability matrix of G, $R(G)$, satisfies*

$$R(G) = \sum_{i=1}^{d(G)} [A(G)]^i = P_1 \text{ Block Diag } (A_1, A_2, \ldots, A_k) P_2,$$

with $k \geqslant 2$ where $d(G)$ is the diameter of G (i.e. the maximal distance between vertices), $A(G)$ is the adjacency matrix of G, $P_{1,2}$ are permutation matrices, and A_i $(i = 1, 2, \ldots, k)$ are some real matrices. Then, a necessary condition for the existence of stable state-trajectories of $\mathbf{D_s}$ and associate stable walks $W(x(x_0, \mathbf{D_s}))$ of G is that $\mathrm{Card}(V^s \cup V^{cs} \cup V^r) \geqslant k$. If $\mathrm{Card}(V^s \cup V^{cs} \cup V^r) = k_0 < k$ then there are at least $(k - k_0)$ with all their vertices in V^u where any bounded initial condition leads to a unstable state-trajectory of $\mathbf{D_s}$ and an associate unstable walk $W(x(x_0, \mathbf{D_s}))$ of G.

Proof. The reachability matrix possesses k cells implying the existence of k connected components of the non-connected (since $k > 1$) graph G. The existence of (at least) a stable terminal or a marked (proper or reduced) vertex per connected component G_i (a subgraph of G) for stability of state-trajectories and associated walks with any initial conditions is necessary, since they may correspond to parameterizations at unstable vertices, from Claim 6. □

A necessity- and sufficiency-type condition of stability for a non-connected graph is directly obtained from Claim 4 as follows:

Claim 8. *Assume that Claim 1 holds so that there are G_i $(i=1, 2, \ldots, k)$ connected components of G with $\mathrm{Card}(\bar{V}^s) \geqslant k \geqslant \mathrm{Card}(\bar{V}^{s_i}) \geqslant 1$ $(i = 1, 2, \ldots, k)$ with $V \supset \bar{V}^s = \bigcup_{1 \leqslant i \leqslant k} (\bar{V}^{s_i})$, $\bar{V}^s := V^s \cup V^{cs} \cup V^r$, $V \supset V^i \supset \bar{V}^{s_i} := V^{s_i} \cup V^{cs_i} \cup V^{r_i}$ $(i = 1, 2, \ldots, k)$ with all the set inclusions being proper and all the set unions being disjoint. Then, a sufficient condition of existence of stable state-trajectories of $\mathbf{D_s}$ and associate stable walks $W(x(x_0, \mathbf{D_s}))$ of G for initial conditions for parameterizations at any vertex is that walks in each connected component G_i $(i = 1, 2, \ldots, k)$ satisfies Claim 6.*

If $V^s \neq \emptyset$ and either Conditions C1 or C2 in Claim 6 stands by replacing V/V^u by *$V/(V^u \cup V^{cs} \cup V^r)$ then the stability of state-trajectories of $\mathbf{D_s}$, and associate stable walks $W(x(x_0, \mathbf{D_s}))$ of G, for initial conditions for parameterizations at any vertex is global asymptotic.*

The two following results are concerned with walks involving Eulerian cycles (i.e. walks with the same start and end vertex where edge/arc is not repeated by the same walk) or Hamiltonian cycles (i.e. walks with the same start and end vertex where each vertex is visited once, the start/end one excepted).

Claim 9. *Assume that G is connected and that the power of each $v \in V$ is $\mathrm{po}(v) = 2i_v$, some $i_v \in \mathbf{Z}^+$. Assume also that $\bar{V}^s = V^s \cup V^{cs} \cup V^r \neq \emptyset$ and consider a walk $W(x(x_0, \mathbf{D_s}))$ with (at least one) Eulerian cycles of the form $W = (W_c, W')$ with W_c being non-empty connected and containing all the Eulerian cycles of W and W' being connected if non-empty. Then, the following items hold*:

(i) *The walk $W(x(x_0, \mathbf{D_s}))$, and thus its associate state-trajectory of $\mathbf{D_s}$, is globally stable if and only if W' is non-empty with a terminal vertex in $\bar{V}^s$ or if W_c has a terminal vertex in $\bar{V}^s$ or a non-empty subset of $\bar{V}^s$ with infinitely many marks.*

(ii) *The walk $W(x(x_0, \mathbf{D_s}))$, and thus its associate state-trajectory of $\mathbf{D_s}$, is globally asymptotically stable if $V^s \neq \emptyset$ and Item* (i) *holds by replacing $\bar{V}^s$ with V^s.*

Proof. Since $\deg(v) = 2i_v$, G is Eulerian and the considered walk W is decomposed into two disjoint connected subwalks: W_c, that contains all the Eulerian cycles and possible trails and paths in-between them as well as the start vertex by construction, and W', with no cycles (since all of them are in W_c). The proof follows directly from Claim 6 since $L_w(W') = \infty \Leftrightarrow (W' \neq \emptyset \wedge L_w(W_c) < \infty)$ and $L_w(W_c) = \infty \Leftrightarrow (L(W') = \emptyset \Rightarrow W' = \emptyset)$ since both W_c and W' are connected.

Claim 10. *Assume that G is connected with $\mathrm{Card}(V) \geqslant n \geqslant 3$ and $V \supset \bar{V}^s = V^s \cup V^{cs} \cup V^r \neq \emptyset$, and that some of the constraints below holds*:

(a) $\deg(v) \geqslant \frac{n}{2}, \ \forall v \in V$.

(b) *For any two non-adjacent vertices v and v', $\deg(v) + \deg(v') \geqslant n$.*

Consider a walk $W(x(x_0, \mathbf{D_s}))$ with (at least one) Hamiltonian cycle of the form $W = (W_c, W')$ with W_c being non-empty connected and containing all the Hamiltonian cycles of W and W' being connected if non-empty. Then, the following items hold:

(i) *The walk $W(x(x_0, \mathbf{D_s}))$, and thus its associate state-trajectory of $\mathbf{D_s}$, is globally stable if and only if W' is non-empty with a terminal vertex in $\bar{V}^s$ or if W_c has a terminal vertex in $\bar{V}^s$ or a non-empty subset of $\bar{V}^s$ with infinitely many marks.*

(ii) *The walk $W(x(x_0, \mathbf{D_s}))$, and thus its associate state-trajectory of $\mathbf{D_s}$, is globally asymptotically stable if $V^s \neq \emptyset$ and Item* (i) *holds by replacing $\bar{V}^s$ with V^s.*

Proof. It follows directly from Claim 6 following a similar reasoning as that of the proof of Claim 9 since under any of the constraints (a) or (b) the graph is Hamiltonian so that it contains at least a Hamiltonian cycle. □

Claim 9 is directly extendable to ℓ-rotational k-cycle systems, [16], which satisfy less restrictive conditions than Eulerian/Hamiltonian cycles. Assume that the set of switches in-between parameterizations is disposed with walks being a set $\beta_k^{(\ell)} := (G, C_k^{(\ell)}) = ((V, E), C_k^{(\ell)})$ of ℓ-rotational k-cycles $B_k := (\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_0\})$ over the k-trail B of graph G. (i.e. $B := (v_0, v_1, \ldots, v_{k-1}) \Rightarrow (v_0+\ell, v_1+\ell, \ldots, v_{k-1}+\ell)$), with $V := (Z_{v-1} \cup \{\infty\}) \ni v_i$ (assuming $Z_{v-1} := \{z \in \mathbf{Z}_0^+ : z \leqslant v-1\}$ and the composition rule $\infty + 1 = \infty$) being the pair-wise distinct vertices and denote a k-cycle system. The subsequent result summarizes properties for existence, non-existence and design of 1-rotational k-cycle systems proved in [21,16,5] with the use of Skolem sequences (see [32,5]).

Lemma 1. *The following results hold*:

- *There exists a rotational cycle of G in $\beta_k(1)$ if V satisfies the congruence relation $v \equiv 3$ or $v \equiv 9 \pmod{24}$.*
- *There is no rotational cycle of G in $\beta_k^{(1)}$ with k even. If $v \equiv 1 \pmod{k}$ then there exists a rotational cycle of G in $\beta_k^{(1)}$ if and only if k is odd and composite.*

- *There exists a rotational cycle of G in* $\beta_k^{(1)}$ *if* $\mathrm{Card}(V) = 2kn + 1$ *with k odd and composite.*
- *There exists a rotational cycle of G in* $\beta_k^{(1)}$ *for any admissible pair* (i, k) *with k odd and* $i \in \{k+1, k+2, \ldots, 3k-1\} - \{2k+1\}$.

Claim 9 extends as follows to ℓ-rotational k-cycle systems by taking into account that terminal vertices (except infinity) do not appear in rotational k-cycles.

Claim 11. *Assume that the set of switches in-between parameterizations is disposed with walks* $W(x(x_0, \mathbf{D_s}))$ *being a set* $\beta_k^{(\ell)} := (G, C_k^{(\ell)}) = ((V, E), C_k^{(\ell)})$ *of* ℓ*-rotational k-cycles* $B_k := (\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_0\})$ *over the k-trail B of graph G. Assume also that the subset of vertices* $\bar{V}^s = V^s \cup V^{cs} \cup V^r \neq \emptyset$*. Then, the following items hold*:

(i) *The walk* $W(x(x_0, \mathbf{D_s}))$, *and thus its associate state-trajectory of* $\mathbf{D_s}$, *is globally stable if and only if there is a non-empty subset of* $\bar{V}^s$ *with infinitely many marks.*
(ii) *The walk* $W(x(x_0, \mathbf{D_s}))$, *and thus its associate state-trajectory of* $\mathbf{D_s}$, *is globally asymptotically stable if and only if* $V^s \neq \emptyset$ *and there is a non-empty subset of* V^s *with infinitely many marks.*

Claim 11 might easily be re-formulated in particular for 1-rotational systems by taking into account Lemma 1 with conditions of existence or non-existence for design of such cycles.

5. Simple formulation variations

Basically, only reflexive edges/arcs have been considered in the basic formulation of the previous sections. Thus, the binary relation between vertices is irreflexive (except, eventually, for some terminal vertices), it is not anti-symmetric and it is trivially transitive. Thus, all walk in the graph satisfies in a natural way, a strict order relation in-between vertices. If all (terminal and non-terminal) vertices in a walk have loops with weights being equal to the corresponding residence time at each associate parameterization and, furthermore, all switches to the preceding vertex are allowed, then the symmetric property holds so that the binary relation is an equivalence relation since it satisfies the reflexive, symmetric and transitive properties. In such a case edges/arcs incident with consecutive vertices are unweighted while only describing the transitions to each successor vertex (i.e. transitions in-between parameterizations themselves) since the residence times at each parameterization is always associated with an auto-loop for both terminal and non-terminal vertices.

A third problem description is as follows: the auto-loop vertices might be eliminated if the binary relation R is defined as $v_i R v_j := \{v_i, v_j\}$ if either $v_i = v_j \in V$ (i.e. for two identical vertices) or $e_{ji}/a_{ji} = \{v_i, v_j\}$ and the extended set of vertices $V_{\mathrm{ext}} := V \cup \{\infty\}$ is introduced with the infinity vertex describing the equilibrium. Then, a terminal vertex of the main formulation in Sections 3 and 4 will translate into a vertex with a weighted auto-loop associated with a residence time interval at a parameterization plus a linking edge/arc from such a vertex to the infinity vertex. All the remaining residence time intervals in a walk will be associated as well with auto-loops with weighted lengths at the corresponding vertices, the edges/arcs not being auto-loops only reflecting links to each successor vertex in the same walf. If transitions from all parameterizations to the preceding one are always allowed, the binary relation is still of equivalence type.

Note that the three descriptions are equivalent in the sense that they allow the description of the same problem with no ambiguity and with the same expected performances. Only minor modification in the Axioms and associate result are required to pass from any of the descriptions to the two remaining ones. We decided to formulate the basic theory with the absence of as many auto-loops as possible in order to be able, if desired, to represent graphically the graphs in the simpler possible way especially if there are many vertices most of them not being terminal. Fig. 1 figures out the philosophy of the proposed formalism. In the graph of five vertices, vertex 4 is terminal and it then possess an auto-loop with infinity weighted length and all the remaining weighted lengths are the residence times at the output vertex being incident with the corresponding edge while it is also an indicator of the successor vertex in the represented walk.

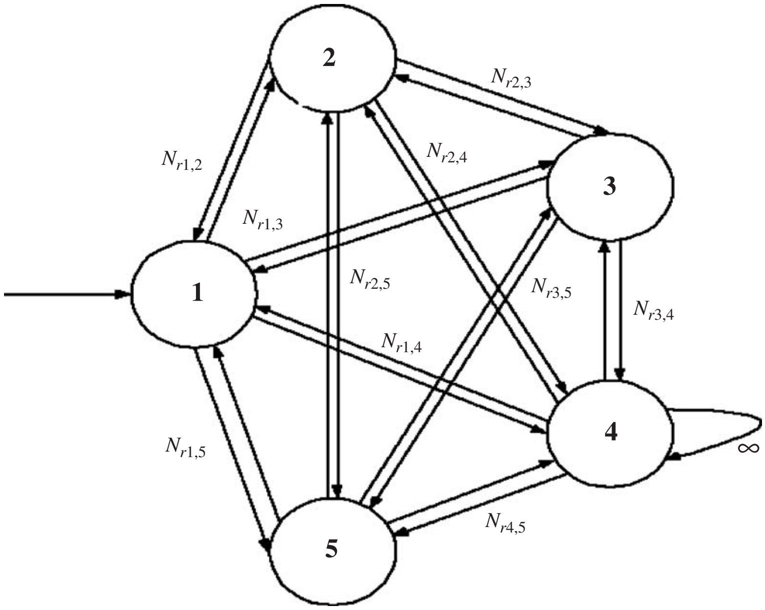

Fig. 1. Auto-loop with infinity weighted length only at terminal vertices.

6. A case study: a parallel multiestimation scheme for adaptive control

Parallel continuous and discrete multiestimation schemes have been successfully used to improve the identification performances compared to that obtained from the use of single estimators to parameterize the adaptive controller, [3,27,2,18]. In such a way, the transient tracking error performance becomes usually improved significantly compared with that obtained from the use of a single estimator to parameterize the adaptive controller. In the context of the proposed formulation, the constraints of the parallel multiestimation structure to parameterize the adaptive controller will be the following:

- All the single pairs of estimators/adaptive controller associate parameterizations are individually adaptive global asymptotic stabilizers if no switches in-between them are implemented. All the adaptive controller parameterizations are pole-placement-based and obtained from the corresponding polynomial diophantine equation whose right-hand side datum is the denominator polynomial of the reference model transfer function. This property has been proved from analytical studies, (see, for instance, [2,18]), under simple assumptions on the plant like all unstable zeros, if any, have to be transmitted to the reference model which has to be stable by obvious reasons. Furthermore, in the non-ideal case of noisy plant or presence of unmodeled dynamics, an absolute upper-bound of the contributions of those terms to the output should be known for all time to be used in the estimation scheme via a deadbeat zone to freeze the estimation when such a bound exceeds a prescribed time-varying threshold related to an absolute upper-bound of the adaptation error, [3,27,2] .
- All switches in-between the various estimators have to respect a minimum residence time of stay at each estimator/adaptive controller parameterization pair so that the closed-loop stability be guaranteed.
- The vertices of the associate graph describe each estimator/associate controller adaptive parameterization pair. All those vertices are stable since each single estimator/controller parameterization with all the remaining ones being switched off guarantees closed-loop stability. The switching process stops when the performance index of a estimator is always less than the remaining ones but this is not always guaranteed because of the uncertainties so that in the general setting up problem, it is not known a priori if terminal vertices are reachable or not in a certain walk.

If the second description is chosen then auto-loops at all vertices in a walk take place which take into account the residence times at each estimator/adaptive controller pair.

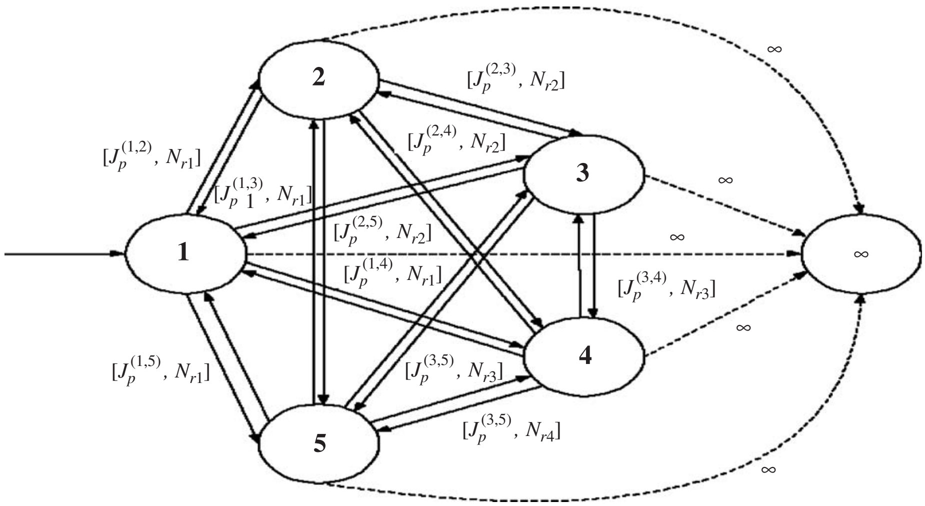

Fig. 2. No auto-loop and infinity vertex of the extended graph describing terminal vertices.

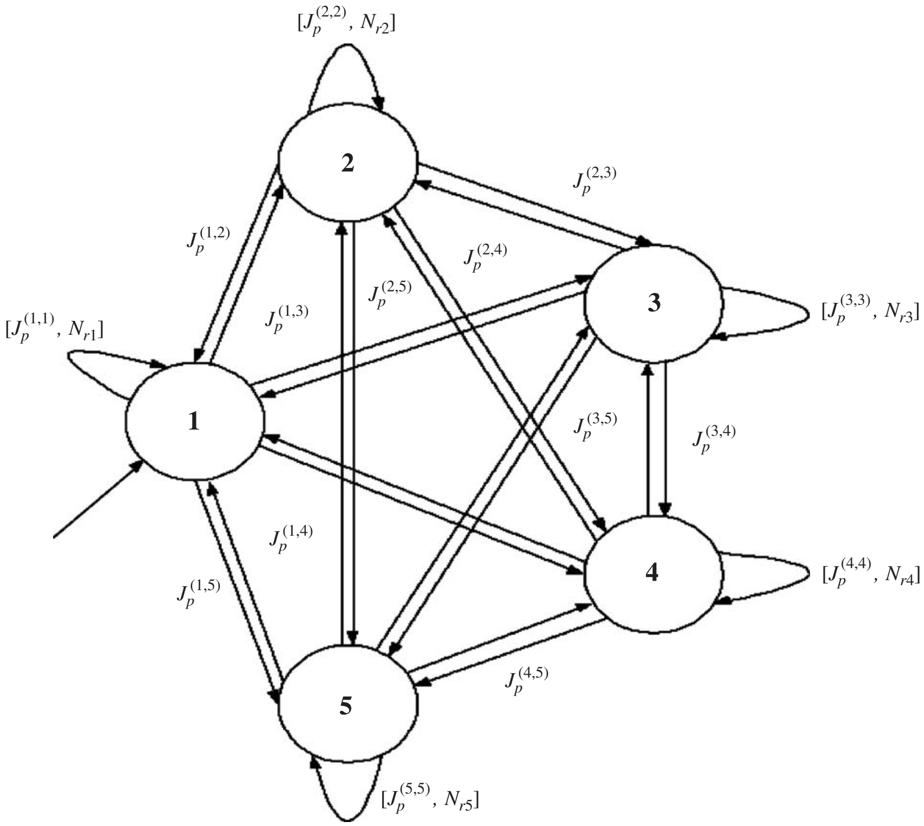

Fig. 3. Auto-loop for the residence time at all vertices.

Now, a discrete time invariant unknown plant is considered whose parameters are estimated via a multiestimation parallel scheme, each estimation scheme which parameterizes the adaptive controller. The chosen estimator/controller parameterization pair is chosen at each time by a supervisor which implements switches in-between the various pairs while respecting a minimum residence time for closed-loop stabilization purposes. Consider the following unstable

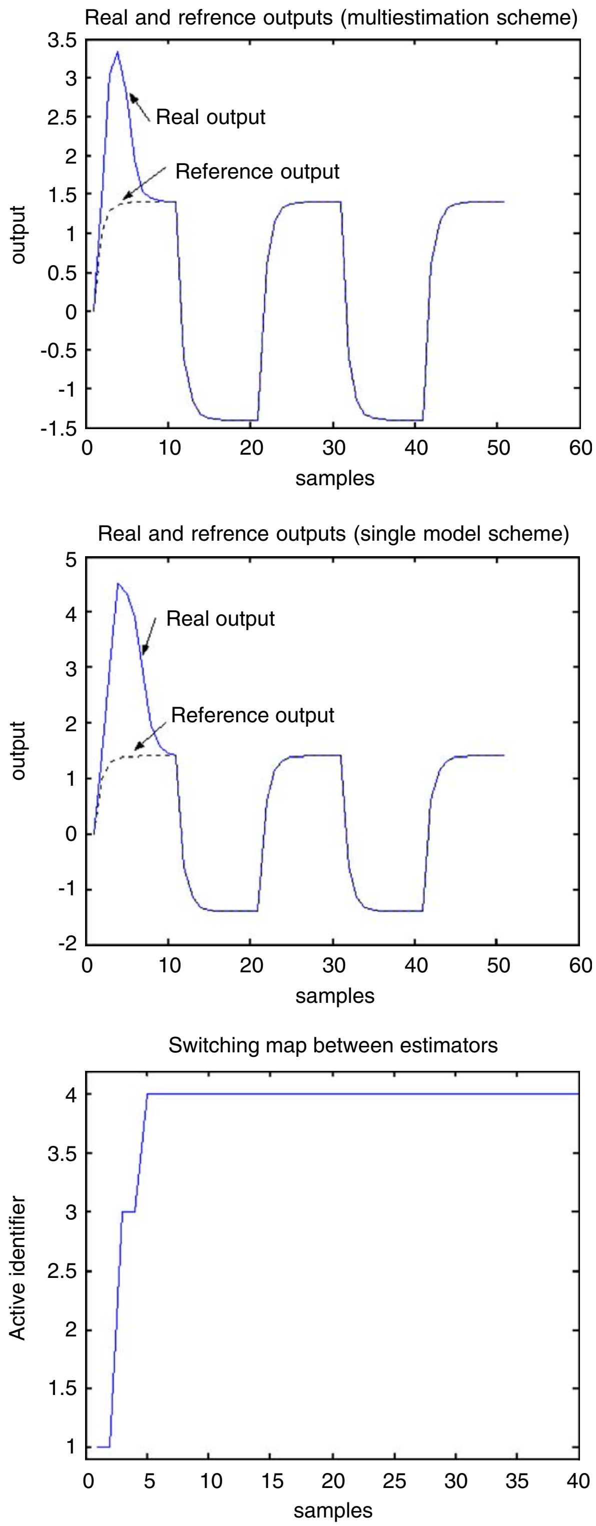

Fig. 4. Results for the parallel versus single multiestimation scheme for adaptive control.

and unknown for controller design plant and reference model specifying the suited closed-loop behavior

$$H(z)=\frac{z^2-0.6z+0.0875}{z^3-1.9z^2+0.73z-0.195}; \quad H_m(z)=\frac{z^2-0.32z+0.0255}{z^3-0.6z^2+0.11z+0.006}.$$

Five recursive estimators are used to run in a parallel multiestimation model for the unknown plant parameters which are all of least-squares-type with initial covariance matrix $P_0^{(i)}=10^5 I_{6\times 6};\ \forall i=1,2,\ldots,5$. The initial values of the estimates are

$$\hat{\theta}_0^{(1)\mathrm{T}}=[-0.85\ \ 0.2\ \ -0.1\ \ 0.7\ \ -0.35\ \ 0.075],$$

$$\hat{\theta}_0^{(2)\mathrm{T}}=[-1\ \ 0.4\ \ -0.4\ \ 0.9\ \ -0.45\ \ 0.084],$$

$$\hat{\theta}_0^{(3)\mathrm{T}}=[-1.5\ \ 0.6\ \ -0.3\ \ 1\ \ -0.55\ \ 0.086],$$

$$\hat{\theta}_0^{(4)\mathrm{T}}=[-2\ \ 0.8\ \ -0.2\ \ 1.2\ \ -0.65\ \ 0.088],$$

$$\hat{\theta}_0^{(5)\mathrm{T}}=[-2.5\ \ 1\ \ -0.15\ \ 1.5\ \ -0.75\ \ 0.088].$$

The minimum number of residence samples is $N_r=2$ which multiplied by the design sampling period gives the minimum residence time. The merit figure to decide switches in-between estimators is given by the identification performance index $J_p^{(i)}(k)=\sum_{\ell=1}^{k}\lambda^{k-\ell}(y_\ell-\hat{y}_\ell^{(i)})^2$ which involves a forgetting factor $\lambda=0.95$. The reference input is a square wave with amplitude ± 1 and a period of 20 samples. The first estimator which parameterizes the adaptive controller is the first one. The subsequent Fig. 4 show the plant and reference outputs for both the parallel multiestimation scheme and a single one based on the first estimator The switching map versus time is also shown which ends at the terminal vertex 4, associated with the fourth estimation/adaptive controller parameterization pair according to the formulation of Section 4 which is figured out via the walk in Fig. 3 or the corresponding alternative equivalent topological structures of Figs. 1 and 2. It is seen that the tracking performance of the parallel scheme becomes improved with respect to the single one (Fig. 4).

7. Conclusions

This paper has considered the problem of describing through graph theory the stability of topological configurations consisting of switches in-between several parameterizations of linear time-varying dynamic systems. This kind of situations exist in nature as well as in some practical computational problems like, for instance, that of the use of parallel multiestimation scheme to obtain a corresponding set of parallel adaptive controller parameterizations from which one is selected on-line which provides with the best identification performance. The selection of the estimation/adaptive controller parameterization pair is modified through time and used to generate the control signal to the plant input. Each of those parameterizations is associated with a vertex of a graph or digraph while edges/arcs linking vertices represent transitions associated with switches which, if weighted, contain information about the residence times at each parameterizations. A minimum residence time at certain stable parameterization is requested via tests performed at certain time intervals in order to keep the closed-loop stability of the whole parameterization. The stability results are obtained from a previous set of axioms which are built from several previous background results which were obtained from analytical stability studies (mainly based on Lyapunov theory) as well as from intuition dictates as well as from basic technical considerations for an appropriate setting up of the topological results. A case study concerning parallel multiestimation for pole-placement-based adaptive control has also been presented as an example of the developed formalism.

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