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Note

A Multidimensional Generalization of Slater's Inequality

JOSIP E. PEČARIĆ

Department of Mathematics and Physics, Faculty of Civil Engineering, University of Beograd, 11000 Beograd, Yugoslavia

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Slater [1] proved the following companion to Jensen's inequality for convex functions:

Suppose that f is convex and nondecreasing (nonincreasing) on (a, b). Then for $x_1,..., x_n \in (a, b)$, $p_1,..., p_n \geqslant 0$, $P_n = p_1 + \cdots + p_n > 0$, and $p_1 f'_+(x_1) + \cdots + p_n f'_+(x_n) \neq 0$, we have

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leqslant f\left(\sum_{i=1}^{n} p_i x_i f'_+(x_i) \Big/ \sum_{i=1}^{n} p_i f'_+(x_i) \right). \tag{1}$$

An integral analog of this result is also valid. Both results remain true if at any occurrence of $f'_+(x)$ we write instead any value in the interval $[f'_-(x), f'_+(x)]$.

The following simple generalization of this result was given in [2]:

Suppose that f is a convex function on (a, b). If for $x_1,..., x_n \in (a, b)$, $p_1,..., p_n \geqslant 0$, and $p_1 f'_+(x_1) + \cdots + p_n f'_+(x_n) \neq 0$, we have

$$\sum_{i=1}^{n} p_i x_i f'_+(x_i) \Big/ \sum_{i=1}^{n} p_i f'_+(x_i) \in (a, b),$$

then (1) is valid.

Note that a similar companion inequality to Jensen–Steffensen's inequality was also given in [2]. Some other inequalities, complementary to Jensen's inequality for convex functions, are given in [3 and 4] (see also [5 and 6]).

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All these results hold for convex functions of one variable. However, we shall show that a generalization of Slater's inequality to convex functions of several variables is also valid.

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If $x, y \in R^m$, say, $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$, then $\langle x, y \rangle = x_1 y_1 + \cdots + x_m y_m$. We shall say that a real function f is convex on an open set I ($I \subseteq R^m$) if the following inequality holds: $f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y)$, $\forall x, y \in I$ and $\forall \lambda \in [0, 1]$.

THEOREM. *Let $f: I \to R$ ($I \subseteq R^m$) be a convex function, and let $x_1, \ldots, x_n \in I$, $p_1, \ldots, p_n \geqslant 0$, $P_n > 0$. If $A \in I$ exists such that*

$$\left\langle A, \sum_{k=1}^{n} p_k f'_+(x_k) \right\rangle \geqslant \sum_{k=1}^{n} p_k \langle x_k, f'_+(x_k) \rangle, \tag{2}$$

where $f'_+(x) = (f'_{1+}(x), \ldots, f'_{m+}(x))$ and $f'_{1+}, \ldots, f'_{m+}$ are right partial derivatives of f, then

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leqslant f(A). \tag{3}$$

Proof. If f is convex on I, then

$$f(A) \geqslant f(x_k) + \langle A - x_k, f'_+(x_k) \rangle,$$

i.e.,

$$f(A) \geqslant f(x_k) + \langle A, f'_+(x_k) \rangle - \langle x_k, f'_+(x_k) \rangle$$

for $k = 1, \ldots, n$.

Multiply the kth inequality by p_k and add the inequalities thus obtained; we obtain

$$\begin{aligned} f(A) P_n &\geqslant \sum_{k=1}^{n} p_k f(x_k) + \left\langle A, \sum_{k=1}^{n} p_k f'_+(x_k) \right\rangle - \sum_{k=1}^{n} p_k \langle x_k, f'_+(x_k) \rangle \\ &\geqslant \sum_{k=1}^{n} p_k f(x_k) \end{aligned}$$

since (2) holds.

COROLLARY. *Let f, $x_1 = (x_{11}, \ldots, x_{1m}), \ldots, x_n = (x_{n1}, \ldots, x_{nm})$, $p_1, \ldots, p_n$ satisfy the conditions of the theorem. If f is also nondecreasing (nonin-*

creasing) in each of its m variables and if $p_1 f'_{j+}(x_1) + \cdots p_n f'_{j+}(x_n) \neq 0$ $(j = 1, \ldots, m)$, *then* (3) *is valid if*

$$A = (A_1, \ldots, A_m)$$
$$= \left(\frac{\sum_{k=1}^{n} p_k x_{k1} f'_{1+}(x_k)}{\sum_{k=1}^{n} p_k f'_{1+}(x_k)}, \ldots, \frac{\sum_{k=1}^{n} p_k x_{km} f'_{m+}(x_k)}{\sum_{k=1}^{n} p_k f'_{m+}(x_k)} \right).$$

Proof. Observe that $A \in I$ since A_j is a convex combination of $x_{1j}, \ldots, x_{mj}$. Since

$$\left\langle A, \sum_{k=1}^{n} p_k f'_+(x_k) \right\rangle = \sum_{k=1}^{n} p_k \langle A, f'_+(x_k) \rangle = \sum_{k=1}^{n} p_k \sum_{j=1}^{m} A_j f'_{j+}(x_k)$$
$$= \sum_{j=1}^{m} A_j \sum_{k=1}^{n} p_k f'_{j+}(x_k) = \sum_{j=1}^{m} \sum_{k=1}^{n} p_k x_{kj} f'_{j+}(x_k)$$
$$= \sum_{k=1}^{n} p_k \sum_{j=1}^{m} x_{kj} f'_{j+}(x_k) = \sum_{k=1}^{n} p_k \langle x_k, f'_+(x_k) \rangle,$$

the theorem implies the corollary.

Remark. One can prove the integral analogs of the above results (i.e., generalizations of inequality (4) of [1]).

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