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Functional limit theorems for generalized quadratic variations of Gaussian processes

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Abstract

In this paper, we establish functional convergence theorems for second order quadratic variations of Gaussian processes which admit a singularity function. First, we prove a functional almost sure convergence theorem, and a functional central limit theorem, for the process of second order quadratic variations, and we illustrate these results with the example of the fractional Brownian sheet (FBS). Second, we do the same study for the process of localized second order quadratic variations, and we apply the results to the multifractional Brownian motion (MBM).

MSC: 60F05; 60F17; 60G15; 60G18; 62G05

Keywords: Almost sure convergence; Central limit theorem; Fractional processes; Gaussian processes; Generalized quadratic variations

1. Introduction

The aim of this paper is to provide new results to estimate the regularity parameters of Gaussian processes. Usual examples are the fractional Gaussian processes (family of processes that generalizes the d-dimensional fractional Brownian motion). Since the papers [17,20], it is well known that second order quadratic variations yield strongly consistent and asymptotically normal estimators of the regularity parameters.

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The convergence of the sequence of second order quadratic variations has already been studied in [14]. Then, we have generalized it to the case of irregular subdivisions in [5]. In [4], we have proved an almost sure asymptotic development and a central limit theorem (CLT) for the sequence of second order quadratic variations. These papers are generalizations of known results for the fractional Brownian motion, which can be found for instance in [12]. The second order quadratic variations have been introduced in [7,20].

For the almost sure convergence of first order quadratic variations, a result has been shown in [3] and extended in [16,22] to a large class of Gaussian processes. In [25], the author has shown a functional CLT for the quadratic variations of the first order, with an application to time deformation. In [10,28], the authors have studied more general quadratic variations under the assumption that the processes are Gaussian and have stationary increments.

The localized quadratic variations have been introduced in [6] to construct estimators of the Hurst function of multifractional Gaussian processes. Then, these results have been generalized in [13,23] to localized quadratic variations of any order and of non-Gaussian processes.

In this paper, we consider a Gaussian process $X = \{X_t; t \in [0, 1]\}$. We denote by M its mean function and R its covariance function

$$\forall t \in [0, 1], \quad M_t = \mathbb{E}X_t, \quad \forall s, t \in [0, 1], \qquad R(s, t) = \mathbb{E}((X_t - M_t)(X_s - M_s)).$$

In the first part of this paper, we want to establish a functional almost sure convergence theorem, and a functional central limit theorem, for the *sequence of processes of second order quadratic variations* $(\{V_n(X)_t; t \in [0, 1]\})_{n\in\mathbb{N}}$, under suitable conditions on X. The *process of second order quadratic variations* $V_n(X) = \{V_n(X)_t; t \in [0, 1]\}$ is defined by

$$\forall t \in [0, 1], \quad V_n(X)_t = \sum_{k=1}^{[(n-1)t]} \left[X_{\frac{k+1}{n}} + X_{\frac{k-1}{n}} - 2X_{\frac{k}{n}}\right]^2, \tag{1}$$

where $[x]$ denotes the integer part of x: $[x] \in \mathbb{Z}$ and $[x] \le x < [x] + 1$. The trajectories of this process belong to $\mathcal{D}([0, 1])$ (the space of real functions defined on $[0, 1]$, which are right-continuous and admit limit from the left at each point) endowed with the Skorohod topology. We also consider the *process of linear interpolations* $v_n(X)$ (as in [25])

$$\begin{aligned} &\forall t \in [0, 1[, \quad v_n(X)_t = V_n(X)_t + ((n-1)t - [(n-1)t])(\Delta X^{(n)}_{[(n-1)t]+1})^2, \\ &v_n(X)_1 = V_n(X)_1, \end{aligned} \tag{2}$$

where, for $1 \le k \le n-1$, $\Delta X^{(n)}_k$ is the *second order increment of X at point k/n with mesh* $1/n$

$$\Delta X^{(n)}_k = X_{\frac{k+1}{n}} + X_{\frac{k-1}{n}} - 2X_{\frac{k}{n}}. \tag{3}$$

The trajectories of $v_n(X)$ belong to $\mathcal{C}([0, 1])$ (the space of continuous real functions defined on $[0, 1]$) endowed with the topology of uniform convergence. The aim of the first part of this paper is to study the convergence of the process $(V_n(X)_t)_{t\in[0,1]}$ (resp. $(v_n(X)_t)_{t\in[0,1]}$) in the space $\mathcal{D}([0, 1])$ (resp. $\mathcal{C}([0, 1])$) when $n \to +\infty$.

Next, we illustrate the result with the example of the fractional Brownian sheet (FBS). We explain why in [14] the authors have not obtained a CLT for all possible values of the Hurst indices of the FBS. Indeed, the limit is always a Gaussian law but the bias term may become infinite. Moreover, we correct an error done in [14], about the domain of possible values for the Hurst indices, in the case where the authors have obtained a classical CLT.

In the second part of this paper, we adapt the preceding results for the *sequence of processes of second order localized quadratic variations* $(V_n^{\mathrm{loc}}(X))_{n\in\mathbb{N}}$, under conditions on X. The *process of second order localized quadratic variations* $V_n^{\mathrm{loc}}(X) = \{V_n^{\mathrm{loc}}(X)_t;\, t \in]0,1[\}$ is defined by

$$\forall t \in]0,1[, \quad V_n^{\mathrm{loc}}(X)_t = \sum_{k\in\mathcal{V}_n^{\epsilon_n}(t)} \left[X_{\frac{k+1}{n}} + X_{\frac{k-1}{n}} - 2X_{\frac{k}{n}}\right]^2, \tag{4}$$

where $\mathcal{V}_n^{\epsilon_n}(t)$ is a neighbourhood of t defined by

$$\mathcal{V}_n^{\epsilon_n}(t) = \left\{k;\, 1 \le k \le n-1 \text{ and } \left|\frac{k}{n} - t\right| \le \epsilon_n\right\}, \tag{5}$$

and $\epsilon_n = f(n)$, with $f :]0,+\infty[\longmapsto]0,+\infty[$ such that $\lim_{x\to+\infty} f(x) = 0$.

Next, we apply these results to the multifractional Brownian motion (MBM), under the condition that its Hurst function is three times continuously differentiable. Please note that the functional limit theorem corrects an error made in [13]. Indeed, we show that the limit process for the CLT of localized quadratic variations of the MBM is a white noise (up to a deterministic multiplicative normalization), whereas in [13] the limit process is a continuous Gaussian process.

2. Notations

To prove the weak convergence of the processes $V_n(X)$ and $v_n(X)$, we use the same assumptions as those of Bégyn [4], Cohen et al. [14]. First, let us state some notations. We define the second order increments for the covariance function R, for $0 < h < 1$ and $h \le t \le 1-h$:

$$\delta_1^h R(s,t) = R(s+h,t) + R(s-h,t) - 2R(s,t),$$
$$\delta_2^h R(s,t) = R(s,t+h) + R(s,t-h) - 2R(s,t).$$

If $1 \le j,k \le n-1$, one sets

$$d_{jk}^{(n)} = \mathbb{E}(\Delta X_j^{(n)} \Delta X_k^{(n)}), \quad j,k = 1,\ldots,n-1,$$

and, whenever it is possible, we note d_{jk} for $d_{jk}^{(n)}$.

We define the following quantities that appear in the formulas for the asymptotic variance:

If $l \in \mathbb{Z}$ and $\gamma \in]0,1[\cup]1,2[$:

$$\rho_\gamma(l) = \frac{(|l-2|^{2-\gamma} - 4|l-1|^{2-\gamma} + 6|l|^{2-\gamma} - 4|l+1|^{2-\gamma} + |l+2|^{2-\gamma})}{(\gamma-2)(\gamma-1)\gamma(\gamma+1)}. \tag{6}$$

If $l \in \mathbb{Z}$ and $\gamma = 1$:

$$\rho_1(l) = \frac{1}{2}(|l-2|\log|l-2| - 4|l-1|\log|l-1| + 6|l|\log|l| \\ - 4|l+1|\log|l+1| + |l+2|\log|l+2|), \tag{7}$$

with the convention that $x\log x = 0$ if $x = 0$.

One can check that $\exists K > 0, \forall l \ge 2, |\rho_\gamma(l)| \le K l^{-2-\gamma}$. For $\gamma \in]0,2[$, one sets

$$\|\rho_\gamma\|^2 = \sum_{l=2}^{+\infty} \rho_\gamma(l)^2. \tag{8}$$

We use the following definition of a regularly varying function:

Definition 1. A Borelian function $\psi :]0, a[\longrightarrow \mathbb{R}$ ($a > 0$) is regularly varying of index $\beta \in \mathbb{R}$ if $\psi(h) = h^\beta L(h)$, where L is a slowly varying function

$$\forall \lambda > 0, \quad \lim_{x \to 0^+} \frac{L(\lambda x)}{L(x)} = 1.$$

The convergence in law will be noted with the symbol $\stackrel{(\mathcal{L})}{\longrightarrow}$ and, through all the paper, K will denote a generic positive constant, whose value does not matter.

3. Functional convergences of $V_n(X)$ and $v_n(X)$

Let us give a functional version of Theorem 5 in [4]: the weak convergence in $\mathcal{D}([0,1])$ of the process of second order quadratic variations $\{V_n(X)_t; t \in [0,1]\}$. Let us note that Theorem 5 in [4] yields the conclusion of Theorem 2 in the case $t = 1$.

For sake of completeness we recall the assumptions of Theorem 5 in [4].

Theorem 2. *Let X be a centred Gaussian process, satisfying:*

(1) *R is continuous on $[0,1]^2$;*
(2) *Let $T = \{0 \le t \le s \le 1\}$. We assume that the derivative $\frac{\partial^4 R}{\partial s^2 \partial t^2}$ exists on $]0,1]^2 \setminus \{s = t\}$, and that there exist a continuous function $C : T \mapsto \mathbb{R}$, a real $\gamma \in]0,2[$ and a positive slowly varying function $L :]0,1] \mapsto \mathbb{R}$ such that*
$$\forall (s,t) \in \overset{\circ}{T}, \quad \frac{(s-t)^{2+\gamma}}{L(s-t)} \frac{\partial^4 R}{\partial s^2 \partial t^2}(s,t) = C(s,t), \tag{9}$$
where $\overset{\circ}{T}$ denotes the interior of T, i.e. $\overset{\circ}{T} = \{0 < t < s < 1\}$.
(3) *We assume that there exist $q+1$ functions ($q \in \mathbb{N}$) $g_0, g_1, \ldots, g_q$ from $]0,1[$ to $\mathbb{R}$, q real numbers $0 < \nu_1 < \cdots < \nu_q$ and a function $\phi :]0,1[\mapsto]0,+\infty[$ such that*
   (a) *if $q \ge 1$ then $\forall 0 \le i \le q-1$, g_i is Lipschitz on $]0,1[$,*
   (b) *g_q is $1/2 + \epsilon_q$-Hölderian on $]0,1[$ with $0 < \epsilon_q \le 1/2$,*
   (c) *there exists $t \in]0,1[$ such that $g_0(t) \ne 0$;*
   (d) *one has*
$$\lim_{h \to 0^+} \frac{1}{\sqrt{h}} \left(\sup_{h \le t \le 1-h} \left| \frac{\left(\delta_1^h \circ \delta_2^h R\right)(t,t)}{h^{2-\gamma} L(h)} - g_0(t) - \sum_{i=1}^q g_i(t) \phi(h)^{\nu_i} \right| \right) = 0, \tag{10}$$
   where, if $q = 0$, then $\sum_{i=1}^q g_i(t)\phi(h)^{\nu_i} = 0$ and where, if $q \ne 0$, then $\lim_{h \to 0^+} \phi(h) = 0$;
   (e) *there exists a bounded function $\widetilde{g} :]0,1[\longrightarrow \mathbb{R}$ such that:*
$$\lim_{h \to 0^+} \sup_{h \le t \le 1-2h} \left| \frac{(\delta_1^h \circ \delta_2^h R)(t+h,t)}{h^{2-\gamma} L(h)} - \widetilde{g}(t) \right| = 0. \tag{11}$$

Then, one has almost surely and uniformly in $t \in [0,1]$:

$$\lim_{n \to +\infty} \frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)} V_n(X)_t = \int_0^t g_0(x) \mathrm{d}x. \tag{12}$$

Moreover, the process

$$\left\{\sqrt{n}\left(\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}V_n(X)_t - \int_0^t g_0(x)\mathrm{d}x - \sum_{i=1}^{q}\int_0^t g_i(x)\mathrm{d}x \cdot \phi\left(\frac{1}{n}\right)^{\nu_q}\right)\right\}_{t\in[0,1]}, \tag{13}$$

converges in law, when $n \to +\infty$*, in the space* $\mathcal{D}([0,1])$*, towards a Gaussian process* $Z = \{Z_t; t \in [0,1]\}$ *defined by*

$$\forall t \in [0,1], \quad Z_t = \int_0^t \sqrt{2g_0(x)^2 + 4\widetilde{g}(x)^2 + 4\|\rho_\gamma\|^2 C(x,x)^2}\mathrm{d}W_x, \tag{14}$$

and W *is the standard Brownian motion.*

Remark. (i) Since the functions g_0, $\widetilde{g}$ and C are bounded, the Kolmogorov criterion yields that the process Z takes its values in the space $\mathcal{C}([0,1])$. Moreover, let us note that Z has independent increments.

(ii) The assumptions are the same as in Theorems 3 and 5 in [4], except for the sequence of subdivisions and for the function ϕ. In [4], the assumptions are stronger because we wanted to show an almost sure asymptotic development.

(iii) Assumption (10) yields that the functions g_i, $0 \le i \le q$, are continuous on $]0,1[$.

Proof of theorem 2. To simplify notations, choose the convention $\nu_0 = 0$ and set for all $t \in [0,1]$

$$b_n(t) = \sum_{i=0}^{q}\int_0^t g_i(x)\mathrm{d}x \cdot \phi\left(\frac{1}{n}\right)^{\nu_i}, \tag{15}$$

$$T_n(t) = \sqrt{n}\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}V_n(X)_t, \tag{16}$$

$$\widetilde{T}_n(t) = T_n(t) - \mathbb{E}(T_n(t)). \tag{17}$$

Step 1. We start with the proof of (12). In the case $t = 1$, it is done in [4], and one can check that the same arguments hold for any $t \in [0,1]$. Then, uniformity in (12) is a simple consequence of Helly theorem (see [11] pp. 114–115).

Step 2. Next we prove (13).

We split the proof into three steps: first the convergence when $n \to +\infty$ of the finite-dimensional margins of the process $\widetilde{T}_n$ towards margins of Z, next the tightness of the family $(\widetilde{T}_n)_{n\in\mathbb{N}}$ in the space $\mathcal{D}([0,1])$, and last the convergence announced in (13).

Step 2.1. Let $0 \le s \le t \le 1$. One sets

$$\sigma_{s,t} = \int_s^t 2g_0(x)^2 + 4\widetilde{g}(x)^2 + 4\|\rho_\gamma\|^2 C(x,x)^2\mathrm{d}x. \tag{18}$$

One has

$$(\widetilde{T}_n(t) - \widetilde{T}_n(s)) \xrightarrow{(\mathcal{L})} \mathcal{N}(0,\sigma_{s,t}), \tag{19}$$

when $n \to +\infty$. Moreover,

$$\lim_{n\to+\infty}\mathrm{Var}(\widetilde{T}_n(t) - \widetilde{T}_n(s)) = \sigma_{s,t}. \tag{20}$$

In the proof of Theorem 5 in [4], one has shown (19) and (20) in the case $s = 0$ and $t = 1$ but one can check that the same arguments hold for any $0 \le s \le t \le 1$. So the proof of (19) and (20) is a straightforward consequence of the proof of Theorem 5 in [4].

Now we prove the convergence of the finite-dimensional margins of $\widetilde{T}_n$ towards margins of Z.

- First we show that, for all $t \in [0, 1]$, $\widetilde{T}_n(t)$ converges towards Z_t, when $n \to +\infty$. This is a consequence of (19) with $s = 0$.
- We show that, for all $s, t \in [0, 1]$, $(\widetilde{T}_n(s), \widetilde{T}_n(t))$ converges in law towards (Z_s, Z_t) when $n \to +\infty$. It is clear that one can assume that $s \le t$ without loss of generality. One considers the one-dimensional variable,

$$S_n(\lambda, \mu) = \frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}(\lambda V_n(X)_s + \mu V_n(X)_t),$$

where λ and μ are nonnegative real numbers.

First one has to study the asymptotic property of $\mathrm{Var}(S_n(\lambda, \mu))$. One has

$$\mathrm{Var}(S_n(\lambda, \mu)) = \frac{n^{2-2\gamma}}{L\left(\frac{1}{n}\right)^2}(\lambda^2 \mathrm{Var}(V_n(X)_s) + \mu^2 \mathrm{Var}(V_n(X)_t) + 2\lambda\mu \mathrm{Cov}(V_n(X)_s, V_n(X)_t)).$$

Therefore,

$$\begin{aligned}\mathrm{Var}(S_n(\lambda, \mu)) = \frac{1}{n}(&\lambda^2 \mathrm{Var}(\widetilde{T}_n(s)) + \mu^2 \mathrm{Var}(\widetilde{T}_n(t)) + 2\lambda\mu \mathrm{Var}(\widetilde{T}_n(s)) \\ &+ 2\lambda\mu \mathrm{Cov}(\widetilde{T}_n(s), \widetilde{T}_n(t) - \widetilde{T}_n(s))),\end{aligned} \tag{21}$$

However, one has

$$\mathrm{Var}(\widetilde{T}_n(t)) = \mathrm{Var}(\widetilde{T}_n(s)) + \mathrm{Var}(\widetilde{T}_n(t) - \widetilde{T}_n(s)) + 2\mathrm{Cov}(\widetilde{T}_n(s), \widetilde{T}_n(t) - \widetilde{T}_n(s)),$$

and with formula (20) it yields

$$\lim_{n \to +\infty} \mathrm{Cov}(\widetilde{T}_n(s), \widetilde{T}_n(t) - \widetilde{T}_n(s)) = 0. \tag{22}$$

Consequently, formulas (21) and (22) yield

$$\begin{aligned}\lim_{n \to +\infty} n\mathrm{Var}(S_n(\lambda, \mu)) &= \lambda^2 \mathrm{Var}(Z_s) + \mu^2 \mathrm{Var}(Z_t) + 2\lambda\mu \mathrm{Var}(Z_s) \\ &= \mathrm{Var}(\lambda Z_s + \mu Z_t).\end{aligned} \tag{23}$$

Next, one applies the Lindeberg CLT. For that, one considers $S_n(\lambda, \mu)$ as the Euclidean norm of the Gaussian vector $(G_i; 1 \le i \le [(n-1)t])$, with

$$G_i = \sqrt{\lambda + \mu}\sqrt{\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}}\Delta X_i^{(n)}, \quad 1 \le i \le [(n-1)s],$$

$$G_i = \sqrt{\mu}\sqrt{\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}}\Delta X_i^{(n)}, \quad [(n-1)s] + 1 \le i \le [(n-1)t].$$

Therefore, by the classical Cochran theorem, one can find a_n positive real numbers $(\tau_{1,n}, \ldots, \tau_{a_n,n})$ and one a_n-dimensional Gaussian vector ξ_n, such that its components are independent Gaussian variables $\mathcal{N}(0, 1)$ and

$$S_n(\lambda, \mu) = \sum_{i=1}^{a_n} \tau_{i,n} (\xi_n^{(j)})^2.$$

As in the proof of (30) in [5], one can check that

$$\tau_n^* = \max_{1 \le i \le a_n} \tau_{i,n} \overset{n \to +\infty}{=} \mathcal{O}\left(\frac{1}{n}\right).$$

Consequently, with (23), one gets

$$\lim_{n \to +\infty} \frac{\tau_n^*}{\sqrt{\mathrm{Var}(S_n(\lambda, \mu))}} = 0,$$

and so the Lindeberg CLT implies

$$\sqrt{n}(S_n(\lambda, \mu) - \mathbb{E}S_n(\lambda, \mu)) \xrightarrow{(\mathcal{L})} \lambda Z_s + \mu Z_t,$$

when $n \to +\infty$.

Moreover, let us note that

$$\lambda \widetilde{T}_n(s) + \mu \widetilde{T}_n(t) = \sqrt{n}(S_n(\lambda, \mu) - \mathbb{E}S_n(\lambda, \mu)).$$

Therefore,

$$\lambda \widetilde{T}_n(s) + \mu \widetilde{T}_n(t) \xrightarrow{(\mathcal{L})} \lambda Z_s + \mu Z_t,$$

when $n \to +\infty$.

Next, one can check that the arguments of the proof of Lemma 4.3 in [25] also hold for the second order quadratic variations. The Cramer–Wold theorem yields the convergence of the finite-dimensional margins of the process $\widetilde{T}_n$ towards those of Z.

Step 2.2. To prove the tightness of $(\widetilde{T}_n)_{n \in \mathbb{N}}$ in $\mathcal{D}([0, 1])$, one shows that the condition of Theorem 15.6 in [9] is satisfied, i.e. for all $0 \le t_1 \le t \le t_2 \le 1$:

$$\mathbb{E}(|\widetilde{T}_n(t) - \widetilde{T}_n(t_1)|^2 |\widetilde{T}_n(t_2) - \widetilde{T}_n(t)|^2) \le K|t_2 - t_1|^2, \tag{24}$$

for n large enough, where K is a generic positive constant which does not depend on n, t, t_1, t_2 (the value of K may change from one line to another).

Let us note that if $t_2 - t_1 < 1/n$ for n large enough, then one has either $[(n-1)t] = [(n-1)t_1]$ or $[(n-1)t] = [(n-1)t_2]$. So, in this case, one has either $T_n(t) = T_n(t_1)$ or $T_n(t) = T_n(t_2)$ and (24) is true for any positive constant K.

Now let us examine the case $t_2 - t_1 \ge 1/n$. By the Cauchy–Schwarz inequality, it is sufficient to prove that, for all $0 \le x \le y \le 1,\ y - x \ge 1/n$:

$$\mathbb{E}(|\widetilde{T}_n(y) - \widetilde{T}_n(x)|^4) \le K|y - x|^2. \tag{25}$$

One has

$$T_n(y) - T_n(x) = \sqrt{n} \frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)} \sum_{k=[(n-1)x]+1}^{[(n-1)y]} (\Delta X_k^{(n)})^2.$$

Therefore, by the Cochran theorem, one can find $a_n(x,y)$ positive real numbers $(\tau_{1,n}(x,y),\ldots,\tau_{a_n,n}(x,y))$ and one $a_n(x,y)$-dimensional Gaussian vector $\xi_n(x,y)$, such that its components are independent Gaussian variables $\mathcal{N}(0,1)$ and

$$T_n(y)-T_n(x)=\sqrt{n}\sum_{j=1}^{a_n(x,y)}\tau_{j,n}(x,y)\xi_n^{(j)}(x,y)^2,$$

which implies

$$\widetilde{T}_n(y)-\widetilde{T}_n(x)=\sqrt{n}\sum_{j=1}^{a_n(x,y)}\tau_{j,n}(x,y)(\xi_n^{(j)}(x,y)^2-1).$$

As in the proof of Lemma 4.4 in [25], one can check that it yields

$$\mathbb{E}\left(|\widetilde{T}_n(y)-\widetilde{T}_n(x)|^4\right)\le Kn^2\left(\sum_{j=1}^{a_n(x,y)}\tau_{j,n}(x,y)^2\right)^2.$$

Therefore,

$$\mathbb{E}(|\widetilde{T}_n(y)-\widetilde{T}_n(x)|^4)\le Kn^2a_n(x;y)^2\tau_n^*(x,y)^4,$$

where $\tau_n^*(x,y)=\max_{1\le j\le a_n}\tau_{j,n}(x,y)$.
The same proof as in Proposition 30 in [5] yields

$$\tau_n^*(x,y)\le K\frac{1}{n},$$

where K does not depend on (x,y). Moreover, $a_n(x,y)$ is less or equal to the dimension of the vector $(\Delta X_k^{(n)};[(n-1)x]+1\le k\le[(n-1)y])$, which yields

$$a_n(x,y)\le[(n-1)y]-[(n-1)x].$$

Consequently, one has

$$\mathbb{E}(|\widetilde{T}_n(y)-\widetilde{T}_n(x)|^4)\le K\frac{([(n-1)y]-[(n-1)x])^2}{n^2}.$$

Therefore,

$$\mathbb{E}(|\widetilde{T}_n(y)-\widetilde{T}_n(x)|^4)\le K\frac{((n-1)(y-x)+1)^2}{n^2}.$$

Since $y-x\ge 1/n$, (25) is satisfied. This proves the tightness of $(\widetilde{T}_n)_{n\in\mathbb{N}}$ in $\mathcal{D}([0,1])$.

Step 2.3. To prove Theorem 2, one uses the decomposition

$$T_n(t)-\sqrt{n}b_n(t)=\widetilde{T}_n(t)+\mathbb{E}T_n(t)-\sqrt{n}b_n(t). \tag{26}$$

One has

$$\lim_{n\to+\infty}\sup_{0\le t\le 1}\left|\mathbb{E}T_n(t)-\sqrt{n}b_n(t)\right|=0, \tag{27}$$

because one has shown (27) in the proofs of Theorems 3 and 5 in [4] in the case $t=1$ and one can check that the main inequalities in these proofs are uniform in t.

Next, combining Prokhorov theorem with Steps 2.1 and 2.2, and Slutzky lemma with (26) and (27), one gets (13). ■

It is clear that the process Z belongs to the space $\mathcal{C}([0,1])$. It suggests the natural question: is it possible to find a continuous form of the second order quadratic variations process $V_n(X)$, such that the convergence announced in (13) also holds in the space $\mathcal{C}([0,1])$? The answer is affirmative if one considers the process of linear interpolations $v_n(X)$. This is the subject of the next corollary:

Corollary 3. *Under the assumptions of Theorem 2 and the following additional assumptions:*

(1) *the paths of X are $(1-\gamma/2-\beta)$-Hölderian on $]0,1[$ for all $0<\beta<1-\gamma/2$,*

(2)

$$\lim_{n\to+\infty}\sqrt{n}L\left(\frac{1}{n}\right)=+\infty,$$

the process

$$\left\{\sqrt{n}\left(\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}v_n(X)_t-\int_0^t g_0(x)\mathrm{d}x-\sum_{i=1}^{q}\int_0^t g_i(x)\mathrm{d}x\cdot\phi\left(\frac{1}{n}\right)^{\nu_i}\right)\right\}_{t\in[0,1]}, \tag{28}$$

converges in law, when $n\to+\infty$, in the space $\mathcal{C}([0,1])$, towards the process Z defined by formula (14).

Remark. If there exists a constant $K>0$ such that

$$\forall(s,t)\in\overset{\circ}{T},\qquad\left|\frac{\partial^2 R}{\partial s\partial t}(s,t)\right|\le\frac{K}{(s-t)^{\gamma}},$$

then the Kolmogorov criterion yields that the paths of X are $(1-\gamma/2-\beta)$-Hölderian on $]0,1[$, for all $0<\beta<1-\gamma/2$.

Proof of corollary 3. We keep the notations of the proof of Theorem 2. One has

$$\begin{aligned}\sqrt{n}\left(\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}v_n(X)_t-b_n(t)\right)&=\sqrt{n}\left(\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}V_n(X)_t-\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}\mathbb{E}(V_n(X)_t)\right)\\&\quad+\mathbb{E}T_n(t)-\sqrt{n}b_n(t)\\&\quad+((n-1)t-[(n-1)t])\sqrt{n}\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}(\Delta X^{(n)}_{[(n-1)t]+1})^2.\end{aligned} \tag{29}$$

In steps 2.1 and 2.2 of the proof of Theorem 2, one has shown that the first term of the right-hand side of (29) converges towards the process Z when $n\to+\infty$, in $\mathcal{D}([0,1])$. Limit (27) implies that the second term of the right-hand side of (29) converges towards 0 when $n\to+\infty$, uniformly in $t\in[0,1]$. For the third term, one has $\sup_{t\in[0,1]}((n-1)t-[(n-1)t])\le 1$. The assumption on the Hölder regularity of the paths of X yields that this term converges almost surely towards 0 when $n\to+\infty$, uniformly in $t\in[0,1]$.

Consequently, the process

$$\left\{\sqrt{n}\left(\frac{n^{1-\gamma}}{L\left(\frac{1}{n}\right)}v_n(X)_t-b_n(t)\right);t\in[0,1]\right\},$$

converges towards the process Z in $\mathcal{D}([0,1])$, when $n \to +\infty$. Since the considered processes have continuous paths, the convergence also holds in the space $\mathcal{C}([0,1])$. ■

3.1. Case of processes with stationary increments

Under the assumptions of Theorem 2, and the additional assumption that X has stationary increments, one can check that the functions g_0, $\widetilde{g}$ and $t \mapsto C(t,t)$ are constant. Therefore, in this case, the process Z defined in (14) is equal to the standard Brownian motion up to a multiplicative constant.

If X is the standard fractional Brownian motion with Hurst index $H \in]0,1[$ i.e.

$$\forall s,t \in \mathbb{R}, \quad R(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}), \tag{30}$$

one gets

$$\forall t \in [0,1], \quad Z_t = \sigma_{\mathrm{FBM},H} W_t,$$

with

$$\begin{aligned}\sigma^2_{\mathrm{FBM},H} &= 2(4 - 2^{2H})^2 + (2^{2H+2} - 7 - 3^{2H})^2 \\ &\quad + (2H)^2(2H-1)^2(2H-2)^2(2H-3)^2\|\rho_{2-2H}\|^2,\end{aligned} \tag{31}$$

as it has been defined in [4] p.26.

3.2. Application to the fractional Brownian sheet

For $H_1, H_2 \in]0,1[$, $H_1 \le H_2$, one considers the two-dimensional standard fractional Brownian sheet of index (H_1, H_2), denoted by $S^{H_1,H_2} = \{S_t^{H_1,H_2}, t \in \mathbb{R}^2\}$ and defined as the unique centred Gaussian process with covariance function

$$\begin{aligned}\forall u,v \in \mathbb{R}^2, \quad \mathrm{Cov}(S_u^{H_1,H_2}, S_v^{H_1,H_2}) &= \frac{1}{4}(|u_1|^{2H_1} + |v_1|^{2H_1} - |u_1 - v_1|^{2H_1}) \\ &\quad \times (|u_2|^{2H_2} + |v_2|^{2H_2} - |u_2 - v_2|^{2H_2}),\end{aligned} \tag{32}$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. This field has been introduced in [21] and studied in [2, 15].

In [14], the authors have presented estimators of H_1, H_2 and of the field axes, constructed from one realization. For that, they consider the restriction of S^{H_1,H_2} along a radial segment $[AB]$ with length $L > 0$. The distance between the segment and the origin is $L\epsilon > 0$; the angle of the segment with respect to the field axes is α. One assumes that for all $k \in \mathbb{Z}$, $\alpha \ne k\pi/2$. The restriction of S^{H_1,H_2} along $[AB]$ yields a centred Gaussian process $X = \{X_t; t \in [0,1]\}$ with covariance function

$$\begin{aligned}\forall s,t \in [0,1], \qquad R(s,t) &= \frac{1}{4}L^{2(H_1+H_2)}(|s+\epsilon|^{2H_1} + |t+\epsilon|^{2H_1} - |s-t|^{2H_1}) \\ &\quad \times (|s+\epsilon|^{2H_2} + |t+\epsilon|^{2H_2} - |s-t|^{2H_2}) \\ &\quad \times |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_2}.\end{aligned} \tag{33}$$

On the one hand, according to the computations of Cohen et al. [14], the function C, defined on $\overset{\circ}{T}$ by

$$\forall s,t \in \overset{\circ}{T}, \qquad C(s,t) = (s-t)^{4-2H_1} \frac{\partial^4 R}{\partial^2 s \partial^2 t}(s,t),$$

which can be extended to a continuous function on T. Moreover, one has, for all $t \in [0,1]$

$$C(t,t) = L^{2(H_1+H_2)} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_2} H_1(2H_1-1)(2H_1-2)(2H_1-3)(t+\epsilon)^{2H_2},$$

if $H_1 < H_2$, and

$$C(t,t) = 2L^{4H_1} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_1} H_1(2H_1-1)(2H_1-2)(2H_1-3)(t+\epsilon)^{2H_1},$$

if $H_1 = H_2$. Hence, X satisfies assumption (2) of Theorem 2 with $L(h) = 1$ and $\gamma = 2 - 2H_1$.

On the other hand, one can check that

$$\begin{aligned} R(t+\delta_j h, t+\delta_k h) = {} & \frac{1}{4} L^{2(H_1+H_2)} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_2} [4(t+\epsilon)^{2(H_1+H_2)} \\ & + 4(H_1+H_2)(\delta_j+\delta_k)(t+\epsilon)^{2(H_1+H_2)-1} h \\ & - 2|\delta_j-\delta_k|^{2H_1}(t+\epsilon)^{2H_2} h^{2H_1} \\ & - 2|\delta_j-\delta_k|^{2H_2}(t+\epsilon)^{2H_1} h^{2H_2} \\ & - 2H_2(\delta_j+\delta_k)|\delta_j-\delta_k|^{2H_1}(t+\epsilon)^{2H_2-1} h^{2H_1+1} \\ & - 2H_1(\delta_j+\delta_k)|\delta_j-\delta_k|^{2H_2}(t+\epsilon)^{2H_1-1} h^{2H_2+1} \\ & + |\delta_j-\delta_k|^{2(H_1+H_2)} h^{2(H_1+H_2)} \\ & + 2H_1(2H_1-1)(\delta_j^2+\delta_k^2)(t+\epsilon)^{2(H_1+H_2)-2} h^2 \\ & + H_2(2H_2-1)(\delta_j^2+\delta_k^2)(t+\epsilon)^{2(H_1+H_2)-2} h^2 \\ & + 2H_1H_2(\delta_j+\delta_k)^2 h^2] + h^{2+2H_1}\eta_t^{(1)}(h) + h^3 \eta_t^{(2)}(h), \end{aligned}$$

where, for $j,k = 1,2,3$, one sets $\delta_j, \delta_k = -1, 0, 1$. Moreover, one can check that the functions $\eta_t^{(i)}$, $i = 1,2$, are such that

$$\sup_{0<h<1} \sup_{h \le t \le 1-h} |\eta_t^{(i)}(h)| < +\infty.$$

It yields

$$\begin{aligned} & \sup_{h \le t \le 1-h} \left| \frac{\left(\delta_1^h \circ \delta_2^h R\right)(t,t)}{h^{2H_1}} - L^{2(H_1+H_2)} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_2} \right. \\ & \left. \times \; [(4-2^{2H_1})(t+\epsilon)^{2H_2} - (4-2^{2H_2})(t+\epsilon)^{2H_1} h^{2(H_2-H_1)} + 2(4-2^{2(H_1+H_2)}) h^{2H_2}] \right| \\ & \overset{h\to 0^+}{=} \mathcal{O}(h^2). \end{aligned}$$

Let us note that it corrects an error made in [14] formula (40).

Moreover, with the same method, one can prove that

$$\lim_{h\to 0^+} \sup_{h\le t\le 1-h} \left| \frac{(\delta_1^h \circ \delta_2^h R)(t+h,t)}{h^{2H_1}} - \frac{1}{2} L^{2(H_1+H_2)} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_2} \times [(t+\epsilon)^{2H_2}(2^{2H_1+2} - 7 - 3^{2H_1}) + (t+\epsilon)^{2H_1}(2^{2H_2+2} - 7 - 3^{2H_2}) h^{2(H_2-H_1)}] \right| = 0.$$

Thus, X satisfies assumption (3) of Theorem 2 with

$$\begin{aligned}
&q = 2,\\
&\nu_1 = 2(H_2 - H_1),\\
&\nu_2 = 2H_2,\\
&\epsilon_2 = 1/2,\\
&\phi(h) = h,\\
&g_0(t) = L^{2(H_1+H_2)} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_2} (4 - 2^{2H_1})(t+\epsilon)^{2H_2},\\
&g_1(t) = L^{2(H_1+H_2)} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_2} (4 - 2^{2H_2})(t+\epsilon)^{2H_1},\\
&g_2(t) = L^{2(H_1+H_2)} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_2} (4 - 2^{2(H_1+H_2)}),\\
&\widetilde{g}(t) = \frac{1}{2} L^{2(H_1+H_2)} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_2} (2^{2H_1+2} - 7 - 3^{2H_1})(t+\epsilon)^{2H_2},
\end{aligned}$$

in the case $H_1 < H_2$, and

$$\begin{aligned}
&q = 1,\\
&\nu_1 = 2H_1,\\
&\epsilon_1 = 1/2,\\
&\phi(h) = h,\\
&g_0(t) = 2L^{4H_1} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_1} (4 - 2^{2H_1})(t+\epsilon)^{2H_1},\\
&g_1(t) = L^{4H_1} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_1} (4 - 2^{4H_1}),\\
&\widetilde{g}(t) = L^{4H_1} |\cos\alpha|^{2H_1} |\sin\alpha|^{2H_1} (2^{2H_1+2} - 7 - 3^{2H_1})(t+\epsilon)^{2H_1},
\end{aligned}$$

in the case $H_1 = H_2$.
Therefore, one can apply Theorem 2 to X. It yields that, if $0 < H_1 \le H_2 < 1$

$$\left\{ \sqrt{n} \left(n^{2H_1-1} V_n(X)_t - \psi_0(t) - \frac{1}{n^{2(H_2-H_1)}} \psi_1(t) - \frac{1}{n^{2H_2}} \psi_2(t) \right); t \in [0,1] \right\} \xrightarrow{(\mathcal{L})} \{Z_t; t \in [0,1]\},$$

with obvious notations. One sees that this central limit theorem is of a classical form, which means that one has

$$\left\{ \sqrt{n}(n^{2H_1-1} V_n(X)_t - \psi(t)); t \in [0,1] \right\} \xrightarrow{(\mathcal{L})} \{Z_t; t \in [0,1]\},$$

where $\psi(t)$ is not related to n, if and only if $H_2 > H_1 + 1/4$ or $H_1 = H_2 > 1/4$ (here it also corrects an error done in [14] Proposition 2, where the condition is $H_1 = H_2 < 3/4$).

Let us note that one has the same results for the process of linear interpolations $v_n(X)$, because, from the remark of Corollary 3, it is clear that the paths of X are $(H_1 - \beta)$-Hölderian for all $0 < \beta < H_1$.

In the next section, we consider the second order localized quadratic variations to have the same kind of results for multifractional processes. We obtain a weaker convergence in distribution, because the limit process is not smooth.

4. Functional convergence of $V_n^{\mathrm{loc}}(X)_t$

Let $\alpha \in]0, 1[$. In this section, we establish a new version of Theorem 2 for the process of second order localized quadratic variations $(V_n^{\alpha,\mathrm{loc}}(X)_t;\, t \in [0, 1])$, which is defined by

$$\forall t \in]0, 1[, \quad V_n^{\alpha,\mathrm{loc}}(X)_t = \sum_{k \in \mathcal{V}_n^\alpha(t)} \left[X_{\frac{k+1}{n}} + X_{\frac{k-1}{n}} - 2X_{\frac{k}{n}} \right]^2,$$

where $\mathcal{V}_n^\alpha(t)$ is a neighbourhood of t defined by

$$\mathcal{V}_n^\alpha(t) = \left\{ k;\, 1 \le k \le n-1 \text{ and } \left| \frac{k}{n} - t \right| \le \frac{1}{n^\alpha} \right\}. \tag{34}$$

Let us note that the number of elements of this set, noted $\sharp\mathcal{V}_n^\alpha(t)$, satisfies

$$\sharp\mathcal{V}_n^\alpha(t) \overset{n\to+\infty}{\sim} 2n^{1-\alpha}.$$

Theorem 4. *Let X be a centered square integrable process, with Gaussian increments, satisfying*

(1) *R is continuous on $[0, 1]^2$;*

(2) *Let $T = \{0 \le t \le s \le 1\}$. We assume that the derivative $\frac{\partial^4 R}{\partial s^2 \partial t^2}$ exists on $]0, 1]^2 \setminus \{s = t\}$, and that there exist a continuous function $C : T \mapsto \mathbb{R}$ and a continuously differentiable function $\gamma : [0, 1] \mapsto]0, 2[$ such that*

$$\forall (s, t) \in \overset{\circ}{T}, \quad (s-t)^{2+\frac{\gamma_s+\gamma_t}{2}} \frac{\partial^4 R}{\partial s^2 \partial t^2}(s, t) = C(s, t), \tag{35}$$

where $\overset{\circ}{T}$ denotes the interior of T;

(3) *We assume that there exist $q+1$ functions ($q \in \mathbb{N}$) $g_0, g_1, \ldots, g_q$ from $]0, 1[$ to $\mathbb{R}$, q functions $\nu_1, \nu_2, \ldots, \nu_q$ from $]0, 1[$ to $]0, +\infty[$, and a function $\phi :]0, 1[\mapsto]0, +\infty[$ such that*
   (a) *if $q \ge 1$ then $\forall t \in]0, 1[$, $0 < \nu_1(t) < \nu_2(t) < \cdots < \nu_q(t)$;*
   (b) *if $q \ge 1$ then $\forall 0 \le i \le q-1$, g_i is Lipschitz on $]0, 1[$;*
   (c) *g_q is $1/2 + \epsilon_q$-Hölderian on $]0, 1[$ with $0 < \epsilon_q \le 1/2$;*
   (d) *there exists $t \in]0, 1[$ such that $g_0(t) \ne 0$;*
   (e) *For all $t \in]0, 1[$,*

$$\lim_{h\to 0^+} \frac{1}{\sqrt{h}} \left(\sup_{t-h^\alpha \le s \le t+h^\alpha} \left| \frac{\left(\delta_1^h \circ \delta_2^h R\right)(s, s)}{h^{2-\gamma_s}} - g_0(s) - \sum_{i=1}^q g_i(s)\phi(h)^{\nu_i(s)} \right| \right) = 0, \tag{36}$$

where if $q = 0$ then $\forall s \in]0, 1[$, $\sum_{i=1}^q g_i(s)\phi(h)^{\nu_i(s)} = 0$, and where if $q \ne 0$ then $\lim_{h\to 0^+} \phi(h) = 0$.

(f) *there exists a bounded function* $\widetilde{g} :]0,1[\longrightarrow \mathbb{R}$ *such that for all* $t \in]0,1[$*:*

$$\lim_{h\to 0^+} \sup_{t-h^\alpha \le s \le t+h^\alpha} \left| \frac{\left(\delta_1^h \circ \delta_2^h R\right)(s+h,s)}{h^{2-\gamma_s}} - \widetilde{g}(s) \right| = 0. \tag{37}$$

Then, for all $t \in]0,1[$*, one has almost surely*

$$\lim_{n\to+\infty} n^{1-\gamma_t+\alpha} V_n^{\alpha,\mathrm{loc}}(X)_t = 2g_0(t). \tag{38}$$

Moreover, the finite-dimensional laws of the process

$$\left\{ n^{\frac{1-\alpha}{2}} \left(n^{1-\gamma_t+\alpha} V_n^{\alpha,\mathrm{loc}}(X)_t - 2g_0(t) - 2\sum_{i=1}^{q} g_i(t)\phi\left(\frac{1}{n}\right)^{\nu_q(t)} \right) \right\}_{t\in]0,1[}, \tag{39}$$

converge, when $n \to +\infty$*, towards those of a centred Gaussian process* $Z^{\mathrm{loc}} = \{Z_t^{\mathrm{loc}}; t \in]0,1[\}$*, whose covariance function is defined by*

$$\forall s,t \in]0,1[, \quad R(s,t) = \begin{cases} 4g_0(t)^2 + 8\widetilde{g}(t)^2 + 8\|\rho_\gamma\|^2 C(t,t)^2 & \text{if } s=t, \\ 0 & \text{if } s \ne t. \end{cases} \tag{40}$$

Remark. (i) One has stated a result with the convergence of the finite-dimensional laws because the limit process is a white noise and so it is not measurable (see [26] chapter VI exercise (2,18)).

(ii) Assumption (36) yields that the functions g_i, $0 \le i \le q$, are continuous on $]0,1[$.

Proof of theorem 4. We proceed as in the proofs of Theorems 3 and 5 in [4]. In all the proof, we note $\mathcal{V}_n(t)$ for $\mathcal{V}_n^\alpha(t)$, whenever it is possible.

Step 1. We start with the proof of (38).

Step 1.1. Let $t \in]0,1[$. Prove that

$$\lim_{n\to+\infty} \frac{n^{1-\gamma_t+\alpha}}{2} \mathbb{E} V_n^{\alpha,\mathrm{loc}}(X)_t = g_0(t). \tag{41}$$

One has

$$\begin{aligned} \left| \frac{n^{1-\gamma_t+\alpha}}{2} \mathbb{E} V_n^{\alpha,\mathrm{loc}}(X)_t - g_0(t) \right| &= \left| \frac{n^{1-\gamma_t+\alpha}}{2} \sum_{k\in\mathcal{V}_n(t)} d_{kk} - g_0(t) \right| \\ &\le \frac{n^{1+\alpha}}{2} \sum_{k\in\mathcal{V}_n(t)} d_{kk} \left| n^{-\gamma_t} - n^{-\gamma\left(\frac{k}{n}\right)} \right| \\ &\quad + \frac{n^{\alpha-1}}{2} \sum_{k\in\mathcal{V}_n(t)} \left| \frac{d_{kk}}{n^{\gamma\left(\frac{k}{n}\right)-2}} - g_0\left(\frac{k}{n}\right) \right| \\ &\quad + \left| \frac{n^{\alpha-1}}{2} \sum_{k\in\mathcal{V}_n(t)} g_0\left(\frac{k}{n}\right) - g_0(t) \right| \\ &= L_1 + L_2 + L_3. \end{aligned}$$

By a Taylor expansion, one has

$$|n^{-\gamma_t} - n^{-\gamma(\frac{k}{n})}| = \left| \gamma_t - \gamma\left(\frac{k}{n}\right) \right| n^{-\gamma\left(\frac{k}{n}\right)} \mathrm{e}^{-\beta_{k,n}\log n} \log n,$$

where $|\beta_{k,n}| \leq \left|\gamma(t) - \gamma\left(\frac{k}{n}\right)\right|$. Moreover, assumption (36) implies that $n^{2-\gamma\left(\frac{k}{n}\right)} d_{kk}$ is bounded in n uniformly in $k \in \mathcal{V}_n(t)$. It yields

$$L_1 \leq \left|\gamma_t - \gamma\left(\frac{k}{n}\right)\right| \mathrm{e}^{\beta_n^* \log n} \log n,$$

where $\beta_n^* = \sup_{k \in \mathcal{V}_n(t)} |\beta_{k,n}|$. Since γ is Lipschitz, previous inequality yields

$$\lim_{n \to +\infty} L_1 = 0.$$

Moreover, one has

$$L_2 \leq \sup_{k \in \mathcal{V}_n(t)} \left| \frac{d_{kk}}{n^{\gamma\left(\frac{k}{n}\right)-2}} - g_0\left(\frac{k}{n}\right) \right|.$$

With assumption (36), it yields

$$\lim_{n \to +\infty} L_2 = 0.$$

One also has

$$\lim_{n \to +\infty} L_3 = 0,$$

because g_0 is a Lipschitz function.

Step 1.2. Prove that, for all $t \in]0, 1[$, one has almost surely

$$\lim_{n \to +\infty} n^{1-\gamma_t+\alpha} (V_n^{\alpha,\mathrm{loc}}(X)_t - \mathbb{E} V_n^{\alpha,\mathrm{loc}}(X)_t) = 0. \tag{42}$$

Since the localized and normalized second order quadratic variation can be considered as the Euclidean norm of a Gaussian vector, one can apply the classical Cochran theorem, which yields that there exist $\sharp\mathcal{V}_n(t)$ nonnegative real numbers $(\lambda_{k,n}; k \in \mathcal{V}_n(t))$, and a $\sharp\mathcal{V}_n(t)$-dimensional vector $(Y_n^{(k)}; k \in \mathcal{V}_n(t))$ whose components are independent reduced Gaussian variables, such that

$$n^{1-\gamma_t+\alpha} V_n^{\alpha,\mathrm{loc}}(X)_t = \sum_{k \in \mathcal{V}_n(t)} \lambda_{k,n} (Y_n^{(k)})^2. \tag{43}$$

Following [5] formula (25) p. 699, the Hanson and Wright inequality (see [18]) yields that there exists $K > 0$ such that for all $0 < a < 1$

$$\mathbb{P}(n^{1-\gamma_t+\alpha} |V_n^{\alpha,\mathrm{loc}}(X)_t - \mathbb{E} V_n^{\alpha,\mathrm{loc}}(X)_t| > a) \leq 2 \exp\left(-\frac{K a^2}{\lambda_n^*}\right),$$

where $K > 0$ and $\lambda_n^* = \max\{\lambda_{k,n}; k \in \mathcal{V}_n(t)\}$. With the arguments of the proof of (30) p. 701 in [5], and the fact that g_0 is Lipschitz, one can prove that

$$\lambda_n^* \overset{n \to +\infty}{=} \mathcal{O}\left(\frac{1}{n^{1-\alpha}}\right).$$

Thus, if one sets

$$a_n^2 = \frac{2 \log n}{K n^{1-\alpha}},$$

one has

$$\lim_{n\to+\infty} a_n = 0,$$
$$\sum_{n=0}^{+\infty} \mathbb{P}\left(\frac{n^{2-\gamma_t}}{\sharp\mathcal{V}_n(t)\phi\left(\frac{1}{n}\right)^{\nu_q(t)}} |V_n^{\alpha,\mathrm{loc}}(X)_t - \mathbb{E}V_n^{\alpha,\mathrm{loc}}(X)_t| > a_n \right) < +\infty,$$

and the Borel–Cantelli lemma yields (42).

Step 2. Next we prove (39).

Step 2.1. Prove that there exist a constant $K > 0$ and a sequence of positive real numbers $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} a_n = 0$, such that if $0 \le t \le s \le 1$, $j \in \mathcal{V}_n(s)$, $k \in \mathcal{V}_n(t)$ and $j - k \ge 3$ then,

$$|d_{jk}^2 - C(s,t)^2 n^{\gamma_s+\gamma_t-4} \rho_{\frac{\gamma_s+\gamma_t}{2}}(j-k)^2| \le K \frac{n^{\gamma_s+\gamma_t-4}}{(j-k-2)^{\gamma_s+\gamma_t+4}} a_n. \tag{44}$$

We follow the proof of (31) in [4]. One has

$$\begin{aligned} d_{jk} &= \int_{\frac{j}{n}}^{\frac{j+1}{n}} \mathrm{d}u \int_{u-\frac{1}{n}}^{u} \mathrm{d}v \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mathrm{d}x \int_{x-\frac{1}{n}}^{x} \frac{\partial^4 R}{\partial s^2 \partial t^2}(v,y)\mathrm{d}y \\ &= \int_{\frac{j}{n}}^{\frac{j+1}{n}} \mathrm{d}u \int_{u-\frac{1}{n}}^{u} \mathrm{d}v \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mathrm{d}x \int_{x-\frac{1}{n}}^{x} \frac{C(v,y)}{(v-y)^{\frac{\gamma_v+\gamma_y}{2}+2}}\mathrm{d}y. \end{aligned}$$

Therefore,

$$\begin{aligned} &|d_{jk} - C(s,t) n^{\frac{\gamma_s+\gamma_t}{2}-2} \rho_{\frac{\gamma_s+\gamma_t}{2}}(j-k)| \\ &\quad \le \|C\|_\infty \int_{\frac{j}{n}}^{\frac{j+1}{n}} \mathrm{d}u \int_{u-\frac{1}{n}}^{u} \mathrm{d}v \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mathrm{d}x \int_{x-\frac{1}{n}}^{x} \left| \frac{1}{(v-y)^{2+\frac{\gamma_v+\gamma_y}{2}}} - \frac{1}{(v-y)^{2+\frac{\gamma_s+\gamma_t}{2}}} \right| \mathrm{d}y \\ &\qquad + \int_{\frac{j}{n}}^{\frac{j+1}{n}} \mathrm{d}u \int_{u-\frac{1}{n}}^{u} \mathrm{d}v \int_{\frac{k}{n}}^{\frac{k+1}{n}} \mathrm{d}x \int_{x-\frac{1}{n}}^{x} \frac{|C(v,y)-C(s,t)|}{(v-y)^{2+\frac{\gamma_s+\gamma_t}{2}}}\mathrm{d}y, \end{aligned} \tag{45}$$

where $\|C\|_\infty = \sup_{t\in T} |C(s,t)|$.

Taylor formula and the following inequality:

$$\forall w \in \mathbb{R}, \quad \left|\mathrm{e}^w - 1\right| \le 2\mathrm{e}^{\frac{|w|}{2}} \sinh\left(\frac{|w|}{2}\right),$$

yield that one has

$$\left| \frac{1}{(v-y)^{2+\frac{\gamma_v+\gamma_y}{2}}} - \frac{1}{(v-y)^{2+\frac{\gamma_s+\gamma_t}{2}}} \right| \le \frac{2}{(v-y)^{2+\frac{\gamma_s+\gamma_t}{2}}} \mathrm{e}^{\frac{\|\gamma'\|_\infty}{2}\frac{1}{n^\alpha}\log(v-y)} \times \sinh\left(\frac{\|\gamma'\|_\infty}{2} \frac{1}{n^\alpha} \log(v-y) \right).$$

Furthermore, on the set of integration $\{(v, y) : \frac{j-1}{n} \leq u - \frac{1}{n} \leq v \leq u \leq \frac{j+1}{n}, \frac{k-1}{n} \leq x - \frac{1}{n} \leq y \leq x \leq \frac{k+1}{n}\}$, one has

$$\frac{1}{n} \leq \frac{j-k-2}{n} \leq v - y \leq \frac{j-k+2}{n} \leq s - t + \frac{2}{n^\alpha} + \frac{2}{n} \leq K,$$

where K is a positive constant. This proves that the first term of the right-hand side of (45) is bounded by

$$K \frac{n^{\frac{\gamma_s+\gamma_t}{2}-2}}{(j-k-2)^{\frac{\gamma_s+\gamma_t}{2}+2}} a_n^{(1)},$$

where $(a_n^{(1)})_{n\in\mathbb{N}}$ is a sequence of positive real numbers such that $\lim_{n\to+\infty} a_n^{(1)} = 0$.

For the second term of the right-hand side of (45), the uniform continuity of C on T implies that it is bounded by the same expression with a different sequence $(a_n^{(1)})_{n\in\mathbb{N}}$. Then (44) is a direct consequence.

Step 2.2. Let $t \in]0, 1[$. Prove that

$$\lim_{n\to+\infty} n^{3-2\gamma_t+\alpha} \mathrm{Var}\ V_n^{\alpha,\mathrm{loc}}(X)_t = 8g_0(t)^2 + 16\widetilde{g}(t)^2 + 16\|\rho_{\gamma_t}\|^2 C(t,t)^2. \tag{46}$$

We follow the proof of theorem 5 in [4].

Isserlis formulas (see [19]) yield

$$\mathrm{Var}\ V_n^{\alpha,\mathrm{loc}}(X)_t = 2 \sum_{k\in\mathcal{V}_n(t)} d_{kk}^2 + 4 \sum_{k<j;\, j,k\in\mathcal{V}_n(t)} d_{jk}^2. \tag{47}$$

- First, prove that

$$\lim_{n\to+\infty} n^{3-2\gamma_t+\alpha} \sum_{j-k\geq 3;\, j,k\in\mathcal{V}_n(t)} d_{jk}^2 = 2C(t,t)^2 \sum_{l=3}^{+\infty} \rho_{\gamma_t}(l)^2. \tag{48}$$

From (44) with $s = t$ and the dominated convergence theorem, one has

$$\lim_{n\to+\infty} \frac{n^{4-2\gamma_t}}{\sharp\mathcal{V}_n(t)} \sum_{j-k\geq 3;\, j,k\in\mathcal{V}_n(t)} d_{jk}^2 = \lim_{n\to+\infty} \frac{C(t,t)^2}{\sharp\mathcal{V}_n(t)} \sum_{j-k\geq 3;\, j,k\in\mathcal{V}_n(t)} \rho_{\gamma_t}(j-k)^2$$
$$= C(t,t)^2 \sum_{l=3}^{+\infty} \rho_{\gamma_t}(l)^2,$$

which yields (48).

- Following the proofs of (34), (35), (36) in [4] and the preceding proof of (48), one gets

$$\lim_{n\to+\infty} n^{3-2\gamma_t+\alpha} \left(2 \sum_{k\in\mathcal{V}_n(t)} d_{kk}^2 + 4 \sum_{1\leq j-k\leq 2;\, j,k\in\mathcal{V}_n(t)} d_{jk}^2 \right)$$
$$= 8g_0(t)^2 + 16\widetilde{g}(t)^2 + 16\rho_{\gamma_t}(2)^2 C(t,t)^2. \tag{49}$$

Hence, (46) is a consequence of (47)–(49).

Step 2.3. Prove that, if $s, t \in]0, 1[$, $s > t$:

$$\lim_{n\to+\infty} n^{3-\gamma_s-\gamma_t+\alpha} \mathrm{Cov}\ (V_n^{\alpha,\mathrm{loc}}(X)_s, V_n^{\alpha,\mathrm{loc}}(X)_t) = 0. \tag{50}$$

Isserlis formulas (see [19]) yield

$$\mathrm{Cov}(V_n^{\mathrm{loc}}(X)_s, V_n^{\alpha,\mathrm{loc}}(X)_t) = 2 \sum_{j\in\mathcal{V}_n(s)} \sum_{k\in\mathcal{V}_n(t)} d_{jk}^2 \geq 0. \tag{51}$$

Let us note that if $j \in \mathcal{V}_n(s)$ and $k \in \mathcal{V}_n(t)$ then $j - k \geq (s - t - \frac{2}{n^\alpha})n$. Therefore, there exists $K > 0$ such that for large n

$$\forall j \in \mathcal{V}_n(t), k \in \mathcal{V}_n(s), \quad j - k - 2 \geq Kn. \tag{52}$$

Moreover, (44) yields that there exists a constant $K > 0$ such that if $j \in \mathcal{V}_n(s)$ and $k \in \mathcal{V}_n(t)$:

$$d_{jk}^2 \leq K \frac{n^{\gamma_s+\gamma_t-4}}{(j-k-2)^{\gamma_s+\gamma_t+4}},$$

which implies

$$\begin{aligned}
&\frac{n^{4-\gamma_s-\gamma_t}}{\sqrt{\sharp\mathcal{V}_n(s)\sharp\mathcal{V}_n(t)}}\mathrm{Cov}\,(V_n^{\alpha,\mathrm{loc}}(X)_s, V_n^{\alpha,\mathrm{loc}}(X)_t)\\
&\quad\leq \frac{K}{\sqrt{\sharp\mathcal{V}_n(s)\sharp\mathcal{V}_n(t)}} \sum_{j\in\mathcal{V}_n(s)} \sum_{k\in\mathcal{V}_n(t)} \frac{1}{(j-k-2)^{\gamma_s+\gamma_t+4}}\\
&\quad\leq \frac{K}{n^{\gamma_s+\gamma_t+4}} \sqrt{\sharp\mathcal{V}_n(s)\sharp\mathcal{V}_n(t)}\\
&\quad\leq K \frac{n^{1-\alpha}}{n^{\gamma_s+\gamma_t+4}},
\end{aligned}$$

and this last inequality proves (50).

Step 2.4. Prove that the finite-dimensional margins of $\widetilde{T_n}^{\mathrm{loc}}$ converge towards those of Z^{loc} where

$$\forall t \in]0,1[, \quad T_n^{\mathrm{loc}}(t) = n^{\frac{3}{2}-\gamma_t+\frac{\alpha}{2}} V_n^{\alpha,\mathrm{loc}}(X)_t,$$

and

$$\forall t \in]0,1[, \quad \widetilde{T_n}^{\mathrm{loc}}(t) = T_n^{\mathrm{loc}}(t) - \mathbb{E}(T_n^{\mathrm{loc}}(t)).$$

As in the proof of Theorem 2, step 2.1, one only proves the convergence in law of $(\widetilde{T_n}^{\mathrm{loc}}(s), \widetilde{T_n}^{\mathrm{loc}}(t))$ towards $(Z_s^{\mathrm{loc}}, Z_t^{\mathrm{loc}})$ for all $s, t \in]0,1[$, $s > t$. One considers

$$S_n^{\mathrm{loc}}(\lambda,\mu) = \lambda T_n^{\mathrm{loc}}(t) + \mu T_n^{\mathrm{loc}}(s),$$

where λ and μ are nonnegative real numbers.

From (46) and (50), one has

$$\begin{aligned}
\lim_{n\to+\infty} \mathrm{Var}\,(S_n^{\mathrm{loc}}(\lambda,\mu)) &= \lambda^2 \mathrm{Var}\,(Z_s^{\mathrm{loc}}) + \mu^2 \mathrm{Var}\,(Z_t^{\mathrm{loc}})\\
&= \mathrm{Var}\,(\lambda Z_s^{\mathrm{loc}} + \mu Z_t^{\mathrm{loc}}).
\end{aligned} \tag{53}$$

Next, one applies the Lindeberg CLT. For that, one considers $S_n^{\mathrm{loc}}(\lambda,\mu)$ as the Euclidean norm of the Gaussian vector $(G_i; i \in \mathcal{V}_n(t) \cup \mathcal{V}_n(s))$, with

$$G_i = \sqrt{\lambda}\sqrt{n^{\frac{3}{2}-\gamma_t+\frac{\alpha}{2}}}\,\Delta X_i^{(n)}, \quad \text{if } i \in \mathcal{V}_n(t),$$

$$G_i = \sqrt{\mu}\sqrt{n^{\frac{3}{2}-\gamma_s+\frac{\alpha}{2}}}\,\Delta X_i^{(n)}, \quad \text{if } i \in \mathcal{V}_n(s).$$

Therefore, by the classical Cochran theorem, one can find a_n nonnegative real numbers $(\tau_{1,n}, \dots, \tau_{a_n,n})$ and one a_n-dimensional Gaussian vector ξ_n, such that its components are independent Gaussian variables $\mathcal{N}(0,1)$ and

$$S_n^{\text{loc}}(\lambda,\mu) = \sum_{i=1}^{a_n} \tau_{i,n} (\xi_n^{(i)})^2.$$

As in the proof of Theorem 6 pp. 24–25 in [4], one can check that:

$$\tau_n^* = \max_{1 \le i \le a_n} \tau_{i,n} \overset{n\to+\infty}{=} \mathcal{O}\left(\frac{1}{n^{\frac{1-\alpha}{2}}}\right).$$

Consequently, with (53), one gets

$$\lim_{n\to+\infty} \frac{\tau_n^*}{\sqrt{\operatorname{Var}(S_n(\lambda,\mu))}} = 0,$$

and so the Lindeberg CLT implies

$$\lambda \widetilde{T_n}^{\text{loc}}(t) + \mu \widetilde{T_n}^{\text{loc}}(s) = S_n^{\text{loc}}(\lambda,\mu) - \mathbb{E}(S_n^{\text{loc}}(\lambda,\mu)) \overset{(\mathcal{L})}{\longrightarrow} \lambda Z_t^{\text{loc}} + \mu Z_s^{\text{loc}},$$

when $n \to +\infty$.

Next, one can check that the arguments of the proof of Lemma 4.3 in [25] also hold for the second order localized quadratic variations. Consequently, the Cramer–Wold theorem yields the convergence of the finite-dimensional margins of the process $\widetilde{T_n}^{\text{loc}}$ towards those of Z^{loc}. ∎

4.1. Application to the multifractional Brownian motion

Let $H : \mathbb{R} \longrightarrow]0,1[$ be a three-times continuously differentiable function. One considers a multifractional Brownian motion $B^{(H)} = \left\{B_t^{(H)}; t \in \mathbb{R}\right\}$ with Hurst function H. It is defined with the following harmonizable representation

$$\forall t \in \mathbb{R}, \quad B_t^{(H)} = \frac{1}{D(H_t)} \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{H_t + \frac{1}{2}}} dW_x, \tag{54}$$

where W is a random Brownian measure on $\mathbb{R}$ (see [8]), and from formula 7.2.13 in [27]:

$$\forall H \in]0,1[, \quad D(H) = \sqrt{\int_{\mathbb{R}} \frac{|e^{ix} - 1|^2}{|x|^{2H+1}} dx} = \sqrt{\frac{\pi}{H\Gamma(2H)\sin(H\pi)}}. \tag{55}$$

Computations yield that $B^{(H)}$ is a centred Gaussian process with covariance function given by (see [1])

$$\forall s,t \in \mathbb{R}, \quad \operatorname{Cov}(B_s^{(H)}, B_t^{(H)}) = R^{(H)}(s,t) = f(H_s,H_t)(|s|^{H_s+H_t} + |t|^{H_s+H_t} - |s-t|^{H_s+H_t}), \tag{56}$$

where for all $x, y \in]0,1[$, $f(x,y) = \frac{D(\frac{x+y}{2})^2}{2D(x)D(y)}$.

This field has been introduced independently in [8,24] and the identification of the Hurst function has been performed in [6,13].

On the one hand, computations yield, for all $s, t \in \overset{\circ}{T}$:

$$\frac{\partial^4 R^{(H)}}{\partial s^2 \partial t^2}(s,t) = (s-t)^{H_s+H_t-3}\psi(s,t) - \frac{1}{2}(H_s+H_t)(H_s+H_t-1) \times (H_s+H_t-2)(H_s+H_t-3)(s-t)^{H_s+H_t-4},$$

where $\psi(s,t)$ is a continuous function on $[0,1]^2$. Therefore, assumption (2) of Theorem 4 is satisfied with $\gamma_t = 2 - 2H_t$.

On the other hand, computations also yield

$$(\delta_1^h \circ \delta_2^h R^{(H)})(s,s) = 2f(H_s, H_{s+h})h^{H_s+H_{s+h}} + 2f(H_{s-h}, H_s)h^{H_{s-h}+H_s} - f(H_{s-h}, H_{s+h})(2h)^{H_{s-h}+H_{s+h}} + \eta_s(h),$$

where

$$\begin{aligned}\eta_s(h) = {} & f(H_{s+h}, H_{s+h})(s+h)^{2H_{s+h}} - 2f(H_s, H_{s+h})(s^{H_s+H_{s+h}} + (s+h)^{H_s+H_{s+h}}) \\ & - 2f(H_{s-h}, H_s)((s-h)^{H_{s-h}+H_s} + s^{H_{s-h}+H_s}) + 4f(H_s,H_s)s^{2H_s} \\ & + f(H_{s-h}, H_{s+h})((s-h)^{H_{s-h}+H_{s+h}} + (s+h)^{H_{s-h}+H_{s+h}}) \\ & + f(H_{s-h}, H_{s-h})(s-h)^{2H_{s-h}}.\end{aligned}$$

One can check that $\eta_s(0) = \eta_s'(0) = \eta_s''(0) = 0$. Therefore Taylor formula implies that

$$\eta_s(h) = \int_0^h \frac{(h-x)^2}{2}\eta_s^{(3)}(x)\mathrm{d}x,$$

and since for all $t \in]0,1[$, $(s,x) \mapsto \eta_s(x)$ is three times differentiable on $[t-h^\alpha, t+h^\alpha] \times [0,h]$ (for h small enough), one has for all $t \in]0,1[$:

$$\sup_{t-h^\alpha \le s \le t+h^\alpha} |\eta_s(h)| \overset{h\to 0^+}{=} \mathcal{O}(h^3).$$

Consequently, for all $t \in]0,1[$:

$$\lim_{h\to 0^+} \frac{1}{\sqrt{h}}\left(\sup_{t-h^\alpha \le s \le t+h^\alpha}\left|\frac{(\delta_1^h \circ \delta_2^h R^{(H)})(s,s)}{h^{2H_s}} - (4-2^{2H_s})\right|\right) = 0.$$

And with the same method, one can prove that for all $t \in]0,1[$:

$$\lim_{h\to 0^+} \sup_{t-h^\alpha \le s \le t+h^\alpha}\left|\frac{(\delta_1^h \circ \delta_2^h R^{(H)})(s+h,s)}{h^{2H_s}} - \frac{1}{2}(2^{2H_s+2} - 7 - 3^{2H_s})\right| = 0.$$

Therefore, $B^{(H)}$ satisfies assumption (3) of Theorem 4 with

$$\begin{aligned}& q = 0, \\ & \epsilon_0 = 1/2, \\ & g_0(t) = 4 - 2^{2H_t}, \\ & \widetilde{g}(t) = \frac{1}{2}(2^{2H_t+2} - 7 - 3^{2H_t}).\end{aligned}$$

Consequently, Theorem 4 yields that, for all $t \in]0,1[$, one has almost surely

$$n^{2H_t-1+\alpha} V_n^{\alpha,\mathrm{loc}}(B^{(H)})_t \overset{n\to+\infty}{=} 2(4 - 2^{2H_t}). \tag{57}$$

Moreover, the finite-dimensional laws of the process

$$\left\{n^{\frac{1-\alpha}{2}}(n^{2H_t-1+\alpha}V_n^{\alpha,\mathrm{loc}}(B^{(H)})_t - 2(4-2^{2H_t})); t \in [0,1]\right\}, \tag{58}$$

converge, when $n \to +\infty$, towards those of a centred Gaussian process $Z^{\mathrm{loc}} = \{Z_t^{\mathrm{loc}}; t \in]0,1[\}$, whose covariance function is defined by

$$\forall s,t \in]0,1[, \quad R(s,t) = \begin{cases} 2\sigma^2_{\mathrm{FBM},H_t} & \text{if } s = t, \\ 0 & \text{if } s \neq t, \end{cases} \tag{59}$$

where $\sigma^2_{\mathrm{FBM},H_t}$ has been defined in (31).

Let us note that the last formula corrects an error in formula (10) p. 7 in [13], in the case $s \neq t$.

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