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Maps preserving zeros of a polynomial ☆

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ABSTRACT

Let $\mathcal{A}$ be an algebra and let $f(x_1, \ldots, x_d)$ be a multilinear polynomial in noncommuting indeterminates x_i. We consider the problem of describing linear maps $\phi : \mathcal{A} \to \mathcal{A}$ that preserve zeros of f. Under certain technical restrictions we solve the problem for general polynomials f in the case where $\mathcal{A} = M_n(F)$. We also consider quite general algebras $\mathcal{A}$, but only for specific polynomials f.

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1. Introduction

Let F be a field, let $F\langle X\rangle$ be the free algebra generated by the set $X = \{x_1, x_2, \ldots\}$ of countably many noncommuting indeterminates, and let $f = f(x_1, \ldots, x_d) \in F\langle X\rangle$ be a nonzero polynomial. We say that a map ϕ from an F-algebra $\mathcal{A}$ into itself *preserves zeros of* f if for all $a_1, \ldots, a_d \in \mathcal{A}$,

$$f(a_1, \ldots, a_d) = 0 \implies f(\phi(a_1), \ldots, \phi(a_d)) = 0.$$

The list of all maps on $\mathcal{A}$ that preserve zeros of f must certainly contain scalar multiples of automorphisms, for some polynomials it must also contain scalar multiples of antiautomorphisms (say, for $f = x_1x_2 + x_2x_1$), and for some of them even all maps of the form

$$\phi(x) = \alpha\theta(x) + \mu(x), \tag{1}$$

where $\alpha \in F$, $\theta : \mathcal{A} \to \mathcal{A}$ is either an automorphism or an antiautomorphism, and μ is a linear map from $\mathcal{A}$ into its center (say, for $f = x_1x_2 - x_2x_1$). Our goal is to show that under certain restrictions – in particular, we will confine ourselves to linear maps ϕ and multilinear polynomials f – the standard example (1) is also the only possible example of a map preserving zeros of f. We will not bother with the question for which polynomials (1) can be simplified.

For certain simple polynomials, especially for $f = x_1x_2$ and $f = x_1x_2 - x_2x_1$, our problem has a long and rich history; see, for example [1,7] for historic comments and references. So far not much is known for general polynomials. For them the problem was explicitly posed by Chebotar et al. [9] for the matrix algebra $\mathcal{A} = M_n(F)$, and some partial solutions were obtained in two recent papers: [13] considers, in particular, the case where the sum of coefficients of f is a nonzero scalar (without assuming the linearity of ϕ), and [10] handles Lie polynomials of degree at most 4. Let us also mention a related, yet considerably simpler, problem of describing linear maps that preserve all values of f, i.e., $\phi(f(a_1, \ldots, a_d)) = f(\phi(a_1), \ldots, \phi(a_d))$ for all $a_i \in \mathcal{A}$. This problem can be solved at a high level of generality by using functional identities, although for finite dimensional algebras (including $M_n(F)$) the obtained results are not optimal; see [4] and also [7, Section 6.5].

One of the most fascinating approaches to linear preserver problems on matrix algebras was developed by Platonov and Đoković [16]. It is based on linear algebraic groups. In Section 2 we will see that this approach is applicable to our problem. In the matrix algebra $\mathcal{A} = M_n(F)$ we will be able to consider general multilinear polynomials f; however, we will be forced to impose several technical restrictions some of which might be superfluous. The general problem from [9] is therefore not yet completely solved.

In Section 3 we will prove three results giving solutions to our problem for some special polynomials, but in the context of rather general classes of prime algebras and/or C^*-algebras. More precisely, we will show that for these polynomials the problem can be reduced to some still nontrivial, but already solved problems. For polynomials that are not covered in our considerations, or at least cannot be handled by similar methods, the problem seems to be very intriguing.

2. The matrix algebra case

The main goal of this section is to prove Theorem 2.2. First we will survey the necessary tools needed in the proof.

2.1. Remarks on free algebras

Let F be a field and let $X = \{x_1, x_2, \ldots\}$ be a set of countably many noncommuting indeterminates. The free algebra $F\langle X\rangle$ consists of polynomials in $x_1, x_2, \ldots$ We say that $f = f(x_1, \ldots, x_d) \in F\langle X\rangle$ is a *multilinear polynomial* if it is of the form

$$f = \sum_{\sigma\in S_d} \lambda_\sigma x_{\sigma(1)} \ldots x_{\sigma(d)},$$

where $\lambda_\sigma \in F$ and S_d is the symmetric group of degree d. A nonzero polynomial $f = f(x_1, \ldots, x_d) \in F\langle X\rangle$ is said to be a *polynomial identity* of an F-algebra $\mathcal{A}$ if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in \mathcal{A}$. For example, $\mathcal{A}$ is a commutative algebra if and only if $[x_1, x_2] = x_1x_2 - x_2x_1$ is its polynomial identity. By the famous Amitsur–Levitzki theorem, the matrix algebra $M_n(F)$ has a polynomial identity of degree $2n$. On the other hand, $M_n(F)$ does not have polynomial identities of degree $< 2n$; the proof of that will be used in our arguing.

Let $F\langle X\rangle_0$ denote the subalgebra of $F\langle X\rangle$ generated by 1 and all polynomials of the form $[x_{k_1}, [x_{k_2}, \ldots, [x_{k_{r-1}}, x_{k_r}] \ldots]]$. That is to say, $F\langle X\rangle_0$ is the subalgebra generated by 1 and all Lie polynomials of degree ≥ 2. Defining the partial derivative $\frac{\partial f}{\partial x_i}$ of $f \in F\langle X\rangle$ in a self-explanatory manner it is easy to see that $\frac{\partial f}{\partial x_i}$ is always 0 if $f \in F\langle X\rangle_0$. Moreover, if $\mathrm{char}(F) = 0$, then this property is characteristic for elements from $F\langle X\rangle_0$ [12, Proposition 3]. Note that if $f = f(x_1, \ldots, x_d)$ is a multilinear polynomial, its partial derivative can be simply obtained by formally replacing x_i by 1:

$$\frac{\partial f}{\partial x_i} = f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d).$$

2.2. The Platonov–Đoković theory

Let K be an algebraically closed field of characteristic 0. We will write M_n for $M_n(K)$. We have $M_n = M_n^0 \oplus K \cdot 1$, where 1 is the identity matrix, and M_n^0 is the space of all $x \in M_n$ with $\mathrm{tr}(x) = 0$. Let $O(n^2)$ be the subgroup of $GL(n^2)$ which preserves the nondegenerate symmetric bilinear form $\mathrm{tr}(xy)$, $x, y \in M_n$. The subgroup of $O(n^2)$ consisting of operators which fix the identity matrix 1 will be denoted by $O(n^2 - 1)$. The identity components of $O(n^2)$ and $O(n^2 - 1)$, i.e., subgroups consisting of matrices whose determinant is 1, will be denoted by $SO(n^2)$ and $SO(n^2 - 1)$, respectively.

By G we denote the subgroup of $GL(n^2)$ consisting of all similarity transformations $x \mapsto axa^{-1}$ with $a \in GL(n)$. Next, by P we denote the subgroup of $GL(n^2)$ which acts trivially on M_n^0 and M_n/M_n^0, and by Q the subgroup of $GL(n^2)$ which acts trivially on $K1$ and $M_n/K1$. Thus, Q consists of all transformations $x \mapsto x + f(x)1$, where f is a linear functional on M_n such that $f(1) = 0$. Let T denote the subgroup of $GL(n^2)$ which acts by scalar transformations on M_n^0 and $K1$, and set $T_1 = T \cap SL(n^2)$.

By τ we denote the transposition map. However, we will write x' for the transpose of x. Note that the group $GQT\langle\tau\rangle$ consists of all invertible linear transformations $\sigma : M_n \to M_n$ that take one of the forms $\sigma(x) = \alpha axa^{-1} + f(x)1$ or $\sigma(x) = \alpha ax'a^{-1} + f(x)1$, where $\alpha \in K^*$, $a \in GL(n)$, and f is a linear functional on M_n such that $f(1) \neq -\alpha$.

The algebra of all linear transformations on M_n can be identified with the tensor product algebra $M_n \otimes M_n^{opp}$, where M_n^{opp} is the opposite algebra of M_n, via the action $(a \otimes b)(x) = axb$, $a, b, x \in M_n$.

With respect to the notations just introduced, the following theorem can be extracted from [16, Theorems A and B].

Theorem 2.1 (Platonov–Đoković). *Let Γ be a proper connected algebraic subgroup of $SL(n^2)$, $n \neq 4$, containing G. Then Γ is one of the groups:*

(a) G, GQ, GT_1, GQT_1,
(b) $SO(n^2 - 1)$, $SO(n^2 - 1)T_1$, $SO(n^2 - 1)P$, $SO(n^2 - 1)Q$,
$SO(n^2 - 1)PT_1$, $SO(n^2 - 1)QT_1$, $SL(n^2 - 1)$, $SL(n^2 - 1)T_1$,
$SL(n^2 - 1)P$, $SL(n^2 - 1)Q$, $SL(n^2 - 1)PT_1$, $SL(n^2 - 1)QT_1$,
$tSO(n^2)t^{-1}$ *for some* $t \in T_1$,
(c) GP, GPT_1,
(d) $t\Big(SL(n) \otimes SL(n)^{opp}\Big)t^{-1}$ *for some* $t \in T_1$.

Moreover, if Γ is one of the groups listed in (a), *then its normalizer in $GL(n^2)$ is a subgroup of $GQT\langle\tau\rangle$.*

Let us point out that all groups listed in (b) contain $SO(n^2-1)$. For $tSO(n^2)t^{-1}$, $t \in T_1$, this can be easily checked, while for others this is entirely obvious. Conversely, only the groups from (b) contain $SO(n^2-1)$.

2.3. Main theorem

Let $f = f(x_1, \dots, x_d) \in F\langle X\rangle$ be a nonzero multilinear polynomial of degree d. Our goal is to show that under suitable assumptions a linear map $\phi : M_n(F) \to M_n(F)$ that preserves zeros of f is of the standard form (1). In the present setting this can be more specifically described as

$$\phi(x) = \alpha axa^{-1} + f(x)1 \quad \text{or} \quad \phi(x) = \alpha ax'a^{-1} + f(x)1, \tag{2}$$

where $\alpha \in F^*$, $a \in GL(n, F)$, and f is a linear functional on $M_n(F)$ such that $f(1) \neq -\alpha$.

If d was $\geq 2n$, then, by the Amitsur–Levitzki theorem, f could be a polynomial identity, making the assumption that ϕ preserves zeros of f meaningless. We will therefore assume that $d < 2n$. Further, we will assume that $n \neq 2, 4$. It is well-known that the $n = 2$ case must be excluded when dealing with the polynomial $x_1x_2 - x_2x_1$. On the other hand, it seems possible that the exclusion of $n = 4$ is unnecessary. We need it in order to apply Theorem 2.1. Another assumption that we have to require is that $\mathrm{char}(F) = 0$. This one is also used because of applying Theorem 2.1 and is possibly redundant. Further, we will assume that ϕ is bijective. This is a usual and certainly necessary assumption in this context (cf. [8] that deals with the polynomial $x_1x_2 - x_2x_1$ without assuming bijectivity). Finally, we will assume that $\phi(1) \in F \cdot 1$; the (un)necessity of this assumption will be discussed in the next subsection.

Let us make a few comments and introduce some notations before stating and proving the theorem. We have to warn the reader that, just as in [16], we are assuming a basic familiarity with the concepts related to linear algebraic groups. A good general reference is Borel's book [5].

We are going to consider a bijective linear map ϕ on $M_n(F)$ that preserves the set of zeros of f,

$$S_F = \{(a_1, \dots, a_d) \in M_n(F)^d \mid f(a_1, \dots, a_d) = 0\}.$$

This is an algebraic set. Indeed, considering S_F as a subset of $(F^{n^2})^d$, it is equal to the vanishing set of polynomials $\{f((x_{ij}^1), \dots, (x_{ij}^d))_{st} \mid 1 \leqslant s, t \leqslant n\}$. Using [16, Lemma 3] (or [11, Lemma 1]) it can be therefore deduced that ϕ^{-1} also preserves S_F, i.e., it also satisfies the condition we are interested in (and so, in fact, $\phi(S_F) = S_F$). Accordingly, the set of all linear maps satisfying this condition is an algebraic group. The goal of our theorem is to describe those of its elements that also preserve scalar matrices.

By K we denote an algebraic closure of F. Since $\phi \in GL(n^2, F) \subseteq GL(n^2)$ $(= GL(n^2, K))$ preserves S_F, it also preserves its Zariski closure S in K^{n^2d}. This is an algebraic set, and therefore, by the same argument as above,

$$\widetilde{G} = \{\psi \in GL(n^2, K) \mid \psi(S) \subseteq S\}$$

is a group (and $\psi(S) = S$ for every $\psi \in \widetilde{G}$). By M we denote the (algebraic) subgroup of $GL(n^2)$ consisting of all maps that preserve scalar matrices. Thus ϕ is contained in the algebraic group $\widetilde{G} \cap M$.

For every algebraic group L defined over F we denote by L_F the group of F-rational points of L. We have $(GQT)_F = G_FQ_FT_F$ and $(GQT\langle\tau\rangle)_F$ consists of elements in $GL(n^2, F)$ that are of the form (2); cf. [16, p. 176]. Thus, if one can establish that

$$\widetilde{G} \cap M \subseteq GQT\langle\tau\rangle, \tag{3}$$

then ϕ, which is defined over F, lies in $G_FQ_FT_F\langle\tau\rangle$ and is therefore of the standard form (2).

Note that S_F is invariant under the G_F-action given by

$$g \cdot (a_1, \dots, a_d) := (g(a_1), \dots, g(a_d)).$$

Hence its closure S is also invariant under G_F, so that $G_F \subseteq \widetilde{G}$. Since $\mathrm{char}(F) = 0$ and G is connected, the rational points G_F are Zariski-dense in G [5, Corollary 18.3]. From this one infers that $G = \overline{G_F} \subseteq \widetilde{G}$;

moreover, $G \subseteq \widetilde{G} \cap M$. In a similar fashion, by first noticing that S_F is closed under multiplication by nonzero scalars in F we see that $\widetilde{G} \cap M$ is closed under multiplication by nonzero scalars in K; that is, if $a \in \widetilde{G}$ and $\lambda \in K^*$, then $\lambda a \in \widetilde{G} \cap M$.

Let us also mention that if H is an arbitrary algebraic group, then its identity component (i.e., the connected component with respect to Zariski topology that contains the identity) is also an algebraic group, and moreover, it is a normal subgroup of H [5, Proposition 1.2].

We now have enough information to prove the following theorem.

Theorem 2.2. *Let F be a field with $\mathrm{char}(F) = 0$, let $f \in F\langle X\rangle$ be a multilinear polynomial of degree $d \geq 2$, and let $\phi : M_n(F) \to M_n(F)$ be a bijective linear map that preserves zeros of f and satisfies $\phi(1) \in F \cdot 1$. Assume that $n \neq 2, 4$ and $d < 2n$. Then ϕ is of the standard form (2).*

Proof. As noticed above, it suffices to establish (3). We claim that it is enough to prove

$$SO(n^2 - 1) \not\subseteq \widetilde{G}. \tag{4}$$

Indeed, assume (4) holds. Consider $H = (\widetilde{G} \cap M) \cap SL(n^2)$ and let H_1 be the identity component of H. Then H_1 is an algebraic group, it is connected, and, since $G \subseteq \widetilde{G} \cap M$, it contains G. Therefore H_1 is one of the groups listed in Theorem 2.1. As $H_1 \subseteq \widetilde{G} \cap M$ and (4) holds, we may exclude the possibilities listed in (b). Furthermore, as the groups from (c) and (d) are not contained in M, H_1 must be one of the groups listed in (a). Theorem 2.1 now tells us that the normalizer of H_1 in $GL(n^2)$ is a subgroup of $GQT\langle\tau\rangle$. Since H_1 is a normal subgroup of H it follows that H is contained in $GQT\langle\tau\rangle$. Now pick $\alpha \in \widetilde{G} \cap M$. As mentioned above, $\widetilde{G} \cap M$ is closed under multiplication by nonzero scalars. Therefore $\det(\alpha)^{-1}\alpha \in H \subseteq GQT\langle\tau\rangle$. As $GQT\langle\tau\rangle$ is also closed under multiplication by nonzero scalars it follows that $\alpha = \det(\alpha)\big(\det(\alpha)^{-1}\alpha\big) \in GQT\langle\tau\rangle$. This proves (3).

Thus, let us prove (4). Assume first that d is an even number. Set $k = \frac{d}{2} + 1$ and note that $k \leq n$. Consider the sequence of d matrix units

$$e_{11},\ e_{12},\ e_{22},\ e_{23},\ e_{33},\ e_{34}, \ldots, e_{k-1,k-1},\ e_{k-1,k}. \tag{5}$$

The product of these matrices in an arbitrary order except in the given one is equal to zero. Therefore, for an appropriate permutation $(a_1, \ldots, a_d)$ of the matrices (5) (corresponding to a nonzero coefficient of f) we have $f(a_1, \ldots, a_d) \neq 0$. Now define a linear transformation θ on $M_n(K)$ according to

$$\theta(e_{12}) = e_{21}, \quad \theta(e_{21}) = e_{12}, \quad \theta(e_{11}) = e_{33}, \quad \theta(e_{33}) = e_{11},$$

and θ fixes all other matrix units. A bit tedious but straightforward verification shows that θ lies in $SO(n^2 - 1)$. Now, θ maps the matrices from (5) into the matrices

$$e_{33},\ e_{21},\ e_{22},\ e_{23},\ e_{11},\ e_{34}, \ldots, e_{k-1,k-1},\ e_{k-1,k}.$$

Their product in an arbitrary order is 0, so that $f(\theta(a_1), \ldots, \theta(a_d)) = 0$. This implies that $\theta \notin \widetilde{G}$. Namely, if θ was in $\widetilde{G}$ then θ^{-1} would map S_F into S which is contained in the set of zeros of f. Thus (4) is proved in this case.

The case where d is odd requires only minor modifications. One has to consider the matrix units

$$e_{11},\ e_{12},\ e_{22},\ e_{23},\ \ldots, e_{k-1,k},\ e_{k,k},$$

where $k = \frac{d+1}{2} \leq n$, and then follow the above argument. □

2.4. Preserving scalar matrices

It seems plausible that the assumption from Theorem 2.2 that $\phi(1) \in F \cdot 1$ can be removed. To this end one should examine carefully the groups from (c) and (d). However, apparently this would require a detailed and tedious analysis making the proof much lengthier. We have therefore decided to omit this problem in its full generality here, and perhaps return to it in a more technical paper. We will now restrict our attention to polynomials from $F\langle X\rangle_0$, which are of special interest in view of [10].

For these polynomials the argument based on the Platonov–Đoković theory is rather short. However, we will use an alternative approach, based on the following elementary lemma which is perhaps of independent interest.

Lemma 2.3. *Let $f \in F\langle X\rangle$, where F is an arbitrary field, be a multilinear polynomial of degree d. Let $n \geq 2$ be such that $d < 2n$. If $c \in M_n(F)$ satisfies*

$$f(c, a_2, \ldots, a_d) = f(a_1, c, a_3, \ldots, a_d) = \cdots = f(a_1, \ldots, a_{d-1}, c) = 0 \tag{6}$$

for all $a_1, \ldots, a_d \in M_n(F)$, then $c \in F \cdot 1$.

Proof. Pick an arbitrary rank one idempotent $e \in M_n(F)$. Then the algebra $(1 - e)M_n(F)(1 - e)$ is isomorphic to $M_{n-1}(F)$, so it contains matrix units h_{ij}, $1 \leq i, j \leq n - 1$, i.e., elements satisfying $h_{ij}h_{kl} = \delta_{jk}h_{il}$ and $\sum_{k=1}^{n-1} h_{kk} = 1 - e$.

Without loss of generality we may assume that $x_1x_2 \ldots x_d$ is a monomial of f. We set $(s, t) := (\frac{d}{2} - 1, \frac{d}{2})$ if d is even and $(s, t) := (\frac{d-1}{2}, \frac{d-1}{2})$ if d is odd. In any case we have $t \leq n - 1$. Examining all possible monomials of f one easily notices that

$$e \cdot f(e, c, h_{11}, h_{12}, h_{22}, h_{23}, \ldots, h_{st}) \cdot h_{t1} = ech_{11}.$$

Since $f(e, c, h_{11}, h_{12}, h_{22}, h_{23}, \ldots, h_{st}) = 0$ by our assumption, we thus have $ech_{11} = 0$. Similarly, by permuting the h_{ij}'s, we see that $ech_{kk} = 0$ for every k. Accordingly, $ec(1 - e) = 0$. In a similar fashion, by using $f(a_1, \ldots, a_{d-2}, c, a_d) = 0$, we get $(1 - e)ce = 0$. Hence it follows that c commutes with every rank one idempotent e. But then $c \in F \cdot 1$. □

Corollary 2.4. *Let F be a field with $\mathrm{char}(F) = 0$, let $f \in F\langle X\rangle_0$ be a multilinear polynomial of degree $d \geq 2$, and let $\phi : M_n(F) \to M_n(F)$ be a bijective linear map that preserves zeros of f. Assume that $n \neq 2, 4$ and $d < 2n$. Then ϕ is of the standard form (2).*

Proof. In view of Theorem 2.2 it suffices to prove that $c := \phi(1)$ lies in $F \cdot 1$. This is an immediate consequence of Lemma 2.3. Namely, since $f \in F\langle X\rangle_0$ we have

$$f(1, b_2, \ldots, b_d) = f(b_1, 1, b_3, \ldots, b_d) = \cdots = f(b_1, \ldots, b_{d-1}, 1) = 0$$

for all $b_i \in M_n(F)$, and hence (6) follows. □

3. Some special polynomials

In this section we will consider some special multilinear polynomials

$$f(x_1, x_2, \ldots, x_d) = \sum_{\sigma \in S_d} \lambda_\sigma x_{\sigma(1)}x_{\sigma(2)} \ldots x_{\sigma(d)} \tag{7}$$

for which our problem can be handled in rather general classes of algebras. Specifically, we will consider polynomials f satisfying one of the following conditions:

$$\frac{\partial^{d-1} f}{\partial x_2 \partial x_3 \ldots \partial x_d} \neq 0, \tag{A}$$

$$\frac{\partial^{d-1} f}{\partial x_2 \partial x_3 \ldots \partial x_d} = 0 \quad \text{and} \quad \frac{\partial^{d-2} f}{\partial x_3 \partial x_4 \ldots \partial x_d} \neq 0, \tag{B}$$

$$\frac{\partial^{d-1} f}{\partial x_2 \partial x_3 \ldots \partial x_d} = 0 \quad \text{and} \quad f(x, x, \ldots, x, y) \neq 0. \tag{C}$$

The conditions (B) and (C) are independent. For example,

$$x_1x_2x_3 + x_3x_1x_2 + x_2x_3x_1 - x_2x_1x_3 - x_1x_3x_2 - x_3x_2x_1$$

(i.e., the standard polynomial of degree 3) satisfies (B) and does not satisfy (C), while

$$x_1(x_2x_3 - x_3x_2) - (x_2x_3 - x_3x_2)x_1$$

satisfies (C) and does not satisfy (B).

3.1. Polynomials satisfying (A)

We begin with an elementary lemma.

Lemma 3.1. *Let f be a multilinear polynomial satisfying (A). Suppose $\mathcal{A}$ is a unital algebra and $\phi : \mathcal{A} \to \mathcal{A}$ is a linear map preserving zeros of f and satisfying $\phi(1) \in F^*1$. If $a, b \in A$ are such that $ab = ba = 0$, then $\phi(a)\phi(b) + \phi(b)\phi(a) = 0$.*

Proof. Without loss of generality we may assume that $\phi(1) = 1$. Namely, if $\phi(1) = \lambda 1$ with $0 \neq \lambda \in F$, then we can replace ϕ by $\lambda^{-1}\phi$ which also preserves zeros of f and does map 1 into 1.

From $ab = ba = 0$ we infer

$$f(a, b, 1, \ldots, 1) = f(b, a, 1, \ldots, 1) = 0,$$

and hence

$$f(\phi(a), \phi(b), 1, \ldots, 1) = f(\phi(b), \phi(a), 1, \ldots, 1) = 0.$$

We write f as in (7). Note that (A) simply means that

$$\lambda := \sum_{\sigma \in S_d} \lambda_\sigma \neq 0.$$

Since

$$\begin{aligned} &f(\phi(a), \phi(b), 1, \ldots, 1) + f(\phi(b), \phi(a), 1, \ldots, 1) \\ &\quad = \sum_{\sigma^{-1}(1)<\sigma^{-1}(2)} \lambda_\sigma \phi(a)\phi(b) + \sum_{\sigma^{-1}(2)<\sigma^{-1}(1)} \lambda_\sigma \phi(b)\phi(a) \\ &\qquad + \sum_{\sigma^{-1}(1)<\sigma^{-1}(2)} \lambda_\sigma \phi(b)\phi(a) + \sum_{\sigma^{-1}(2)<\sigma^{-1}(1)} \lambda_\sigma \phi(a)\phi(b) \\ &\quad = \lambda\Big(\phi(a)\phi(b) + \phi(b)\phi(a)\Big), \end{aligned}$$

it follows that $\phi(a)\phi(b) + \phi(b)\phi(a) = 0$. $\square$

Recall that a *Jordan epimorphism* on an algebra $\mathcal{A}$ is a surjective linear map θ satisfying $\theta(a^2) = \theta(a)^2$ for every $a \in \mathcal{A}$.

Theorem 3.2. *Let f be a multilinear polynomial of degree $d \geq 2$ satisfying (A), and let $\mathcal{A}$ be a unital C^*-algebra. If a continuous surjective linear map $\phi : \mathcal{A} \to \mathcal{A}$ preserves zeros of f and satisfies $\phi(1) \in \mathbb{C}^* \cdot 1$, then ϕ is a scalar multiple of a Jordan epimorphism.*

Proof. The conclusion of Lemma 3.1 makes it possible for us to directly apply [2, Theorem 3.3]. The statement of this theorem together with a well-known fact that Jordan epimorphisms preserve unities [15, Corollary 3, p. 482] immediately gives the desired conclusion. $\square$

Corollary 3.3. *Assume the conditions of Theorem 3.2. If $\mathcal{A}$ is a prime algebra, then ϕ is a scalar multiple of either an epimorphism or an antiepimorphism.*

Proof. If $\mathcal{A}$ is prime, then epimorphisms or antiepimorphisms are the only Jordan epimorphisms by Herstein's theorem [14]. $\square$

3.2. Polynomials satisfying (B)

The treatment of (B) is similar to that of (A).

Lemma 3.4. *Let f be a multilinear polynomial satisfying* (B)*. Suppose $\mathcal{A}$ is a unital algebra and $\phi : \mathcal{A} \to \mathcal{A}$ is a linear map preserving zeros of f and satisfying $\phi(1) \in F^*1$. If $a, b \in A$ are such that $ab = ba = 0$, then $\phi(a)\phi(b) = \phi(b)\phi(a)$.*

Proof. We can reword (B) as

$$\sum_{\sigma \in S_d} \lambda_\sigma = 0 \quad \text{and} \quad \mu := \sum_{\sigma^{-1}(1) < \sigma^{-1}(2)} \lambda_\sigma \neq 0.$$

Therefore

$$\sum_{\sigma^{-1}(2) < \sigma^{-1}(1)} \lambda_\sigma = -\mu.$$

We may assume, for the same reason as in the proof of Lemma 3.1, that $\phi(1) = 1$. If $a, b \in \mathcal{A}$ are such that $ab = ba = 0$, then

$$f(a, b, 1, \ldots, 1) = 0,$$

and hence

$$f(\phi(a), \phi(b), 1, \ldots, 1) = 0.$$

Since

$$f(\phi(a), \phi(b), 1, \ldots, 1) = \sum_{\sigma^{-1}(1) < \sigma^{-1}(2)} \lambda_\sigma \phi(a)\phi(b) + \sum_{\sigma^{-1}(2) < \sigma^{-1}(1)} \lambda_\sigma \phi(b)\phi(a),$$

it follows that $\mu\Big(\phi(a)\phi(b) - \phi(b)\phi(a)\Big) = 0$, i.e., $\phi(a)$ and $\phi(b)$ commute. □

Theorem 3.5. *Let f be a multilinear polynomial of degree $d \geq 2$ satisfying* (B)*, and let $\mathcal{A}$ be a unital prime C^*-algebra that is not isomorphic to $M_2(\mathbb{C})$. If a continuous bijective linear map $\phi : \mathcal{A} \to \mathcal{A}$ preserves zeros of f and satisfies $\phi(1) \in \mathbb{C}^* \cdot 1$, then there exist $\alpha \in \mathbb{C}$, an automorphism or an antiautomorphism θ of $\mathcal{A}$, and a linear functional f on $\mathcal{A}$ such that $\phi(a) = \alpha\theta(a) + f(a)1$ for all $a \in \mathcal{A}$.*

Proof. Lemma 3.4 makes it possible for us to apply [2, Corollary 3.6], which immediately gives the result. □

3.3. Polynomials satisfying (C)

The condition (C) means that there exist $\lambda_1, \ldots, \lambda_d \in F$, not all zero, such that

$$\sum_{i=1}^{d} \lambda_i = 0 \quad \text{and} \quad f(x, x, \ldots, x, y) = \sum_{i=1}^{d} \lambda_i x^{d-i} y x^{i-1}.$$

The simplest case where $f = x_1x_2 - x_2x_1$ was considered in [6, Theorem 2]. This result was one of the earliest applications of functional identities. Incidentally, [6, Theorem 2] was used in the proof of [2, Corollary 3.6], and therefore indirectly also in the proof of Theorem 3.5. What we would now like to show is that using the advanced theory of functional identities one can handle, in a more or less similar fashion, a more general situation where f satisfies (C).

Functional identities can be informally described as identical relations on rings that involve arbitrary ("unknown") functions. The goal is to describe these functions, or, when this is not possible, to determine the structure of the ring in question. For a full account on functional identities, as well as to some other notions that will appear below, we refer to the book [7].

Theorem 3.6. *Let f be a multilinear polynomial of degree $d \geq 2$ satisfying (C), let $\mathrm{char}(F) \neq 2, 3$, and let $\mathcal{A}$ be a centrally closed prime F-algebra with $\dim_F \mathcal{A} > d^2$. If a bijective linear map $\phi : \mathcal{A} \to \mathcal{A}$ preserves zeros of f, then there exist $\alpha \in F$, an automorphism or an antiautomorphism θ of $\mathcal{A}$, and a linear functional f on $\mathcal{A}$ such that $\phi(a) = \alpha\theta(a) + f(a)1$ for all $a \in \mathcal{A}$.*

Proof. As $f(x, x, \ldots, x, x^2)$ is obviously 0 if f satisfies (C), we have

$$f(\phi(a), \phi(a), \ldots, \phi(a), \phi(a^2)) = 0$$

for all $a \in \mathcal{A}$, i.e.,

$$\sum_{i=1}^{d} \lambda_i \phi(a)^{d-i} \phi(a^2) \phi(a)^{i-1} = 0.$$

A complete linearization of this identity leads to a situation where [7, Theorem 4.13] is applicable under suitable assumptions on $\mathcal{A}$ and ϕ. In view of [7, Theorems 5.11 and C.2], these assumptions are fulfilled in our case since ϕ is surjective and $\dim_F \mathcal{A} > d^2$. The conclusion is that $\phi(ab + ba)$ is a quasi-polynomial. As $\mathrm{char}(F) \neq 2$, this is equivalent to the existence of $\lambda \in F$ and maps $\mu, \nu : A \to F$ (with μ linear) such that

$$\phi(a^2) = \lambda\phi(a)^2 + \mu(a)\phi(a) + \nu(a)$$

for every $a \in \mathcal{A}$. Since ϕ is also injective and $\mathrm{char}(F) \neq 3$, the result now follows from [6, Theorem 2]. $\square$

It is worth pointing out that all prime C^*-algebras are centrally closed [3, Proposition 2.2.10]. Let us also mention that infinite dimensional algebras are not excluded in Theorem 3.6; only algebras of “small" dimension $\leq d^2$ are.

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