Commutator theory, action groupoids, and an intrinsic Schreier–Mac Lane extension theorem

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Abstract

Beyond groups of automorphisms in the category $Gp$ of groups and Lie-algebras of derivations in the category $K$-$Lie$ of Lie algebras, there are structures of internal groupoids (called action groupoids) in both categories. They allow a synthesis of the notion of obstruction to extensions. This leads, in any pointed protomodular category $C$ with split extension classifiers, to a general treatment of non-abelian extensions which can be understood as morphisms in a certain groupoid $\text{Tors} C$.

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0. Introduction

It is classical [25] that any extension of groups:

$$1 \longrightarrow K \overset{k}{\longrightarrow} X \overset{f}{\longrightarrow} Y \longrightarrow 1$$

determines, via conjugation in the group $X$, a group homomorphism $\phi : Y \rightarrow \text{Aut} K / \text{Int} K$. It is called the abstract kernel of this extension and allows the recovery of the set $\text{Ext}_\phi (Y, K)$ of all isomorphic classes of extensions with abstract kernel $\phi$. With the combinatorial notion of...
obstruction, it was possible indeed to show that there is on Ext_φ(Y, K) a simply transitive action of the abelian group Ext_φ(Y, ZK), where ZK is the center of K and the ZK-module structure φ is given by the restriction φ(y) of the automorphism φ(y) to ZK. Conversely an abstract kernel φ was shown to have an extension if one of its obstructions is the cochain (in the group H^3(φ, ZK)) identically 0.

The recent introduction of the notion of split extension classifier, see [5,6], in the context of protomodular categories [10] now allows us to show that the previous result on extensions fully holds in any pointed protomodular category C with split extension classifiers, provided it is exact (in the sense of Tierney, see Section I.3.1 in [1]). On the one hand, this explains, among other things, the very well observed, but unexplained up to now, parallelism of treatment of cohomology theory for groups and for Lie K-algebras: both categories Gp and K-Lie are exact protomodular with split extension classifiers (the group of automorphisms and the Lie-algebra of derivations respectively). On the other hand, it is not only a simple generalization of the result in Gp, but also sheds new light on the structure of extensions: they will appear as morphisms in a certain groupoid Tors C, which will give a stronger meaning to the group action we recalled above. Let us briefly sketch the steps.

The first point is that the split extension classifier of an object X of C actually underlies a groupoid structure D_1X [3], the so-called action groupoid:

\[
\begin{array}{ccc}
D_1X & \xrightarrow{d_1} & DX \\
\delta_2 & \xrightarrow{d_0} & D_1X && D_1X & \xleftarrow{s_0} & DX.
\end{array}
\]

This groupoid retains all that remains of the abelianness of X: for instance, the kernel of its normalization j_X: X → DX is the center ZX of the object X. This implies in particular that the category C is arithmetical (i.e. has no non-trivial abelian object) if and only if the map j_X is a monomorphism for any object X. There are non-classical examples of such protomodular categories [4] given by the dual of some toposes [14]. In this restricted context, the treatment of extensions, on the model of groups with trivial center (i.e. X → Aut X is a monomorphism), is fairly easy; there is at most one object in Ext_φ(Y, K) (see [3]).

The second point, here in the non-arithmetical context, will be to point out the strong relationship between extension and commutator theory in the sense of [16]. Let us start with any extension (f, k) with abstract direction φ: Y → Q_K, where q: D_K → Q_K is the coequalizer of the pair (d_0, d_1) above (i.e. where Q_K plays the role of Aut K / Int K). The following pullback diagram:

\[
\begin{array}{ccc}
D_1K & \xrightarrow{d_1\phi} & D_1K \\
\delta_2 & \xrightarrow{d_0} & D_1K \\
D_K & \xrightarrow{d_0} & DK \\
\phi & \xrightarrow{q} & Q_K
\end{array}
\]
defines an internal groupoid $D_1\phi$ which will be the categorical expression of the obstruction. We can then show that the extension $(f, k)$ gives rise to a regular epic factorization $f_\phi : X \twoheadrightarrow D\phi$ such that $[R[f_\phi], R[f]] = 0$.

This leads to the third point. We call pretorsor any pair $X \ll W \rightarrow Y$ of regular epimorphisms such that $[R[f], R[g]] = 0$. As far as we know, this notion was introduced as a concept in [24], under the name of pregrouploid. It was further studied in [22] under the name of herdoid. Because of the strong relationship of this notion with the notion of (bi-)torsor and the too “static” name of pregrouploid, in contrast to the dynamic aspect we are going to emphasize hereafter, we preferred here the name pretorisor.

On the one hand, we notice that a pretorisor is nothing other than a special kind of profunctor (introduced in [2] under the name of “distributeur” and thought of as a “generalized functors”). On the other hand, we observe that:

1. pretorsors are stable under composition of profunctors
2. because of the symmetrical definition, any pretorisor $(f, g)$ defines a pretorisor $(f, g)^*$ in the inverse direction which is actually the inverse (with respect to composition) of the pretorisor $(f, g)$ in question.

This means that pretorsors make up a bigroupoid, whose isomorphism classes produce a groupoid denoted by $\text{Tors}_C$.

Thanks to the second point above associating with any extension $(f, k)$ the pretorisor $(f_\phi, f)$, we get that $\text{Ext}_{\phi}(Y, K)$ is nothing other than a certain “hom,” namely $\text{Tors}_C(D_1\phi, E_1\phi)$, in the groupoid $\text{Tors}_C$. In the same way, $\text{Ext}^\phi_\phi(Y, ZK)$ is nothing other than the “hom” $\text{Tors}_C(E_1\phi, E_1\phi)$ of endomaps at $E_1\phi$. This last “hom” is naturally endowed with a group structure since $\text{Tors}_C$ is a groupoid; and obviously it has a canonical simply transitive action on $\text{Tors}_C(D_1\phi, E_1\phi)$. This is the point which organizes the structure of the set $\text{Ext}_{\phi}(Y, K)$ when it is non-empty.

We recalled above the cohomological cochain condition, in the category $\text{Gp}$, related to the existence of an extension with a given abstract kernel $\phi$. We shall give here an interpretation dealing with the second cohomology group (and not the third): the shifting of grading is explained by the fact that the answer in $\text{Gp}$, as described in [25], was relative to the forgetful functor $U : \text{Gp} \rightarrow \text{Set}$, while our answer is intrinsic to the category $C$ itself.

There are different ways to realize cohomology groups. One way is by means of simplicial objects as in [19,20]. But also it can be done by means of $n$-groupoids [9]: any abelian group $A$ in a finitely complete category $\mathcal{E}$ determines an abelian group $K_n(A)$ inside the category $n$-Grd$\mathcal{E}$ of internal $n$-groupoids in $\mathcal{E}$, and, when $\mathcal{E}$ is exact, the $(n + 1)$-th group $H^{n+1}_\mathcal{E}A$ can be realized by the component classes of certain $K_n(A)$-torsors, i.e. of certain $n$-groupoids endowed with $K_n(A)$ actions. In particular, the elements of the group $H^2_{\mathcal{E}}\mathcal{X}_1(A)$ are nothing other than the component classes of aspherical groupoids $\mathcal{X}_1$ (i.e. connected groupoids such that $X_0$ has a global support) with global direction $K_1(A)$. Accordingly, the groupoid $D_1\phi$ appears in a very natural way as an element of the cohomology group $H^2_{\mathcal{C}/\mathcal{Y}}E_1\phi$. We show that the triviality of this element guarantees, according to the classical model, the existence of an extension with abstract direction $\phi$. 
1. Commutator theory and torsor

Given any efficiently regular Mal’cev category $\mathbb{E}$ (see Definition 1.2), the aim of this section is to introduce and investigate the groupoid $\text{Tors} \mathbb{E}$.

1.1. Internal groupoids

Let $\mathbb{E}$ be a finitely complete category, and $\text{Grd} \mathbb{E}$ denote the category of internal groupoids in $\mathbb{E}$. An internal groupoid $Z_1$ in $\mathbb{E}$ will be presented (see [8]) as a reflexive graph $Z_1 \Rightarrow Z_0$ endowed with an operation $\zeta_2$:

![Diagram](Diagram.png)

making the previous diagram satisfy all the simplicial identities (including the ones involving the degeneracies). By $R[z_0]$ we denote the kernel equivalence relation of the map $z_0$. In the set theoretical context, this operation $\zeta_2$ associates the composite $\psi.\phi^{-1}$ with any pair $(\phi, \psi)$ of arrows with same domain. The groupoid $Z_1$ is said to be totally disconnected when we have $z_0 = z_1$, i.e. when it is the same thing as an internal group inside the category $\mathbb{E}/Z_0$. Recall that an internal functor $f_1 : Z_1 \rightarrow W_1$ is a discrete fibration between two groupoids whenever any commutative square of the underlying diagram defining the functor $f_1$ is a pullback.

We denote by $(\cdot)_0 : \text{Grd} \mathbb{E} \rightarrow \mathbb{E}$ the forgetful functor: it is a fibration. Any fiber $\text{Grd}_X \mathbb{E}$ above an object $X$ has an initial object $\Delta_X$, namely, the discrete equivalence relation on $X$, and a final object $\nabla_X$, namely, the indiscrete equivalence relation on $X$.

1.2. Connected equivalence relations

Let $R$ and $S$ be two equivalence relations on an object $X$ in any finitely complete category $\mathbb{E}$. Let us recall the following definition from [16]:

**Definition 1.1.** A connector on the pair $(R, S)$ is a morphism

$$p : R \times_X S \rightarrow X, \quad (xRySz) \mapsto p(x, y, z)$$

which satisfies the identities:

1. $xSp(x, y, z)$,
2. $p(x, y, y) = x$,
3. $p(x, y, p(y, u, v)) = p(x, u, v)$,

1’. $zRp(x, y, z)$,
2’. $p(y, y, z) = z$,
3’. $p(p(x, y, u), u, v) = p(x, y, v)$. 

In set theoretical terms, Condition 1 means that, with any triple \( x R y S z \), we associate a square:

\[
\begin{array}{c}
\xymatrix{ 
\ast 
\ar[r]^{S} 
\ar[d]_{R} & p(x, y, z) 
\ar[d]^{R} 
\ar[r]_{y} & \ast 
\ar[r]_{z} & \ast 
}
\end{array}
\]

More acutely, any connected pair produces the following double equivalence relation where any commutative square is a pullback. This means that any of the associated internal functors is a discrete fibration:

\[
\begin{array}{c}
\xymatrix{ 
R \times_X S 
\ar[r]^{p_1} 
\ar[d]_{(d_0, p_0, p)} & S 
\ar[d]_{d_1} 
\ar[r]_{d_0} & X.
}
\end{array}
\]

**Examples.**

1. A typical example of connector is produced by a given discrete fibration \( f_1 : R \to Z_1 \), where \( R \) is an equivalence relation. For that consider the following diagram:

\[
\begin{array}{c}
\xymatrix{ 
R[f_1] 
\ar[r]^{p_1} 
\ar[d]_{R(d_0)} & R 
\ar[r]^{f_1} & Z_1 
\ar[d]_{d_1} 
\ar[r]_{d_0} & Z_1 
\end{array}
\]

It is clear that, since \( f_1 \) is a discrete fibration, \( R[f_1] \) is isomorphic to \( R[f_0] \times_X R \) and that the map

\[
p : R[f_1] \to R \quad d_1
\]

determines a connector.

2. Given any groupoid \( Z_1 \), we do have such a discrete fibration \( R[z_0] \to Z_1 \):

\[
\begin{array}{c}
\xymatrix{ 
R[z_0] 
\ar[r]^{\xi_2} 
\ar[d]_{p_1} & Z_1 
\ar[d]_{p_0} 
\ar[r]^{z_1} & Z_0.
}
\end{array}
\]
This implies that we have a connector on the pair \((R[z_0], R[z_1])\). The converse is true as well, see [16,18]; given a reflexive graph:

\[
\begin{array}{c}
Z_1 \\
\downarrow^z_{z_0} \\
\downarrow^z_{z_0} \\
Z_0
\end{array}
\]

any connector on the pair \((R[z_0], R[z_1])\) determines a groupoid structure.

Let us recall also:

**Proposition 1.1.** Suppose \(p\) is a connector for the pair \((R, S)\). Then the following reflexive graph underlies a groupoid we shall denote by \(R \triangleright S\):

\[
R \times_X S \xrightarrow{\ C} X
\]

When \(R \cap S = \Delta X\), the groupoid \(R \triangleright S\) is an equivalence relation.

**Proof.** Thanks to the Yoneda embedding, it is enough to prove it in \(Set\). This is straightforward by just setting:

\((zRuSv)(xRySz) = xRp(u, z, y)Sv\).

The inverse of the arrow \(xRySz\) is \(zRp(x, y, z)Sx\). \(\square\)

Next we have from [7]:

**Proposition 1.2.** Given any discrete fibration \(f_1 : R \rightarrow Z_1\), the associated internal functor \(\mathcal{R}[f_0] \triangleright R \rightarrow R \rightarrow Z_1\) is fully faithful (or, in other words, cartesian with respect to the fibration \((\mathcal{R})_0 : Grd \mathcal{E} \rightarrow \mathcal{E}\)).

Recall that a category \(\mathcal{E}\) is Mal’cev [17,18] when it is finitely complete and such that any reflexive relation is actually an equivalence relation. The category \(Grp\) of groups and \(R\)-Lie of Lie \(R\)-algebras are Mal’cev. When \(\mathcal{E}\) is finitely complete, the category \(Grp \mathcal{E}\) of internal groups in \(\mathcal{E}\) is Mal’cev. In any Mal’cev category, the conditions (2) and (2’) imply the others, and moreover a connector is necessarily unique when it exists. Accordingly, the existence of a connector becomes a property for the pair \((R, S)\). Moreover, in a Mal’cev category, any internal category is necessarily an internal groupoid.

**Example 1.1.** By Propositions 3.6, 2.12 and Definition 3.1 in [26], two relations \(R\) and \(S\) in a Mal’cev variety \(\mathcal{V}\) are connected if and only if \([R, S] = 0\) (see [28]). Accordingly, when \(\mathcal{E}\) is Mal’cev, we shall write \([R, S] = 0\) whenever \(R\) and \(S\) are connected equivalence relations.
Example 1.2. We say that a map \( f : X \to Y \) has an abelian kernel relation when \([R[f], R[f]] = 0\). It has a central kernel relation when \([\nabla X, R[f]] = 0\). An object \( X \) is said to be abelian when \([\nabla X, \nabla X] = 0\).

1.3. Pretorsors

We shall need now \( \mathbb{E} \) efficiently regular. Let us recall the following:

**Definition 1.2.** A category \( \mathbb{E} \) is said to be regular when the regular epimorphisms are stable under pullbacks and the effective equivalence relations have quotients [1]. It is said to be efficiently regular [11] when, moreover, any equivalence relation \( T \) on an object \( X \) which is a subobject \( j : T \to R \) of an effective equivalence relation on \( X \) by an effective monomorphism in \( \mathbb{C} \) (which means that \( j \) is the equalizer of some pair of maps in \( \mathbb{C} \)), is itself effective.

Efficiently regular categories are stable under formation of slice and coslice categories. The category \( \mathbb{GpTop} \) (respectively \( \mathbb{AbTop} \)) of (respectively abelian) topological groups is efficiently regular, but not exact. A finitely complete regular additive category \( \mathbb{A} \) is efficiently regular if and only if the kernel maps are stable under composition. The main point here is that when \( \mathbb{E} \) is efficiently regular and when there is a discrete fibration \( S \to R \) between two equivalence relations, then \( S \) is effective as soon as \( R \) is effective. Let us introduce now the first of our main tools, see also [24] (pregroupoid) and [22] (herdoid), and more recently [23,27].

**Definition 1.3.** A pretorsor in an efficiently regular category \( \mathbb{E} \) is a pair of regular epimorphisms \( X \xleftarrow{f} W \xrightarrow{g} Y \) such that \([R[f], R[g]] = 0\).

Now consider the following diagram where the upper left-hand part is the double equivalence relation determined by the connector on the pair \((R[f], R[g])\):

\[
\begin{array}{ccc}
R[g] & \times_W R[f] & \xrightarrow{p_1} R[f] \\
p_0 & \pi_1 & \pi_0 \\
R[g] & W & \xrightarrow{g} Y \\
f_1 & d_1 & y_0 & y_1 \\
X_1 & \xrightarrow{x_0} X & \xrightarrow{f} Y \\
& d_0 & & \\
\end{array}
\]

Since any functor associated with this double equivalence relation is a discrete fibration, the upper horizontal equivalence relation and the vertical one on the left-hand side are effective and admits quotients, respectively \( g_1 \) and \( f_1 \). They produce two reflexive graphs \( Y_1 \) and \( X_1 \). These graphs are actually underlying groupoids since \( R[f] \) and \( R[g] \) are groupoids, see also [24]. Let
us denote by $\text{PrtE}$ the category of pretorsors, where a morphism of pretorsors is given by a commutative diagram:

\[
\begin{array}{ccc}
X & \xleftarrow{f} & W \\
\downarrow x & & \downarrow w \\
X' & \xleftarrow{f'} & W'.
\end{array}
\begin{array}{ccc}
& & \xrightarrow{g} \\
& & \downarrow y \\
& & Y
\end{array}
\begin{array}{ccc}
& & \xleftarrow{g'} \\
& & \downarrow y' \\
& & Y'.
\end{array}
\]

The previous diagrammatic construction gives rise to a pair of functors:

\[
\begin{array}{c}
\text{PrtE} \\
\xleftarrow{\delta_1} \\
\xrightarrow{\delta_0} \\
\text{GrdE}
\end{array}
\]

with $\delta_0(f, g) = X_1$ and $\delta_1(f, g) = Y_1$. On the other hand, we have a functor

\[
j : \text{GrdE} \to \text{PrtE};
\]

\[
Z_1 \mapsto (z_0, z_1)
\]

such that $\delta_0.j = 1 = \delta_1.j$ thanks to the following diagram associated with the groupoid $Z_1$ in question:

\[
\begin{array}{ccc}
R^2[z_0] & \xleftarrow{s_1} & R[z_0] \\
p_0 & \xleftarrow{p_1} & p_0 \\
R[z_0] = R[z_1] & \xleftarrow{s_0} & Z_1 \\
p_0 & \xleftarrow{p_1} & p_0 \\
Z_1 & \xleftarrow{s_0} & Z_0.
\end{array}
\]

Whence we obtain a reflexive graph in $\text{Cat}$:

\[
\begin{array}{c}
\text{PrtE} \\
\xleftarrow{j} \\
\xrightarrow{\delta_0} \\
\text{GrdE}
\end{array}
\]

The major aim of our first section will be to show that this reflexive graph underlies a bigroupoid structure. We shall denote now a pretorsor in the following way: $X_1^{(f,g)} \sim Y_1$.

**Example 1.3.** If $X$ and $Y$ have global supports, then we have $\nabla X^{(p_X,p_Y)} \sim \nabla Y$. 
Example 1.4. With any regular epimorphism $f : X \to Y$ is associated a pretorsor $R[f] \cong \Delta Y$.

Example 1.5. In [11], with any abelian object $X$ in $\mathbb{E}$ having global support, was associated an abelian group object $dX$, called the direction of $X$:

$$
\begin{array}{cccccc}
X \times X \times X & \xleftarrow{p_2} & X \times X & \xrightarrow{q_X} & dX \\
p_0 & \downarrow s_0 & (p_0.p_0.\pi) & \downarrow s_0 & \eta_X \\
X \times X & \xleftarrow{p_0} & X & \xrightarrow{1} & \\
\end{array}
$$

where $q_X$ is the quotient of the upper equivalence relation determined by the internal Mal’cev operation $\pi$ associated with the connector on the pair $(\nabla X, \nabla X)$. Actually any abelian object with global support produces a pretorsor

$$
K_1 dX \cong \tau_X \Rightarrow \tau_X 
$$

where $\tau_X : X \to 1$ is the terminal map and $K_1 dX$ denotes the (totally disconnected and with only one object) groupoid determined by the internal group $dX$.

Proof. The map $q_X$ which is the coequalizer of the upper pair

$$
\begin{array}{cccccc}
X \times X \times X & \xleftarrow{p_2} & X \times X & \xrightarrow{q_X} & dX \\
p_0 & \downarrow (p_0.p_0.\pi) & \downarrow p_0 & \eta_X \\
X \times X & \xleftarrow{p_0} & X & \xrightarrow{1} & \\
\end{array}
$$

is still the coequalizer of the left-hand side pair, since the isomorphism

$$
\gamma : X \times X \times X \to X \times X \times X;
$$

$$(x, y, z) \mapsto (x, p(x, y, z), z)$$

is such that $p_2.\gamma = (\pi, p_1.p_0)$ and $(p_0.p_0.\pi).\gamma = p_0$. $\square$

We shall need also:

Proposition 1.3. Suppose $\mathbb{E}$ is efficiently regular. Suppose moreover $(1_X, \theta, 1_Y) : (f, g) \to (f', g')$ is a morphism of pretorsors such that $\delta_0((1_X, \theta, 1_Y)) = 1_{X_1}$ and $\delta_1((1_X, \theta, 1_Y)) = 1_{Y_1}$. Then $\theta$ is an isomorphism in $\mathbb{E}$.
Proof. Consider the following diagram:

$$
\begin{array}{c}
\text{Proof. Consider the following diagram:}
\\
\begin{array}{ccc}
\text{R[}f\text{]} & \xrightarrow{g_1} & \text{R[}f'\text{]} & \xrightarrow{g'_1} & Y_1 \\
\\
\text{d_0} & & \text{d_1} & & \text{d_0}
\\
W & \xrightarrow{\theta} & W' & \xrightarrow{g_0'} & Y_0.
\\
\text{f} & & \text{f'} & & \text{1}_X
\\
X & & X
\end{array}
\end{array}
$$

The functor $R[f] \to R[f']$ determined by $\theta$ is a discrete fibration since both $g_1$ and $g'_1$ are. According to the Barr–Kock theorem in regular categories, the lower square is a pullback, which implies that $\theta$ is an isomorphism. □

1.4. Correlation between domain and codomain

Our present aim will be to show how strongly the domain and the codomain of a pretorsor are correlated. Let us begin by the following very general lemma:

Lemma 1.1. Let $\mathcal{E}$ be a finitely complete category, $R$ an equivalence relation on an object $X$ and $f_1 : R \to \mathcal{Z}_1$ any discrete fibration. When $\mathcal{Z}_1 = S$ is itself an equivalence relation, then $R[f_0] \cap R = \Delta X$. If moreover $\mathcal{E}$ is regular and $f_0$ is a regular epimorphism, the converse is true; namely, $\mathcal{Z}_1$ is an equivalence relation when $R[f_0] \cap R = \Delta X$.

Proof. Thanks to the Yoneda embedding, it is sufficient to prove the first assertion in Set. Suppose $xRx'$ and $f_0(x) = f_0(x')$. Then $f_1(x, x')$ is an endomap. Since $\mathcal{Z}_1 = S$, this endomap is necessarily $1_{f_0(x)}$. Since $f_1$ is a discrete fibration and we have also $f_1(x, x) = 1_{f_0(x)}$, then we have certainly $x = x'$. Thanks to the Barr embedding, the converse can be proved in Set. Suppose we have two parallel maps $(\beta_1, \beta_2) : z \Rightarrow z'$ in $\mathcal{Z}_1$. Since $f_0$ is a regular epimorphism, there is an $x \in X$ such that $f_0(x') = z'$. Since $f_1$ is a discrete fibration, there are $x_i$ such that $x_iRx'$ and $f(x_i, x') = \beta_i$, which implies $f_0(x_i) = z$. Accordingly, we have $x_1Rx_2$ and $\beta_2.f_1(x_1, x_2) = \beta_1$. But $R[f_0] \cap R = \Delta X$. So we have $x_1 = x_2$, and $\beta_1 = \beta_2$. Accordingly, the groupoid $\mathcal{Z}_1$ is an equivalence relation. □

Proposition 1.4. Let $(f, g)$ be a pretorsor in an efficiently regular Mal’cev category $\mathcal{E}$. Then $\delta_0(f, g)$ is an equivalence relation if and only if $\delta_1(f, g)$ is an equivalence relation. This is the case if and only if $(f, g)$ is a relation, i.e. the pair $(f, g)$ is jointly monic.

Proof. This is a straightforward consequence of the previous lemma, since the maps $f$ and $g$ are regular epimorphisms, and since the pair $(f, g)$ is jointly monic if and only if $R[f] \cap R[g] = \Delta X$. □
Now let \( Z_1 \) be any internal groupoid. When \( E \) is Mal’cev and regular, the canonical decomposition of the map \((z_0, z_1) : Z_1 \to Z_0 \times Z_0\) gives rise to an equivalence relation \( \Sigma Z_1:\n\)

\[
Z_1 \to \Sigma Z_1 \to Z_0 \times Z_0
\]

which is nothing other than the support of the object \( Z_1 \) in the fiber \( \text{Grd}_{Z_0}E \) with respect to the fibration \((0)_0 : \text{Grd}E \to E\).

**Definition 1.4.** The groupoid \( Z_1 \) is said to have effective support when the equivalence relation \( \Sigma Z_1 \) is effective (i.e. the kernel relation of some map).

It is clear that when the Mal’cev category \( E \) is not only efficiently regular, but also exact, any groupoid has effective support. When \( Z_1 \) is a groupoid with effective support, we denote by \( q_{Z_1} : Z_0 \to \pi_0 Z_1 \) the quotient of this effective support.

**Proposition 1.5.** Suppose \( E \) is Mal’cev and efficiently regular. Let \((f, g)\) be a pretorsor. Then \( X_1 \) has effective support if and only if \( Y_1 \) has effective support. If \( Y_1 \) has effective support, there is unique dotted arrow which completes the following square:

\[
\begin{array}{ccc}
W & \xrightarrow{g} & Y \\
\downarrow{f} & & \downarrow{q_{Y_1}} \\
X & \xrightarrow{\pi_0 X_1} & \pi_0 Y_1.
\end{array}
\]

It is the quotient map \( q_{X_1} \), and it produces a regular pushout (i.e. such that the factorization of the pair \((f, g)\) through the pullback is a regular epimorphism).

**Proof.** Suppose the groupoid \( Y_1 \) has effective support. Let us consider the following diagram:

\[
\begin{array}{ccc}
R[g] \times_W R[f] & \xrightarrow{p_1} & R[f] \\
\downarrow{p_0} & & \downarrow{g_1} \\
R[g] & \xrightarrow{\pi_1} & Y_1 \\
\downarrow{f_1} & & \downarrow{\sigma_{Z_1}} \\
X_1 & \xrightarrow{\pi_0 X_1} & \Sigma Y_1
\end{array}
\]

There is a regular epimorphism \( q \) which completes the square since both \( f \) and \( q_{Z_1} \) are coequalizers. Since \( f_1 \) is a regular epimorphism, the map \( q \) coequalizes the pair \((x_0, x_1)\), which determines a factorization \( \sigma : X_1 \to R[q] \). The fact that \( g_1 \) is a regular epimorphism implies that \( q \) is actually
the coequalizer of the pair \((x_0, x_1)\). On the other hand, the factorization \(R(g) = \sigma Z_1, g_1\) is also a regular epimorphism. Next consider the following diagram:

\[
\begin{array}{ccccccc}
R[R(g)] & \xrightarrow{d_1} & R[f] & \xrightarrow{d_1} & \Sigma Y_1 \\
\downarrow{R(d_0)} & & \downarrow{R(d_1)} & & \downarrow{d_1} \\
R[g] & \xrightarrow{s_0} & W & \xrightarrow{g} & Y \\
\downarrow{R(f)} & & \downarrow{d_0} & & \downarrow{d_0} \\
R[q] & \xrightarrow{d_1} & X & \xrightarrow{q} & \pi_0 Y_1. \\
\downarrow{d_0} & & \downarrow{d_0} & & \downarrow{d_0} \\
Z_0 & \xrightarrow{\pi_0 Z_1} & \pi_0 Z_1.
\end{array}
\]

According to Theorem 3.1 (= denormalized 3 × 3 lemma) in [13], the factorization \(R(f) = \sigma . f_1\) is still a regular epimorphism, and consequently \(\sigma\) is a regular epimorphism. This implies first that \(R[q]\) is the support of the groupoid \(X_1\) (which has thus an effective support), and secondly that we have \(q = q X_1\). Moreover, following this same theorem, the square completed by \(q\) is a regular pushout. □

Recall that, in a Mal’cev category \(E\), any internal groupoid is abelian [12] and that, when \(E\) is moreover efficiently regular, a direction was defined for any aspherical groupoid (i.e. connected groupoid with object of objects having global support). It becomes now a particular case of the following generalization:

**Definition 1.5.** Suppose \(E\) is Mal’cev and efficiently regular. The global direction \(d_1 Z_1\) of a groupoid \(Z_1\) with effective support is the totally disconnected groupoid produced on the right-hand side by the following pushout (which necessarily exists in the Mal’cev efficiently regular category \(E\)):

\[
\begin{array}{cccc}
R[(z_0, z_1)] & \longrightarrow & d_1 Z_1 \\
p_0 \downarrow{p_0} & & \downarrow{p_1} \\
Z_1 & \longrightarrow & \pi_0 Z_1. \\
\downarrow{(z_0, z_1)} & & \downarrow{Z_0} \times Z_0
\end{array}
\]

Notice that the maps \(p_0\) and \(p_1\) produce the same retraction of \(\pi_0 Z_1 \rightarrow d_1 Z_1\) since the lower horizontal map coequalizes these maps. Notice also that any of the two dotted downward squares are pullbacks. Next we have:

**Proposition 1.6.** Suppose \(E\) is Mal’cev and efficiently regular. Let \(f, g\) be a pretorsor such that \(Y_1\) has effective support. Then the global directions of \(X_1\) and \(Y_1\) are the same.
Proof. We just showed that when $Y_1$ has effective support, this is also the case for $X_1$ and that we have $\pi_0X_1 = \pi_0Y_1$. Moreover, according to Proposition 1.2, the functors $R[g] \circ R[f] \to Y_1$ and $R[g] \circ R[f] \to X_1$ are $\pi_0$-cartesian, which, according to Proposition 10 in [12] (applied in the category $E/\pi_0Y_1$), means that $Y_1$ and $X_1$ have the same global direction.

In the same way:

Proposition 1.7. Suppose $E$ is Mal’cev and efficiently regular. Let $(f, g)$ be a pretorsor. We get $R[f] \subseteq R[g]$ if and only if $Y_1$ is totally disconnected. Then $X_1$ has effective support, and $Y_1$ is the global direction of $X_1$.

Proof. Suppose $Y_1$ is totally disconnected. Since we have $y_0 = y_1$, the map $g$ coequalizes the equivalence relation $R[f]$. Accordingly, we get $R[f] \subseteq R[g]$. Conversely suppose we have $R[f] \subseteq R[g]$. Since $f$ is a regular epimorphism, we get a factorization $\tau : X \to Y$ such that $g = \tau \cdot f$. It is a regular epimorphism, since so is $g$. The end of the proof is a consequence of the following proposition applied to the category $D = E/Y$.

Proposition 1.8. Suppose $E$ is Mal’cev and efficiently regular. Let $f : W \to X$ be a regular epimorphism with central kernel relation such that $W$ has a global support. In other words, suppose $(f, \tau_W)$ is a pretorsor. Then $A_1 = \delta_1(f, \tau_W)$ is totally disconnected, and the groupoid $X_1 = \delta_0(f, \tau_W)$ has effective support. Moreover $A_1$ is the global direction of $X_1$.

Proof. Consider the following diagram:

Then we have $A_0 = 1$ and the groupoid $A_1$ is trivially totally disconnected. So it has effective support. And trivially, it is equal to its direction. According to Proposition 1.6, the groupoid $X_1$ has effective support and its direction is the same as that of $A_1$, namely $A_1$ itself.

1.5. Pretorsors and profunctors

Let $X_1$ and $Y_1$ be two internal categories in any finitely complete category $E$. Recall that a profunctor $X_1 \to Y_1$ is a discrete fibration above the category $X_1^{op} \times Y_1$. Equivalently, it is a pair $(f, g) : W \to X \times Y$ of maps together with a left action of $X_1$ on $f$ and a right action of $Y_1$ on $g$ which commute with each other. Profunctors were introduced in [2] under the name of
“distributeurs” and thought of as “generalized functors.” It is quite remarkable that the following diagram associated with the pretorsor \((f, g)\):

\[
\begin{array}{c}
\begin{array}{c}
R[g] \times_W R[f] \\
p_0 \downarrow \\
R[g]
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\pi_0 \\
d_0
\end{array} \\
\begin{array}{c}
\pi_1 \\
d_0
\end{array}
\end{array} \\
\begin{array}{c}
\begin{array}{c}
Y_1 \\
y_1
\end{array} \\
\begin{array}{c}
Y \\
g
\end{array}
\end{array}
\end{array}
\]

defines a profunctor \(X_1 \rightarrow Y_1\); since any lower square and any right-hand side square is a pullback, the map \(d_0 : R[g] \rightarrow W\) (respectively \(d_1 : R[f] \rightarrow W\)) underlies a left (respectively right) action of \(X_1\) on \(f\) (respectively of \(Y_1\) on \(g\)). The commutativity of the square:

\[
R[g] \times_W R[f] \xrightarrow{\pi_0} R[f] \\
\downarrow \pi_1 \\
R[g] \xrightarrow{d_0} W
\]

guarantees that the two actions commute.

When the ground category \(E\) has pullback stable coequalizers of reflexive pairs, there is a composition of profunctors. It is usually denoted by \(\otimes\). Let \(Y_1 \rightarrow Z_1\) be another profunctor given by left and right actions on \(Y \xleftarrow{h} V \xrightarrow{l} Z\). The top \(U\) of the pair \(X \xleftarrow{f} U \xrightarrow{g} Z\) underlying the composite \((h, l) \otimes (f, g)\) is the quotient of \(W \times_Y V\) by the equivalence relation \(T\) described in the following diagram:

\[
\begin{array}{c}
x \xleftarrow{w} y \\
x' \xrightarrow{w'} y'
\end{array}
\]

where \(x \rightarrow y\) denotes an object \(w \in W\) such that \(f(w) = x\) and \(g(w) = y\), and \(\gamma\) denotes an arrow in \(Y_1\). In other words \((w, v)T(w', v')\) if and only if there is an arrow \(\gamma\) in the category \(Y_1\) such that \(\gamma.w = w'\) and \(v'.\gamma = v\). Composition of profunctors is associative and unitary up to (coherent) isomorphism; see for instance [21] for the technical details.
Since pretorsors are special kinds of profunctors, we can compose them. Since moreover we are dealing with groupoids, the last equation also can be written \( v' = v.\gamma^{-1} \), and the composition is defined by the following diagram:

![Diagram]

By \( R^{op}[l] \) we mean the dual of the equivalence relation \( R[l] \). The two dotted pullbacks above actually underlie the following pullback (on the left-hand side) inside the category \( \text{Grd} \mathbb{E} \). It produces the equivalence relation \( S \) on the top \( W \times_Y V \) of the right-hand side pullback in \( \mathbb{E} \):

![Pullback Diagram]

The arrow \( t_w \) denotes the internal functor induced by the twisting isomorphism

\[
t_w : R[l] \to R[l]; \\
(v, v') \mapsto (v', v)
\]

The top groupoid \( S \) in the left-hand side pullback is an equivalence relation since it is the domain of a discrete fibration (\( p_{1V} : S \to R^{op}[l] \) is the pullback of the discrete fibration \( g_{1} \)) with codomain an equivalence relation. This equivalence relation \( S \) is given by the pair \( (\theta_0, \theta_1) \) and means that we have \( (w, v)S(w', v') \) if and only if there is a map \( \gamma \) in the category \( Y_1 \) such that \( w' = \gamma.w \) and \( v' = v.\gamma^{-1} \).

When moreover \( \mathbb{E} \) is efficiently regular, \( S \) is effective, since \( R^{op}[l] \) is effective and \( p_{1V} \) a discrete fibration. It admits a pullback stable quotient \( \theta : W \times_Y V \to U \) and two factorizations
\( \bar{f} : U \rightarrow X, \bar{l} : U \rightarrow Z \). These two maps complete two pullbacks since both functors \( \bar{p}_W \) and \( \bar{p}_V \) are discrete fibrations:

\[
\begin{array}{c}
W & \xleftarrow{p_W} & W \times_Y V & \xrightarrow{p_V} & V \\
\downarrow f & & \downarrow \theta & & \downarrow l \\
X & \xleftarrow{\bar{f}} & U & \xrightarrow{\bar{l}} & Z
\end{array}
\]

We clearly have \( [R[p_W], R[p_V]] = 0 \), since the pair \((p_W, p_V)\) comes from a pullback. Since \( f \) and \( \theta \) are regular epimorphisms, we have \( \theta(R[p_W]) = R[\bar{f}] \). Similarly we have \( \theta(R[p_V]) = R[\bar{l}] \). Accordingly, \([R[\bar{f}], R[\bar{l}]] = [\theta(R[p_W])], \theta(R[p_V])] = \theta([R[p_W], R[p_V]]) = 0 \), and

\[
(\bar{f}, \bar{l}) = (h, l) \otimes (f, g) : X_1 \rightarrow Z_1
\]

underlies a pretorsor. In other words:

**Proposition 1.9.** Suppose \( \mathbb{E} \) is Mal’cev and efficiently regular. Then pretorsors are stable under composition of profunctors.

We shall call this restricted composition the *Baer sum* of the two pretorsors. This terminology is justified by the fact that this composition extends the Baer sum of exact sequences with abelian kernel relation: an exact sequence with abelian kernel relation:

\[
1 \rightarrow A \xrightarrow{k} X \xrightarrow{f} Y \rightarrow 1
\]

can indeed be thought of as a pretorsor \( Y \xleftarrow{f} X \xrightarrow{f} Y \), since \([R[f], R[f]] = 0 \). See Section 3.1 below for the details about the composition.

The composition of profunctors determines a so-called bicategory [2] whose objects are the internal categories, whose maps are the profunctors. And 2-cells are the morphisms of profunctors. Let us denote by \( \mathbb{Prt}\mathbb{E} \) the full sub-bicategory of pretorsors.

**Theorem 1.1.** Suppose \( \mathbb{E} \) is Mal’cev and efficiently regular. The bicategory \( \mathbb{Prt}\mathbb{E} \) is actually a bigroupoid.
Proof. We must show that any pretorsor $X_1^{(f,g)} \Rightarrow Y_1$ has an inverse up to isomorphism. Once again let us consider the diagram associated with this pretorsor:

$$\begin{array}{ccc}
R[g] \times_W R[f] & \xrightarrow{p_1} & R[f] \\
p_0 & & d_0 \\
\downarrow & & \downarrow \\
R[g] & \xrightarrow{d_0} & W
\end{array}$$

Since any commutative square is a pullback, the map $d_0 : R[f] \rightarrow W$ produces a left action of $Y_1$ on $g$, while $d_1 : R[g] \rightarrow W$ produces a right action of $X_1$ on $g$. The commutativity of the square:

$$\begin{array}{ccc}
R[g] \times_W R[f] & \xrightarrow{p_1} & R[f] \\
p_0 & & d_0 \\
\downarrow & & \downarrow \\
R[g] & \xrightarrow{d_1} & W
\end{array}$$

guarantees that the two actions commute. We shall denote by $Y_1^{(f,g)^*} \Rightarrow X_1$ this pretorsor. Let us show that $(f, g)^* \otimes (f, g) \simeq 1_{X_1}$. First we are interested in the following pullback:

$$\begin{array}{ccc}
W \times_Y W = R[g] & \xrightarrow{d_1} & W \\
d_0 & & g \\
\downarrow & & \downarrow \\
W & \xrightarrow{g} & Y
\end{array}$$

and in the equivalence relation $S$ on this object. Since the category $\mathcal{E}$ is regular, the Barr embedding theorem allows us to argue as if we were in $Set$. This equivalence relation $S$ is described by the following diagram:
where \((w, \bar{w})S(w', \bar{w}')\) if and only if there is a map \(\gamma\) such that \(\gamma \cdot w = w'\) and \(\gamma^{-1} \cdot \bar{w}' = \bar{w}\), and consequently \(\bar{w}' = \gamma \cdot \bar{w}\). Accordingly, we get \(f_1(w, w') = f_1(\bar{w}, \bar{w}')\), and \(f_1 : R[g] \to X_1\) is the quotient of this equivalence relation. The commutativity of the following diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{d_0} & R[g] & \xrightarrow{d_1} & W \\
\downarrow f & & \downarrow f_1 & & \downarrow f \\
X & \xleftarrow{x_0} & X_1 & \xrightarrow{x_1} & X
\end{array}
\]

shows that \((f, g)^* \otimes (f, g) = (x_0, x_1) = 1_{X_1}\).

We shall denote by \(\text{Tors} E\) the groupoid associated with the previous bigroupoid. Its objects are the groupoids, its maps are the class of pretorsors up to the isomorphisms described in Proposition 1.3.

2. Split extension classifier and action groupoid

The aim of this section is to introduce and investigate the notion of split extension classifier. This will allow us, in the next section, to connect extensions with pretorsors.

2.1. Split extension classifier

We shall suppose now that \(\mathbb{C}\) is a pointed protomodular category \([10]\), which implies that \(\mathbb{C}\) is Mal’cev. All the examples of Mal’cev categories given above are actually protomodular, in particular the categories \(G_{\text{p}}\) and \(R\)-Lie. Recall the following definition from \([3,5,6]\):

**Definition 2.1.** An object \(X\) of the pointed protomodular category \(\mathbb{C}\) is said to have a *split extension classifier* when there is a split extension:

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma} & D_1X & \xleftarrow{d_0} & DX \\
\end{array}
\]

which is universal in the sense that any other split extension:

\[
\begin{array}{ccc}
X & \xrightarrow{k} & H & \xleftarrow{s} & G \\
\end{array}
\]

determines a unique pair of morphisms \((\chi, \chi_1)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{k} & H & \xleftarrow{s} & G \\
\downarrow 1_X & & \downarrow \chi_1 & & \downarrow \chi \\
X & \xrightarrow{\gamma} & D_1X & \xleftarrow{d_0} & DX.
\end{array}
\]
The category $\mathcal{C}$ will be said to be action representative when such a universal split extension exists for any object $X$.

Of course, the category $\mathcal{C}$ being protomodular and $1_X$ being an isomorphism, the right-hand side commutative square is necessarily a pullback. Moreover the map $\chi_1$ is uniquely determined by $\chi$, since the pair $(k, s)$ is jointly strongly epic (this is actually one of the equivalent definition of protomodularity). This map $\chi$ will be called the classifying map of the split extension.

**Examples.**

(1) A pointed protomodular category $\mathcal{C}$ is additive if and only if it is trivially action representative, in the sense that, for any object $X$, the split extension classifier does exist and is:

\[
\begin{array}{ccc}
X & \xrightarrow{1_X} & X \\
\downarrow{1_X} & \Downarrow{\tau_X} & \downarrow{\alpha_X} \\
0 & & 0
\end{array}
\]

(2) The protomodular category $Gp$ of groups is clearly action representative, with $DX = \text{Aut} X$ the group of automorphisms of $X$.

(3) The protomodular category $R$-$\text{Lie}$ of $R$-Lie algebras is also clearly action representative with $DX = \text{Der} X$ the $R$-Lie algebra of derivations of $X$.

(4) The dual of the category of pointed objects of certain (bi-Heyting) toposes is action representative, see [4].

Now let $X$ be any object with split extension classifier in $\mathcal{C}$. Evidently, each splitting of a given extension determines a different classifying map. Certainly the classifying map of the following split extension must be $0 : X \to DX$:

\[
\begin{array}{ccc}
X & \xrightarrow{r_X=(0,1)} & X \times X \\
\downarrow{l_X=(1,0)} & & \downarrow{p_0} \\
X & & X
\end{array}
\]

By $j_X : X \to DX$ we shall take to mean the classifying map of the following split extension, where, following the simplicial notations, $s_0$ is the diagonal:

\[
\begin{array}{ccc}
X & \xrightarrow{r_X} & X \times X \\
\downarrow{1_X} & & \downarrow{\tilde{j}_X} \\
X & \xrightarrow{\gamma} & D_1X \\
\downarrow{\gamma} & & \downarrow{d_0} \\
X & \xrightarrow{\gamma} & DX
\end{array}
\]

It makes the following diagram commute:
When \( C = Gp \), the map \( j_X \) is precisely the classical group homomorphism \( X \rightarrow \text{Aut} X \).

Actually the split extension classifier of the object \( X \) underlies an internal groupoid structure, see [3]. Here we briefly recall its construction:

**Theorem 2.1.** Let \( C \) be a pointed protomodular category and \( X \) any object with split extension classifier. Then this split extension classifier underlies a structure of groupoid \( D_1X \) such that \( d_1.\gamma = j_X \). We shall call it the action groupoid of the object \( X \).

**Proof.** Consider the following split extension, with \( R[d_0] \) the kernel relation of \( d_0 : D_1X \rightarrow DX \), and \( s_1 \) (according to the simplicial notations) the unique map such that \( p_0.s_1 = s_0.d_0 \) and \( p_1.s_1 = 1_{D_1X} \):

\[
\begin{array}{ccc}
X & \xrightarrow{s_1.\gamma} & R[d_0] \\
& & \xrightarrow{d_0} \\
& & s_0 \\
& & \xleftarrow{d_0} \\
D_1X & \xleftarrow{s_0} & DX.
\end{array}
\]

It determines a unique pair \((d_1, \delta_2)\) of arrows making the following commutative square a pullback:

\[
\begin{array}{ccc}
R[d_0] & \xrightarrow{\delta_2} & D_1X \\
\downarrow d_0 & & \downarrow d_0 \\
D_1X & \xrightarrow{d_1} & DX.
\end{array}
\]

Since any protomodular category \( C \) is Mal’cev, this is sufficient to produce the following groupoid \( D_1X \):

\[
\begin{array}{ccc}
R[d_0] & \xrightarrow{\delta_2} & D_1X \\
\downarrow d_0 & & \downarrow d_0 \\
D_1X & \xleftarrow{s_0} & DX.
\end{array}
\]

Moreover, we have \( d_1.\gamma = j_X \) since these two maps clearly classify the same split extension. \( \square \)

This last equation means that the map \( j_X : X \rightarrow DX \) is the normalization (see Appendix A) of the groupoid \( D_1X \), which is equivalent to saying that the following right-hand side square is a pullback:

\[
\begin{array}{ccc}
ZX & \xrightarrow{\kappa} & X \xrightarrow{\gamma} D_1X \\
\downarrow j_X & & \downarrow (d_0,d_1) \\
I & \xrightarrow{(0,1)} & DX \times DX.
\end{array}
\]

We denote by \( ZX \) the kernel of \( j_X \). Notice it is also, via \( \gamma.\kappa \), the kernel of \( (d_0,d_1) \).
**Example.** When $\mathbb{C} = \mathbb{A}$ is additive, the action groupoid associated with any object $X$ is nothing other than the canonical (internal) abelian group structure on $X$, namely:

$$X \times X \xrightarrow{\tau_X} X \xrightarrow{\alpha_X} 0.$$ 

When the category $\mathbb{C}$ is not additive, the split extension classifier still has a strong connexion with additivity, see [3]:

**Theorem 2.2.** Let $\mathbb{C}$ be a pointed protomodular category and $X$ any object with split extension classifier. The kernel relation $R[j_X]$ is the center of $X$, i.e. the greatest central equivalence relation on $X$. As a consequence, the kernel $Z_X$ of $j_X$ is the center of $X$.

This is a consequence of the fact that $j_X$ is the normalization of the action groupoid, i.e. of the fact that we have a discrete fibration $j_{1X} : \nabla X \to D_1 X$. The previous theorem provides a characterization of abelian objects in $\mathbb{C}$.

**Corollary 2.1.** Let $\mathbb{C}$ be a pointed protomodular category and $X$ any object with split extension classifier. Then the following conditions are equivalent:

1. the object $X$ is abelian;
2. the map $j_X$ is a zero map (or equivalently $R[j_X] = \nabla_X$);
3. $d_0 = d_1$, i.e. $D_1 X$ is totally disconnected.

### 2.2. The universal property of the action groupoid

Let us recall now the universal property of the action groupoid. Suppose we are given an internal groupoid:

$$
\begin{array}{ccc}
R[z_0] & \xrightarrow{p_1} & Z_1 \\
p_0 & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \Quad
There is a unique internal functor $\chi_1 = (\chi_0, \chi_1)$ such that $\chi_1.k_0 = \gamma$:

\[
\begin{array}{c}
R[z_0] \xrightarrow{z_1} Z_1 \xleftarrow{s_0} Z_0 \\
R(\chi_1) \xrightarrow{\delta_1} D_1X \xrightarrow{D_1} DX.
\end{array}
\]

Clearly this functor is a discrete fibration. Now suppose we are given a map $f : X \to Y$ and its kernel $k : K \to X$. Then the previous universal property determines a functor $(\tilde{k}, \tilde{k}_1) : R[f] \to D_1X$:

\[
\begin{array}{c}
R^2[f] \xrightarrow{p_1} R[f] \xrightarrow{p_0} X \\
R(\tilde{k}_1) \xrightarrow{\delta_1} D_1K \xrightarrow{D_1} DK.
\end{array}
\]

We can check that $\tilde{k}.k = j_K$ since the two maps classify the same split extension. When $C = Gp$, this map $\tilde{k}$ is the classical homomorphism $X \to \text{Aut} K$ associated with the normal subgroup $K \to X$.

**Lemma 2.1.** Suppose $C$ is action representative. Then $[R[f], R[\tilde{k}]] = 0$.

**Proof.** This follows from our typical example of connector, since the internal functor $\tilde{k}_1 : R[f] \to D_1K$ is a discrete fibration.

We have moreover:

**Proposition 2.1.** Suppose the category $C$ is action representative. Then a map $f : X \to Y$ has its kernel relation $R[f]$ abelian (or equivalently the map $f$, seen as an object in the slice category $C/Y$, is abelian) if and only if its kernel object $K$ is abelian. The kernel relation $R[f]$ is central if and only if its classifying map $\tilde{k}$ is 0.

**Proof.** The normalization of an abelian equivalence relation gives necessarily an abelian object. The converse is true in any action representative category, see Proposition 4.2 in [3]. Now
suppose $\tilde{k} = 0$. Then consider the right-hand side of the following diagram, completed by the horizontal kernel relations:

\[
\begin{array}{c}
R[\tilde{k}] \\ R(d_0) \\
\downarrow \ \\
X \times X \\
\downarrow \ \\
X
\end{array}
\]

The functor $\tilde{k}_1 : R[f] \to D_1 K$ is a discrete fibration. Accordingly, we have $[R[f], R[\tilde{k}]] = 0$. But $R[\tilde{k}] = R[0] = \nabla X$. Then $[R[f], \nabla X] = 0$, and $R[f]$ is central. Conversely suppose $R[f]$ central. Certainly $R[f]$ is abelian, and $K = A$ is abelian. Then consider the following diagram where the left-hand part is the centralizing double relation determined by the centrality of $R[f]$:

\[
\begin{array}{c}
X \times X \\
\downarrow \ \\
X
\end{array}
\]

Then $\tilde{k}$ coequalizes the two horizontal maps, since they both classify the same split extension, namely:

\[
\begin{array}{c}
A \\
\downarrow \ \\
X \times R[f] \\
\downarrow \ \\
X \times X
\end{array}
\]

Accordingly, $\tilde{k}$ factorizes through 1, and $\tilde{k} = 0$. □

2.3. The comparison $DX \to DZX$

What is rather awkward and uncomfortable is that the objects $DX$ (although having a universal property) do not give rise to any functorial process. However we have some very specific constructions. Let us denote by $\♭G_1 : G_1 \to \nabla G_0$ the terminal map in the fiber Grd$_G_0$ $C$. We shall need now to associate with any groupoid $G_1$ the part which only retains the “endomorphisms” of $G_1$; it is a totally disconnected groupoid given by the following pullback in the category Grd $C$ and called the endosome of the groupoid $G_1$, see [7]:

\[
\begin{array}{c}
\Delta G_0 \\
\downarrow \ \\
G_1 \\
\downarrow \ \\
\nabla G_0
\end{array}
\]

\[
\begin{array}{c}
En_1 G_1 \\
\downarrow \ \\
G_1 \ \\
\downarrow \ \\
\nabla G_0
\end{array}
\]
In particular, the object of morphisms of the groupoid $\text{En}_1 D_1 X$ is given by the following right-hand side pullback in $\mathbb{C}$:

$$
\begin{array}{ccc}
Z X & \xrightarrow{\theta X} & En_1 D_1 X \\
\downarrow & & \downarrow \delta X \\
1 & \xrightarrow{\sigma X} & DX \\
\end{array}
$$

We noticed that the kernel of $(d_0, d_1)$ is $\gamma \kappa$. Therefore there exists a unique map $\theta X$ which makes the left-hand square a pullback. We obtain in this way a split extension $(\theta X, \delta X)$ which determines a classifying map $\zeta : DX \rightarrow DZX$:

$$
\begin{array}{ccc}
En_1 D_1 X & \xrightarrow{\zeta_1} & D_1(ZX) \\
\downarrow \delta X & & \downarrow \sigma X \\
DX & \xrightarrow{\zeta} & DZX \\
\end{array}
$$

When $\mathbb{C} = Gp$, this map $\zeta$ is given by the restriction of any automorphism of $X$ to the center $ZX$. Clearly the restriction of an inner automorphism is trivial. In a similar way, we have, more generally:

**Proposition 2.2.** Suppose $\mathbb{C}$ is action representative. The map $\zeta.j_X$ is such that $\zeta.j_X = 0$.

**Proof.** Let us consider the following diagram:

$$
\begin{array}{ccc}
R[j_X] & \xrightarrow{j_X} & En_1 D_1 X & \xrightarrow{\zeta_1} & D_1(ZX) \\
\downarrow & & \downarrow \delta X & & \downarrow \sigma X \\
X \times X & \xrightarrow{j_X} & D_1 X & \xrightarrow{d_0} & \xrightarrow{s_0} \\
\downarrow p_0 & & \downarrow p_1 & & \downarrow \tilde{d}_1 \\
X & \xrightarrow{j_X} & DX & \xrightarrow{\zeta} & DZX.
\end{array}
$$

The upper left-hand side square is a pullback by the following lemma, so that the whole rectangle is a pullback. Accordingly, the map $\zeta.j_X$ is the classifying map of $j_X$. We noticed that $R[j_X]$ is central. By Proposition 2.1, we have $\zeta.j_X = 0$. \qed

Here is the needed lemma:
Lemma 2.2. Let $\mathcal{E}$ be any finitely complete category. Suppose we are given any internal functor $h_1 : R \to G_1$, where $R$ is an equivalence relation and $G_1$ is an internal groupoid, then the following square is a pullback in the category $\text{Grd} \mathbb{C}$:

$$
\begin{array}{c}
R[h_0] \cap R \\
\downarrow \\
R \\
\downarrow h_1 \\
G_1.
\end{array}
$$

Proof. Straightforward by commutativity of limits, see [7].

Suppose now that the pointed protomodular category $\mathbb{C}$ is moreover efficiently regular and that the action groupoid $D_1X$ has effective support. Denote by $q_X : DX \to Q_X$ the quotient of its support $\Sigma D_1X$, which is the coequalizer of the pair $(d_0, d_1) : D_1X \rightrightarrows DX$ as well, and also the cokernel of $j_X : X \to DX$. We observed that $\zeta \cdot j_X = 0$. Accordingly, there is a unique factorization $\xi : Q_X \to DZX$ such that $q_X \cdot \xi = \zeta$. This map allows us to determine the global direction of the action groupoid $D_1X$:

Proposition 2.3. Let $\mathbb{C}$ be an efficiently regular action representative category. Suppose the action groupoid $D_1X$ has effective support. Then its global direction is nothing other than $\xi^*(D_1ZX)$.

Proof. Consider the following diagram where the right-hand side downward square is a pullback:

Then there is a dotted horizontal arrow which makes the left-hand side downward square a pullback. Since $q_X$ is a regular epimorphism, the left-hand side upward square is a pushout and, according to Proposition 3.4, via Proposition 3.2 and Section 3.2.2 in [7], the central part of the diagram is the global direction of the groupoid $D_1X$. 

3. Extensions

We are now ready to connect extensions with pretorsors.
3.1. Extensions with abelian kernels

First, we shall only consider a regular action representative category $\mathbb{C}$. Suppose $K = A$ is abelian and $f$ is a regular epimorphism. Then since we have $\tilde{k} \cdot k = j_A = 0$ ($A$ abelian) and $f$ is the cokernel of $k$, there is a unique factorization $\phi : Y \to DA$ such that $\phi \cdot f = \tilde{k}$. We shall call it the direction of the extension:

$$
1 \longrightarrow A \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 1.
$$

This map $\phi$ is, in turn, the classifying map of a split extension with abelian kernel $A$:

$$
A \xrightarrow{k_{\phi}} E_{\phi} \xrightarrow{e_{\phi}} Y \xleftarrow{s_{\phi}} Y
$$

which, thanks to Proposition 2.1, is an internal group in the category $\mathbb{C}/Y$ (or equivalently a totally disconnected groupoid in $\mathbb{C}$). This group will be called the direction of the extension in question as well: the following diagram made of pullbacks:

$$
\begin{array}{ccccccccc}
& & & R[f] & \xrightarrow{s_{\phi}} & E_{\phi} & \xrightarrow{\phi} & D_1(A) & \\
& f_{0} & & s_{0} & \downarrow & e_{\phi} & \downarrow & s_{\phi} & d_{0} & \downarrow & s_{0} \\
\downarrow & & & & & & & & & & \\
A & \xrightarrow{k} & X & \xrightarrow{f} & Y & \xrightarrow{\phi} & D(A) & & & &
\end{array}
$$

shows us indeed that the map $\tilde{f}$ is the cokernel of $s_{0} \cdot k : A \to R[f]$ and that consequently we recover the construction of the direction of the extension, as defined in [15]. We shall denote by $E_{1}\phi = K_{1}E_{\phi}$ the totally disconnected groupoid associated with this internal group in $\mathbb{C}/Y$.

This notion of direction allows us to characterize easily the central extensions:

**Proposition 3.1.** Suppose $\mathbb{C}$ is a regular action representative category. An extension with direction $\phi$ is central if and only if $\phi = 0$.

**Proof.** The extension is central if, by definition, $R[f]$ is central. This is the case if and only if its classifying map $\phi \cdot f = 0$. But $f$ is a regular epimorphism. So, $\phi \cdot f = 0$ if and only if $\phi = 0$. $\square$

We shall suppose from now on that $\mathbb{C}$ is efficiently regular. Then, according to [15] without any reference to the existence of the split extension classifier, the set $\text{Ext}_{\phi}(Y, A)$ of isomorphic classes of extensions with direction $E_{\phi}$ is endowed with an abelian group structure given by the
following $3 \times 3$ Baer sum construction, where $(f, k)$ and $(f', k')$ are two extensions with same directions:

\[
\begin{array}{c}
1 \rightarrow A \rightarrow A \times A \rightarrow A \rightarrow 1 \\
1 \rightarrow A \rightarrow X \times_{Y} X' \rightarrow X \otimes_{Y} X' \rightarrow 1 \\
1 \rightarrow 1 \rightarrow Y \rightarrow 1.
\end{array}
\]

The main point is to show that the map $(-k, k')$ is a kernel map. In our words, this is the case because this map is precisely the normalization of the equivalence relation $S$ we defined, in Section 1.5 above, in relation to the definition of the composition of protorsors (with here $R^{op}[f] = R^{op}[f']$). Let us consider indeed the image of the following square:

\[
\begin{array}{c}
S \xrightarrow{P_{1X'}} R^{op}[f'] \\
\downarrow P_{1X} \quad \quad \quad \quad \downarrow \sigma \quad \quad \quad \quad \downarrow \ell' \\
R[f] \xrightarrow{\ell_{1}} E_{1}\phi
\end{array}
\]

by the left exact normalization process (see Appendix A). It is given by the following levelwise pullbacks, provided we noticed that the normalization of the functors $\ell_{1}$ and $\ell'_{1} \sigma$ are given by the pairs $(1_{A}, f)$ and $(-1_{A}, f')$ respectively:

\[
\begin{array}{c}
A \rightarrow A \rightarrow A \\
(-k, k') \downarrow \quad k' \downarrow \quad k \\
X \times_{Y} X' \rightarrow X' \rightarrow X
\end{array}
\]

Accordingly, the normalization of $S$ is necessarily the factorization $(-k, k')$.

Let us end this section by a result which turns out to be determinant later on:
Theorem 3.1. Suppose $\mathbb{C}$ is an efficiently regular action representative category. The group $\text{Ext}_\phi(Y, A)$ of abelian extensions with direction $\phi$ is isomorphic to the group of (endo)maps $\text{Tors}_\mathbb{C}(E_1\phi, E_1\phi)$ of the groupoid $\text{Tors}_\mathbb{C}$.

Proof. Thanks to our previous remark pointing out that $(-k, k')$ is precisely the normalization of $S$, this result is just a reformulation of our previous $3 \times 3$ Baer sum construction in terms of composition of pretorsors. We noticed indeed that an exact sequence $(f, k)$ with abelian kernel relation $R[f]$ is nothing other than a pretorsor $Y \xleftarrow{f} X \xrightarrow{f} Y$, since we do have $[R[f], R[f]] = 0$. □

3.2. Non-abelian extensions; Schreier–Mac Lane theorem

The aim now will be to make explicit the structure of the extensions when the kernel $K$ is no longer abelian. So consider any extension:

\[ 1 \longrightarrow K \xrightarrow{k} X \xrightarrow{f} Y \longrightarrow 1. \]

Suppose moreover that the action groupoid $D_1K$ has effective support. The following diagram induces a factorization $\phi$, we shall call the abstract direction of the extension:

\[
\begin{array}{c}
R[f] \xrightarrow{k_1} D_1K \\
\downarrow d_0 \quad \downarrow d_0 \quad \downarrow d_1 \\
X \xrightarrow{k} DK \\
\downarrow f \quad \downarrow q_K \\
Y \xrightarrow{\phi} Q_K.
\end{array}
\]

It is clear that when $K = A$ is abelian, the abstract direction coincides with the previous notion of direction since we have $Q_A = DA$. In the classical and original context of the category $\text{Gp}$ [25], this map $\phi$ was called abstract kernel.

We shall denote by $\text{Ext}_\phi(Y, K)$ the set of isomorphic classes of extensions with abstract direction $\phi$. We are now going to characterize the structure of this set. At the moment, it is quite natural to introduce the following lower pullback:

\[
\begin{array}{c}
K \xrightarrow{j_K} K \\
\downarrow j_\phi \quad \downarrow j_\phi \\
D\phi \xrightarrow{q_\phi} DK \\
\downarrow d_\phi \quad \downarrow q_K \\
Y \xrightarrow{\phi} Q_K.
\end{array}
\]

This produces a unique map $j_\phi$ such that $d_\phi \cdot j_\phi = j_K$ and $q_\phi \cdot j_\phi = 0$. The kernel of $j_\phi$ is necessarily the kernel $\kappa_K : ZK \hookrightarrow K$ of $j_K$. The map $j_K$ and $j_\phi$ having the same regular epimorphic part,
the map $q_\phi$ is the cokernel of $j_\phi$. Actually, the construction of $D\phi$ underlies a groupoid $D_1\phi$ produced by the following iterated pullbacks on the left-hand side of the following diagram. They make the functor $d_{1\phi}: D_1\phi \to D_1K$ a discrete fibration:

![Diagram](image)

**Proposition 3.2.** Suppose $C$ is an efficiently regular action representative category where the action groupoid $D_1K$ has effective support. The global direction $E_1\phi$ of the groupoid $D_1\phi$ is given by the above dotted pullback.

**Proof.** This is a consequence of the fact that $D_1\phi$ is defined by change of base along $\phi: Y \to Q_K$ of the groupoid $D_1K$ whose global direction is $E_1K$ (see Proposition 2.3). □

Consider now the following diagram:

![Diagram](image)

The functor $d_{1\phi}: D_1\phi \to D_1K$ is a discrete fibration by construction. Let us denote by $f_\phi$ and $f_{1\phi}$ the factorizations induced by $\tilde{k}$ and $\tilde{k}_1$. They make $f_{1\phi}: R[f] \to D_1\phi$ a discrete fibration. Since we have $f = q_\phi.f_\phi$, we have clearly $\pi_0(f_{1\phi}) = 1_Y$.

**Proposition 3.3.** Suppose $C$ is an efficiently regular action representative category where the action groupoid $D_1K$ has effective support. The map $f_\phi$ underlies an extension with abelian kernel relation whose direction is $\zeta.d_\phi$. 
Proof. The fact that $f_\phi$ is a regular epimorphism is a consequence of Lemma A.1 in Appendix A. Let us consider the following diagram:

$$
\begin{array}{ccccccccc}
R[f_\phi] & \longrightarrow & En_1 D_1 \phi & \longrightarrow & En_1 D_1 K & \xrightarrow{\zeta_1} & D_1(ZK) \\
\downarrow & & \downarrow e_1 D_1 \phi & & \downarrow e_1 D_1 K & & \downarrow d_0 \\
R[f] & \longrightarrow & D_1 \phi & \longrightarrow & D_1 K & & \\
p_0 & \downarrow p_1 & d_0 & \downarrow d_{1\phi} & d_0 & \downarrow d_{1} & \\
X & \longrightarrow & D\phi & \longrightarrow & DK & \xrightarrow{\zeta} & DZK. \\
& & \downarrow \tilde{k} & & & & \\
& & & & & & \\
\end{array}
$$

All squares are known to be pullbacks, except two of them which are pullbacks as well: the upper left-hand side one is a pullback by Lemma 2.2, while the upper middle one is a pullback by the nature of the construction of $D_1 \phi$ (which is obtained by left exact change of base along $\phi$). Accordingly, the whole rectangle is a pullback, which means that $\zeta, \tilde{k}$ classifies the kernel relation $R[f_\phi]$. This kernel relation is consequently abelian since the domain of the kernel of $f_\phi$ is certainly $ZK$. Moreover the equality $\zeta.d_{\phi}.f_\phi = \zeta.\tilde{k}$ means that the direction of the extension $f_\phi$ is precisely $\zeta.d_{\phi}$. Completing the previous diagram on the left:

$$
\begin{array}{cccccccccc}
ZK \times ZK & \xrightarrow{\tilde{k}} & R[j_K] & \longrightarrow & R[f_\phi] & \longrightarrow & En_1 D_1 \phi & \longrightarrow & En_1 D_1 K & \xrightarrow{\zeta_1} & D_1(ZK) \\
\downarrow & & \downarrow & & \downarrow e_1 D_1 \phi & & \downarrow e_1 D_1 K & & \downarrow d_0 \\
K \times K & \xrightarrow{\tilde{k}} & R[f] & \longrightarrow & D_1 \phi & \longrightarrow & D_1 K & & \\
p_0 & \downarrow p_1 & p_0 & \downarrow p_1 & d_0 & \downarrow d_{1\phi} & d_0 & \downarrow d_{1} & \\
ZK & \xrightarrow{\kappa_K} & K & \xrightarrow{k} & X & \xrightarrow{f_\phi} & D\phi & \longrightarrow & DK & \xrightarrow{\zeta} & DZK \\
& & \downarrow \tilde{k} & & & & & & & & \\
& & & & & & & & & & \\
\end{array}
$$

produces a dotted arrow which makes a pullback and shows that the kernel map of $f_\phi$ is $k.\kappa_K$. □

It follows that the extension:

$$
1 \longrightarrow ZK \xrightarrow{k.\kappa_K} \bar{X} \xrightarrow{f_\phi} D\phi \longrightarrow 1
$$

belongs to $\text{Ext}_{\zeta.d_{\phi}}(D\phi, ZK)$. And then:

**Proposition 3.4.** Suppose $\mathbb{C}$ is an efficiently regular action representative category where the action groupoid $D_1 K$ has effective support. The pair $(f_\phi, f)$ is a pretorsor $D_1 \phi \Rightarrow d_1(D_1 \phi) = E_1 \phi$. 
Proof. We know that \( f \) and \( f_\phi \) are regular epimorphisms. We noticed that \( [R[\hat{k}], R[f]] = 0 \) (Lemma 2.1). Since \( R[f_\phi] = R[f] \cap R[\hat{k}] \subset R[\hat{k}] \), still we have \( [R[f_\phi], R[f]] = 0 \), and the pair \((f_\phi, f)\) is a pretorsor. Since \( \hat{f}_1 : R[f] \to D_1 \phi \) is a discrete fibration, we know that the domain of this pretorsor is \( D_1 \phi \). On the other hand, since \( R[f_\phi] \subset R[f] \), we know by Proposition 1.7 that its codomain is the global direction of \( D_1 \phi \), namely \( E_1 \phi \).

We just described a clearly injective mapping:

\[
\Theta : \text{Ext}_\phi(Y, K) \to \text{Tors}(D_1 \phi, E_1 \phi); \quad (f, k) \mapsto (f_\phi, f).
\]

**Theorem 3.2.** Suppose \( \mathbb{C} \) is an efficiently regular action representative category where the action groupoid \( D_1 K \) has effective support. The mapping \( \Theta \) is bijective.

Proof. Suppose we have a pretorsor \((\bar{g}, \bar{f}) : D_1 \phi \leadsto E_1 \phi\). Then, according to Proposition 1.7 and since \( E_1 \phi \) is totally disconnected, we have \( R[\bar{g}] \subset R[\bar{f}] \). Accordingly, there is a unique map \( q : D \phi \to Y \) such that \( q.\bar{g} = \bar{f} \). Since \( \bar{g}_1 \) is a regular epimorphism, this map \( q \) is the coequalizer of the pair \((d_0, d_1) : D_1 \phi \rightrightarrows D \phi\), and there is an isomorphism \( \lambda : Y \to Y \) such that \( \lambda.q = q_\phi \). Thus we get \( q_\phi.\bar{g} = \lambda.\bar{f} \). Then consider the following diagram:

\[
\begin{array}{ccccccccc}
R[\bar{f}] & \xrightarrow{\bar{g}_1} & D_1 \phi & \xrightarrow{d_1 \phi} & D_1 K \\
\downarrow{p_0} & & \downarrow{p_1} & & \downarrow{d_0} & & \downarrow{d_1} \\
\bar{X} & \xrightarrow{\lambda.\bar{f}} & D \phi & \xrightarrow{q_\phi} & Q K
\end{array}
\]

Clearly we have \( R[\bar{f}] = R[\lambda.\bar{f}] \). By assumption, the upper left-hand side part underlies a discrete fibration, so that we have a discrete fibration \( d_1 \phi, \bar{g}_1 : R[\bar{f}] \to D_1 K \). This means that the map \( d_\phi.\bar{g} \) classifies \( R[\bar{f}] \) and that there is a kernel map \( \bar{k} : K \to \bar{X} \) of \( \lambda.\bar{f} \) such that \( d_\phi.\bar{g}_1.(0, k) = \gamma : K \to D_1 K \). Then we have certainly \( \bar{k} = d_\phi.\bar{g} \). Accordingly, since the right-hand side lower square commutes by definition, the abstract direction of the following extension:

\[
1 \to K \xrightarrow{\lambda.\bar{f}} \bar{X} \xrightarrow{\lambda.\bar{f}} Y \to 1
\]

is \( \phi \), with moreover \((\lambda.\bar{f})_\phi = \bar{g} \). □

We can now assert our main result:

**Theorem 3.3.** Suppose \( \mathbb{C} \) is an exact action representative category. When \( \text{Ext}_\phi(Y, K) \neq \emptyset \), on the set \( \text{Ext}_\phi(Y, K) \) there is a canonical simply transitive action of the abelian group \( \text{Ext}_{\xi,\phi}(Y, Z K) \).
Proof. Recall that, since \( C \) is supposed to be exact action representative, any action groupoid has effective support. We can apply the last theorem which asserts that, when \( \text{Ext}_\phi(Y, K) \neq \emptyset \), there is a bijection \( \text{Ext}_\phi(Y, K) \cong \text{Tors}_C(D_1 \phi, E_1 \phi) \). So, since \( \text{Tors}_C \) is a groupoid, there is a canonical simply transitive action of the group \( \text{Tors}_C(E_1 \phi, E_1 \phi) \) of endomaps on the codomain \( E_1 \phi \) of this “hom.” But we already noticed in Theorem 3.1 that we have \( \text{Tors}_C(E_1 \phi, E_1 \phi) \cong \text{Ext}_\xi \phi(Y, ZK) \). \( \square \)

3.3. Existence of extension

Given any map \( \phi : Y \to Q_K \), the question now is the existence of an extension:

\[
1 \xrightarrow{k} K \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{l} 1
\]

with abstract direction \( \phi \). In the category \( Gp \), the classical answer is that, to \( \phi \), is assigned a cohomology class of so-called obstructions, and then \( \text{Ext}_\phi(Y, K) \) is shown to be non-empty if and only if this cohomology class is 0. This cohomology class is interpreted as an element of a certain third cohomology group involving \( ZK \). We shall give here an interpretation dealing with the second cohomology group, the shifting of grading being explained by the fact that the answer in \( Gp \) was relative to the forgetful functor \( U : Gp \to \text{Set} \), while our answer is intrinsic to the category \( C \) itself.

There are different ways to realize cohomology groups. One way uses simplicial objects, see [19,20]. Another uses \( n \)-groupoids [9]: any abelian group \( A \) in a finitely complete category \( \mathcal{E} \) determines an abelian group \( K_n(A) \) inside the category of internal \( n \)-groupoids in \( \mathcal{E} \), and, when \( \mathcal{E} \) is exact, the \((n + 1)\)-th group \( H^{n+1}_\mathcal{E} A \) can be realized by the component classes of certain \( K_n(A) \)-torsors, i.e. of certain \( n \)-groupoids endowed with \( K_n(A) \) actions. In particular, \( H^3_\mathcal{E} A \) is given by the component classes of aspherical groupoids \( X_1 \) (i.e. connected groupoids such that \( X_0 \) has a global support) with global direction \( K_1(A) \). In this way, any internal groupoid determines necessarily an element in the second cohomology group with coefficient in its global direction. So we can now make even more precise sense of Proposition 1.6:

**Proposition 3.5.** Suppose \( \mathcal{E} \) is Mal’cev and efficiently regular. Let \((f, g)\) be a pretorsor such that \( Y_1 \) has effective support. Then not only the global directions of \( X_1 \) and \( Y_1 \) are the same, let us say \((v, u) \), \( v : V_1 \to V = \pi_0 Y_1 \), but also the two groupoids \( X_1 \) and \( Y_1 \) determine the same element in the cohomology group \( H^2_{\mathcal{E}/V} V_1 \).

**Proof.** We noticed in the proof of Proposition 1.6 that the functors \( R[g] \# R[f] \to Y_1 \) and \( R[g] \# R[f] \to X_1 \) were \( (0_0) \)-cartesian. This means precisely that \( X_1 \) and \( Y_1 \) are in the same connected component of the \( H^2 \) in question. \( \square \)

In particular, the groupoid \( D_1 \phi \) appears naturally as an element of the cohomology group \( H^2_{\mathcal{E}/Y} E_1 \phi \), where \( E_1 \phi \) is the abelian group of \( C/Y \) given by the following pullback:

\[
\begin{array}{ccc}
E_1 \phi & \to & D_1 ZK \\
\downarrow & & \downarrow \phi \\
Y & \to & Q_K \xrightarrow{\xi} DZK.
\end{array}
\]
By Proposition 3.2 we know indeed that the global direction of $D_1\phi$ is this $E_1\phi$.

**Theorem 3.4.** Suppose $\mathbb{C}$ is an exact action representative category. Given any map $\phi: Y \to QK$, the set of extensions $\text{Ext}_\phi(Y, K)$ is non-empty if and only if the groupoid $D_1\phi$, understood as an element of the cohomology group $H^2_{\mathbb{C}/Y}E_1\phi$, is 0.

**Proof.** The condition is necessary, since we observed that the existence of such an extension produced a pretorsor $(f\phi, f) : D_1\phi \rightsquigarrow d_1(D_1\phi) = E_1\phi$. By Proposition 1.2, we get two $0$-cartesian morphisms:

$$D_1\phi \leftarrow R[f\phi] \nrightarrow R[f] \to E_1\phi.$$ 

This means exactly (see Proposition 10 in [12]) that $D_1\phi$ and $E_1\phi$ are in the same connected component among the $E_1\phi$-torsors, and that consequently $D_1\phi$ is 0 in $H^2_{\mathbb{C}/Y}E_1\phi$.

Conversely suppose that $D_1\phi$ is 0 in $H^2_{\mathbb{C}/Y}E_1\phi$. Recall that, in a exact category $\mathbb{E}$, aspherical groupoid $Z_1$ with direction $K_1A$ is 0 in $H^2_{\mathbb{E}}A$ if and only if there is an object $X$ with global support and $0$-cartesian functor $\nabla X \times K_1A \to Z_1$ (see p. 174 in [9] or Theorem 12 in [12]). This is the same thing as the existence of a functor $\nabla : R[h] \to D_1\phi$ which induces the identity on $Y$.

The category $\mathbb{E} = \mathbb{C}/Y$ being exact, we can construct, according to Theorem 4 in [8], a factorization $l_1 = m_1.n_1 : R[h] \to X_1 \to D_1\phi$ such that $m_1$ is a discrete fibration and $n_1$ is a final functor. For exactly the same reasons as in the proof of Proposition 1.4, since $R[h]$ is an equivalence relation, the groupoid $X_1$ is actually an equivalence relation $S$ on $X = X_0$. Since $n_1$ is a final, the quotient of $R[h]$ and $S$ are the same. Accordingly, we get $S = R[f]$ for some regular epimorphism $f$ such that $f.n = h$. Consider now the following diagram:

$$
\begin{array}{ccccccccc}
R[h] & \xrightarrow{n_1} & R[f] & \xrightarrow{m_1} & D_1\phi & \xrightarrow{d_1\phi} & D_1K \\
p_0 & \downarrow{p_1} & \downarrow{p_1} & \downarrow{p_1} & \downarrow{d_1} & \downarrow{d_0} & \downarrow{d_1} \\
H & \xrightarrow{m} & X & \xrightarrow{m} & D\phi & \xrightarrow{d_\phi} & DK \\
& \downarrow{f} & \downarrow{q_\phi} & \downarrow{q} & \downarrow{q} \\
& \downarrow{h} & \downarrow{q} & \downarrow{q} & \downarrow{q} \\
& Y & \phi & QK. \\
\end{array}
$$

The functors $m_1$ and $d_1\phi$ being discrete fibrations, this is still the case for $d_1\phi.m_1$. Accordingly, the map $d_\phi.m$ is the classifying map of $R[f]$. Thus we get an exact sequence:

$$1 \xrightarrow{k} K \xrightarrow{f} X \xrightarrow{f} Y \xrightarrow{f} 1$$

whose abstract direction is $\phi$, since the map $d_\phi.m = \tilde{k}$ clearly induces the factorization $\phi : Y \to QY$. $\square$
Appendix A

We have gathered in this appendix some technical results about the normalization of internal groupoids. Suppose $\mathcal{E}$ is pointed and finitely complete:

**Definition A.1.** The normalization of the groupoid $Z_1$ is the map $nZ_1 = z_1k_0$ where $k_0$ is the kernel of $z_0$:

$$
\begin{array}{c}
NZ_1 \xrightarrow{k_0} Z_1 \xrightarrow{z_1} Z_0. \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \xrightarrow{\alpha_{Z_0}} Z_0
\end{array}
$$

**Remark.** The following upper extension of the previous diagram shows that this normalization produces an internal functor $n_1Z_1 : \nabla NZ_1 \rightarrow Z_1$. It is actually a discrete fibration, since the right-hand side squares are necessarily pullbacks:

$$
\begin{array}{c}
NZ_1 \times NZ_1 \xrightarrow{R(k_0)} R[z_0] \xrightarrow{\zeta_2} Z_1 \\
\downarrow p_0 \quad \quad \downarrow p_1 \\
NZ_1 \xrightarrow{k_0} Z_1 \xrightarrow{z_0} Z_0. \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \xrightarrow{\alpha_{Z_0}} Z_0
\end{array}
$$

Clearly this normalization process defines a left exact functor $N : \text{Grd}\mathcal{E} \rightarrow \mathcal{E}$ such that $N\Delta X = 0$. We have also $n_1\nabla X = 1_{\nabla X}$ (which implies $N\nabla X = X$) by the following diagram:

$$
\begin{array}{c}
X \times X \xrightarrow{r_X \times X} X \times X \times X \xrightarrow{p_2} X \times X \\
\downarrow p_0 \quad \quad \downarrow p_1 \\
X \xrightarrow{r_X} X \times X \xrightarrow{p_1} X. \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \xrightarrow{\alpha_X} X
\end{array}
$$

**Proposition A.1.** The left exact normalization functor $N$ is:

1. such that the image of any discrete fibration is an isomorphism,
2. a right adjoint to the fully faithful functor $\nabla$.

**Proof.** Straightforward. □
Lemma A.1. Suppose now \( \mathbb{C} \) is a pointed protomodular. Let \( G_1 \) and \( Z_1 \) be two groupoids and \( f_1 : Z_1 \to G_1 \) an internal functor. The normalization \( Nf_1 \) is an isomorphism if and only if \( f_1 \) is a discrete fibration. Suppose moreover \( \mathbb{C} \) efficiently regular and the two groupoids have effective supports. When \( f_1 \) is a discrete fibration and \( \pi_0 f_1 \) an isomorphism, then \( f_0 \) and \( f_1 \) are regular epimorphisms.

Proof. Suppose \( Nf_1 \) is an isomorphism. Then consider the following diagram:

\[
\begin{array}{ccc}
NZ_1 & \xrightarrow{k_0} & Z_1 & \xrightarrow{z_0} & Z_0 \\
Nf_1 & \downarrow & f_1 & \downarrow & f_0 \\
NG_1 & \xrightarrow{k_0} & G_1 & \xrightarrow{g_0} & G_0.
\end{array}
\]

This is a morphism of split exact sequences in a pointed protomodular category. Accordingly, when \( Nf_1 \) is an isomorphism, the right-hand square is a pullback and \( f_1 \) a discrete fibration.

Next, consider the following diagram produced by the canonical epi-mono factorization:

\[
\begin{array}{ccc}
NZ_1 & \xrightarrow{nZ_1} & \text{Im} nZ_1 & \xrightarrow{qZ} & \pi_0 Z_1 \\
Nf_1 & \downarrow & \phi & \downarrow & \pi_0 f_1 \\
NG_1 & \xrightarrow{nG_1} & \text{Im} nG_1 & \xrightarrow{qG} & \pi_0 G_1.
\end{array}
\]

Since the groupoids have effective supports, the middle horizontal monomorphisms are the normalizations of these effective supports. We obtain in this way two horizontal exact sequences on the right-hand side. On the other hand, the factorization \( \phi \) is necessarily a regular epimorphism since \( Nf_1 \) is an isomorphism (\( f_1 \) being a discrete fibration). Accordingly, since the triple \( (\phi, f_0, \pi_0 f_1) \) determines a morphism of exact sequences with \( \pi_0 f_1 \) an isomorphism, the map \( f_0 \) is a regular epimorphism as well. And \( f_1 \) also, since \( f_1 \) is a discrete fibration.

References