Irregular vectors of Hilbert space operators

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A vector \( x \) in a Hilbert space \( \mathcal{H} \) is called irregular for an operator \( T : \mathcal{H} \rightarrow \mathcal{H} \) provided that

\[
\sup_n \|T^nx\| = \infty \quad \text{and} \quad \inf_n \|T^nx\| = 0.
\]

We establish some basic properties of operators having irregular vectors and present examples that highlight the relationship, or lack thereof, between irregularity and hypercyclicity.

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1. Introduction

In this paper we will be concerned with a certain type of behavior of orbits of bounded linear operators on complex separable Hilbert spaces. We will denote such a space by \( \mathcal{H} \) and the algebra of all bounded linear operators on \( \mathcal{H} \) by \( B(\mathcal{H}) \). If \( T \in B(\mathcal{H}) \) we will denote by \( \sigma(T) \) the spectrum and by \( r(T) \) the spectral radius of \( T \). If \( T \in B(\mathcal{H}) \) and \( x \in \mathcal{H} \) then the set

\[
\text{Orb}_T(x) = \{x, Tx, T^2x, T^3x, \ldots\}
\]

is called the orbit of the vector \( x \) under the operator \( T \). If \( \text{Orb}_T(x) \) is dense in \( \mathcal{H} \) then \( T \) is called a hypercyclic operator and \( x \) is called a hypercyclic vector of \( T \). The operators with dense orbits received a lot of attention in the past 20 years. The bibliography in [2] (and the whole text, of course) is probably the best source of information about them. We are looking here at a more general concept that was introduced in [3].

An irregular vector of an operator is a vector such that \( \limsup_n \|T^nx\| = \infty \) and \( \liminf_n \|T^nx\| = 0 \), or, equivalently, \( \sup_n \|T^nx\| = \infty \) and \( \inf_n \|T^nx\| = 0 \). Hypercyclic vectors are irregular, but a hypercyclic operator can have irregular vectors that are not hypercyclic and an operator can have a huge number of irregular vectors and no hypercyclic one.

2. Properties

Some obvious properties of irregular vectors are:

1. \( x \) is an irregular vector of \( T \) if and only if there are two sequences \( k_n \) and \( l_n \) increasing to \( \infty \) such that \( \lim_n T^{k_n}x = 0 \) and \( \lim_n \|T^{l_n}x\| = \infty \).
2. \( x \) is an irregular vector of \( T \) if and only if \( x \) is an irregular vector of \( \alpha T \) for all complex numbers \( \alpha \) of absolute value 1.
3. \( x \) is an irregular vector of \( T \) if and only if \( \alpha x \) is an irregular vector of \( T \) for all non-zero complex numbers \( \alpha \).
4. \( x \) is an irregular vector of \( T \) if and only if \( T^kx \) is an irregular vector of \( T \) for some \( k \) if and only if \( T^kx \) is an irregular vector of \( T \) for every \( k \).

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5. The set of all irregular vectors of \( T \) is invariant under \( T \).

6. If \( x \) is an irregular vector of \( T \) and if and only if \( S^{-1}x \) is an irregular vector of \( S^{-1}TS \) for some invertible operator \( S \) if and only if \( S^{-1}x \) is an irregular vector of \( S^{-1}TS \) for every invertible operator \( S \).

There are some norm and spectral obstructions to the existence of irregular vectors. First, if \( T \) has irregular vectors then \( \|T\| > 1 \) and \( \sup \|T^n\| = \infty \) because otherwise all orbits of \( T \) will be bounded. Next, \( r(T) \geq 1 \) because otherwise all orbits of \( T \) will be convergent to 0. Finally, the spectrum of \( T \) cannot be completely outside the closed unit disc, because otherwise all orbits of \( T \) (except for the orbit of the vector 0) will be convergent (in norm) to \( \infty \).

2.1. Lemma. If \( A \oplus B \) has irregular vectors then at least one of \( A \) or \( B \) has irregular vectors.

Proof. Let \( x \oplus y \) be an irregular vector of \( A \oplus B \). Thus there is a sequence \( k_n \) such that \((A \oplus B)^k(x \oplus y) \to 0\). This means that \((A^k \oplus B^k)(x \oplus y) \to 0\) which means that \( A^kx \to 0 \) and \( B^ky \to 0 \).

In the same time, there is a sequence \( l_n \) such that \( \|k(A \oplus B)^l(x \oplus y)\| \to \infty \) which means that

\[
\|A^l x \oplus B^l y\|^2 \to \infty \iff \|A^l x\|^2 + \|B^l y\|^2 \to \infty
\]

which implies that at least one of them has a subsequence converging to \( \infty \).

Therefore either \( A \) or \( B \) has irregular vectors. \( \square \)

2.2. Theorem. If \( T \) has irregular vectors then the spectrum of \( T \) must intersect the unit circle.

Proof. Suppose not. By the discussion above, \( \sigma (T) \) intersects the closed unit disk. If it does not intersect the unit circle then it must intersect the open unit disk and, in fact, it must have at least one component inside the open unit disk. In the same time, \( \sigma (T) \) intersects the complement of the open unit disk. If it does not intersect the unit circle then it intersects the complement of the closed unit disk and, in fact, it must have at least one component outside the closed unit disk. Therefore \( T \) is similar to an operator of the type \( A \oplus B \), with \( \sigma (A) \) included in the open unit disk and \( \sigma (B) \) included in the complement of the closed unit disk.

By Lemma 2.1, either \( A \) or \( B \) must have irregular vectors, which is a contradiction because \( A \) has all orbits converging to 0 while \( B \) has all non-zero orbits converging in norm to \( \infty \).

Therefore the spectrum of \( T \) must intersect the unit circle. \( \square \)

2.3. Remark. Unlike the case of hypercyclic vectors, it is not true that the existence of an irregular vector implies that every component of the spectrum must intersect the unit circle. For example, if \( A \) is a hypercyclic operator, \( x \) a hypercyclic vector of \( A \), and \( B \) is an operator having the spectrum outside the closed unit disk and disjoint from the spectrum of \( A \), \( x \oplus 0 \) is an irregular vector of \( A \oplus B \) and at least one component of \( \sigma (A \oplus B) \) does not intersect the unit circle.

Moreover, in contrast with hypercyclicity, it is not true that the existence of one irregular vector implies the existence of a dense set of irregular vectors. This can be seen from the same example, for which, for every vector of the type \( z \oplus y \), with \( y \neq 0 \), \( \lim_n \|(A \oplus B)^n(z \oplus y)\| = \infty \).

2.4. Proposition. \( x \) is an irregular vector of \( T \) if and only if \( x \) is an irregular vector of \( T^m \) for every \( m \).

Proof. Let \( x \) be an irregular vector of \( T \). Thus there is a sequence \( k_n \) such that \( T^{k_n}x \to 0 \). Let \( k_n = mq_n + r_n \), with \( r_n \in \{0, 1, \ldots, m - 1\} \). Since there are only \( m \) choices for \( r_n \), one of the values repeats infinitely many times. Therefore we can assume, without loss of generality, that \( r_n = r \) for every \( n \). Thus \( T^{mq_n+r}x \to 0 \) and hence \( T^{m-r}(T^{mq_n+r}x) \to 0 \) which means that \( T^{m(qn+1)}x \to 0 \).

There is also a subsequence \( l_n \) such that \( \|T^{l_n}x\| \to \infty \). Let \( l_n = mq_n + r_n \), with \( r_n \in \{0, 1, \ldots, m - 1\} \) and let \( M = \max \|T\|, \|T^2\|, \ldots, \|T^{m-1}\| \). Then \( \|T^{l_n}x\| = \|T^{mq_n+r}x\| \leq ||T^n|| \|T^{mq_n+r}x\| \leq \|T^{mq_n}x\| \) which implies that \( \|T^{mq_n}x\| \to \infty \).

The converse is obvious. \( \square \)

3. Examples

Where are the irregular vectors coming from when they are not hypercyclic? In a lot of cases they are still connected in one way or another with hypercyclicity. The simplest examples of irregular non-hypercyclic vectors are in the case of operators having hypercyclic restrictions to invariant subspaces. If \( A \) is a hypercyclic operator and \( x \) is a hypercyclic vector of \( A \) then \( x \oplus 0 \) is an irregular non-hypercyclic vector for

\[
T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.
\]

In fact here we can see a more general idea. If \( x \) is an irregular vector of \( A \) then \( x \oplus 0 \) is an irregular vector of \( T \). In particular, this implies that if \( T \) does not have any irregular vectors the same is true for all its restrictions to invariant subspaces.
3.1. Proposition. If $T$ is a hyponormal operator and $x \in H$ then the sequence $\|T^n x\|$ is either increasing or eventually increasing or strictly decreasing.

Proof. By Proposition 2.6 in [4], if $\|x\| \leq \|Tx\|$ then the sequence $\|T^n x\|$ is increasing.

If $\|x\| > \|Tx\|$ then there are two possibilities: either $\|T^n x\| > \|T^{n+1} x\|$ for every $n \geq 1$, in which case the sequence is strictly decreasing or there is $k$ such that $\|T^k x\| < \|T^{k+1} x\|$, in which case, by the same Proposition 2.6 in [4], $\|T^n x\| \leq \|T^{n+1} x\|$ for every $n \geq k$ and thus the sequence is eventually increasing.

3.2. Corollary. Hyponormal operators do not have irregular vectors.

3.3. Corollary. Subnormal operators do not have irregular vectors.

3.4. Corollary. Normal operators do not have irregular vectors.

Before we move further we want to make a remark connected to [4]. Theorem 4.1 states that hyponormal operators are power regular, that is, for every $p$,

$$\|T^n x\| \leq \|T^{n+1} x\|$$

(see Proposition 2.1 in [4] or the proof of Proposition 4.7 in [7]). This implies, if $\|x\| = 1$, that the sequence $\|T^n x\|^{1/n}$ is increasing (see [9, p. 52] how Newton’s theorem implies Maclaurin’s theorem). In the case of $\|x\| > 1$ the sequence is not necessarily monotone. Let $C$ be the discrete Cesàro operator and $x = e_1 - 3e_2 + 2e_3$. It was noted in [5] that the operator is hyponormal (it is actually even subnormal).

3.5. Proposition. An operator on a finite dimensional space cannot have irregular vectors.

Proof. In view of the similarity invariance of the property and of Lemma 2.1, it suffices to notice that a Jordan block cannot have irregular vectors, which is simple to see.

3.6. Proposition. Compact operators do not have irregular vectors.

Proof. Let $K$ be a compact operator. If the spectrum of $K$ is included in the open unit disk then every orbit of $K$ converges to 0. If the spectrum of $K$ is not included in the open unit disk then $K$ is similar to some $A \oplus B$, with the spectrum of $A$ inside the open unit disk and $B$ an operator on a finite dimensional space. Since neither one has irregular vectors, by Lemma 2.1, the same is true for $K$.

A second type of example is coming from [8]. Let $d > 0$. A vector is called $d$-hypercyclic if its orbit intersects any ball of radius $d$. It is clear that a $d$-hypercyclic vector is irregular. Roughly speaking, a $d$-hypercyclic vector has an orbit uniformly spread throughout the space. Theorem 2.6 in [8] shows that twice the backward (unilateral) shift has, for each $d > 0$, $d$-hypercyclic vectors which are not hypercyclic. Notice that, according to Theorem 2.1 in [8], an operator having a $d$-hypercyclic vector is, in fact, hypercyclic but the $d$-hypercyclic vector is not necessarily a hypercyclic vector. Nevertheless, by Proposition 2.5 in [8] any $d$-hypercyclic vector is supercyclic. This means that the set of all scalar multiples of the vectors in its orbit is dense.

What we saw with the previous example is actually a general fact. Recall from [15] that a backward weighted (unilateral) shift of weights $p_1, p_2, p_3, \ldots$ is hypercyclic if and only if $\sup_n p_1p_2\cdots p_n = \infty$.

3.7. Theorem. Every hypercyclic unilateral backward weighted shift has irregular non-hypercyclic vectors.

Proof. Let $T$ be the unilateral backward weighted shift of weights $p_1, p_2, p_3, \ldots$ with respect to the orthonormal basis $(e_n)_{n \geq 1}$ and $(z_n)_{n \geq 1} = \mathbb{Q} \cap (0, \infty)$. By similarity invariance it suffices to consider positive weights. Here $Te_1 = 0$ and, for $n \geq 2$, $Te_n = p_{n-1} e_{n-1}$. For each $n$, let $k_n$ such that

$$\frac{z_n}{p_1p_2\cdots p_{k_n}} \leq \frac{1}{2^{n+1}}, \max\{1, \|T^k\|, \|T^{k_2}\|, \ldots, \|T^{k_{n-1}}\|\}.$$
Let \( x_n = z_n e_1 \) and
\[
x = \sum_{n} \frac{z_n}{p_1 p_2 \cdots p_{k_n}} e_{k_n+1}.
\]
It is clear that \( x \in \mathcal{H} \). Recall that \( \|T^n\| = \sup_n p_n p_{n+1} \cdots p_{n+m-1} \). Then
\[
\|T^k x - x_0\|^2 = \sum_{j=n+1}^\infty \frac{z_j^2}{(p_1 p_2 \cdots p_{k_j})^2} (p_{k_j-k_n+1} \cdots p_{k_j})^2 \leq \sum_{j=n+1}^\infty \frac{z_j^2}{(p_1 p_2 \cdots p_{k_j})^2} \|T^k\|_2^2 \leq \sum_{j=n+1}^\infty \frac{1}{2^n(j+1)} = \frac{1}{3 \cdot 2^n}.
\]
Thus
\[
\|T^k x - x_0\| < \frac{1}{2^n + 1}
\]
and thus \( x \) is an irregular vector of \( T \). The orbit is not dense because all vectors in the orbit have positive components. Similar constructions can be made using dense countable sets of points in any curve connecting 0 and \( \infty \) and in many other sets that have 0 and \( \infty \) as cluster points. \( \Box \)

So far, the examples we showed were either coming from hypercyclicity or were hypercyclic operators having some irregular non-hypercyclic vectors. Next we will show some examples of operators with many irregular vectors and no hypercyclic ones.

3.8. Proposition. \( T \) be the unilateral forward weighted shift of non-zero weights \( (p_n) \) and \( e_1 \) the first vector in the orthonormal basis. Then:

(i) \( \limsup_n \|T^n x\| = \infty \) for every \( x \neq 0 \) \( \iff \limsup_n \|T^n e_1\| = \infty \) \( \iff \limsup_n \prod_{j=1}^n |p_j| = \infty \).

(ii) \( \liminf_n \|T^n x\| = 0 \) for every \( x = \sum_{j=1}^k \alpha_j e_j \) \( \iff \liminf_n \|T^n e_1\| = 0 \) \( \iff \liminf_n \prod_{j=1}^n |p_j| = 0 \).

Proof. In each case the second equivalence is obvious since \( \|T^n e_1\| = \prod_{j=1}^n |p_j| \). Also, in each case one implication is obvious.

The other one comes from the following computation. Let \( x = \sum_{j=1}^\infty \alpha_j e_j \) with \( \alpha_k \neq 0 \). Then
\[
\|T^n x\|^2 = \sum_{j=1}^\infty \alpha_j^2 \|T^n e_j\|^2 = \sum_{j=1}^\infty \left( \sum_{j=1}^k \alpha_j |\alpha_k|^2 \|T^n e_k\|^2 \right) \geq \sum_{j=1}^\infty |\alpha_k|^2 \|T^n e_k\|^2 = \frac{|\alpha_k|^2}{|p_1 \cdots p_{k-1}|^2} \|T^{n+k-1} e_1\|^2.
\]

(ii) The only implication that requires some justification follows from
\[
T^n e_j = \frac{1}{p_1 \cdots p_{j-1}} T^{n+j-1} e_1 = \frac{1}{p_1 \cdots p_{j-1}} T^{j-1} (T^n e_1).
\]

3.9. Proposition. The unilateral forward weighted shift of weights \( 2, 5, 5, 2, 2, 5, 5, 5, 5, 5, \ldots \) has a dense set of irregular vectors, it has vectors that are not irregular and it does not have hypercyclic vectors.

Proof. The last statement is obvious since the adjoint has eigenvalues.

For \( n \geq 1 \), \( T^{n(2n-1)} e_1 = 2^n e_{n(2n-1)+1} \). Thus \( \|T^{n(2n-1)} e_1\| = 2^n \) and so, for every \( x \), \( \sup_n \|T^n x\| = \infty \).

For \( n \geq 1 \), \( T^{n(2n+1)} e_1 = 2^{-n} e_{n(2n+1)+1} \). Thus \( \|T^{n(2n+1)} e_1\| = 2^{-n} \) and so, for every \( x \) in a dense set \( \inf_n \|T^n x\| = 0 \). Therefore \( T \) has a dense set of irregular vectors.

To show that \( T \) has vectors that are not irregular it suffices to find a vector \( x \) such that \( \lim_n \|T^n x\| = \infty \). Let
\[
x = \sum_k \frac{1}{k} e_{k(2k+1)+1}.
\]

Then
\[
\|T^n x\|^2 = \sum_k \frac{1}{k^2} \|T^n e_{k(2k+1)+1}\|^2 \geq \sum_k \frac{1}{k^2} \left( \|T^n e_{k(2k+1)+1}\|^2 \right) \geq \frac{1}{n^2} \|T^n e_{n(n+1)+1}\|^2 = \frac{2^{2n}}{n^2} \rightarrow \infty.
\]

A somehow opposite type of example is coming from [11] (although the existence of its irregular orbits was not noted there). It is due to I. Halperin.

Let \( m_1 = 1 \), \( a_1 = 1 + \frac{1}{2} \), and \( \mathcal{H}_1 = \mathbb{C}^2 \). Suppose that we have selected \( m_{n-1}, a_{n-1} \) and \( \mathcal{H}_{m-1} \). Let \( p_n \) be an integer such that
Let $a_n = 1 + \frac{1}{2^n}$ and let $m_n > m_{n-1}$ be an integer such that $a_n^{m_n} \geq n$. Finally, let $\mathcal{H}_n = C^{m_n+1}$. For each $n$ we define the operator $A_n$ on $\mathcal{H}_n$ given by

$$A_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_0 & 0 & \cdots & 0 & 0 \\ 0 & a_n & \cdots & 0 & 0 \\ \vdots \\ 0 & 0 & \cdots & a_n & 0 \end{pmatrix}.$$ 

Notice that $\|A_n\| \leq 2$ and that $\|A_n^{m_n} e_1\| = \|a_n^{m_n} e_{m_n+1}\| > n \|e_1\|$, which implies that $\|A_n^{m_n}\| \geq n$. Let $A = \bigoplus_{n=1}^{\infty} A_n$ on $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$.

### 3.10. Proposition

The operator $A$ has a dense set of irregular vectors, there are vectors with orbits converging to $0$, and there is no hypercyclic vector.

#### Proof

We have that $\|A\| \leq 2$ and $\|A^{m_n}\| \geq n$. Thus, by Banach–Steinhauss theorem, there is a dense set $M$ such that for each $x \in M$, $\sup_n \|A^n x\| = \infty$.

Next we will show that for every vector $x \in \mathcal{H}$, $\inf_n \|A^n x\| = 0$. Let $x = \bigoplus_{n=1}^{\infty} x_n \in \mathcal{H}$ and $\varepsilon > 0$. Let $n$ such that $\bigoplus_{k=n+1}^{\infty} x_k \| < \frac{\varepsilon}{2}$. If $j \geq n + 1$, then

$$\|A^{m_{n+1}} x_j\| = a_j^{m_{n+1}} \|x_j\| \leq 2 \|x_j\|.$$ 

Therefore $\|A^{m_{n+1}} \bigoplus_{j=n+1}^{\infty} x_j\| \leq \varepsilon$. For $j \leq n$, $\|A^{m_{n+1}} x_j\| = 0$. Hence we get that $\inf_n \|A^n x\| = 0$ which implies that every vector in $M$ is irregular.

If $x = \bigoplus_{k=1}^{\infty} x_k$ it is easy to see that $\lim_n A^n x = 0$.

Finally, writing the operator as a direct sum of matrices on different spaces was just a matter of taste. We can see $A$ as a forward weighted shift on $l^2$ which has a subsequence of $0$ weights. Thus, being a forward weighted shift, $A$ has no hypercyclic vectors. □

A similar example, with the same ingenious use of $0$ weights, can be found in [3, Example 4A, pp. 66–68]. It has exactly the same properties as Halperin’s example.

Another type of irregular orbit comes from $\varepsilon$-hypercyclicity. The concept was introduced in [1]. Let $\varepsilon \in (0, 1)$, fixed. A vector $x$ is called $\varepsilon$-hypercyclic if for every $y \in \mathcal{H}$ there is $n$ such that $\|T^n x - y\| < \varepsilon \|y\|$. Since $\|T^n x\| - \|y\| \leq \|T^n x - y\| < \varepsilon \|y\|$ we get that $\|T^n x\| < (1 + \varepsilon) \|y\|$ and so $\inf_n \|T^n x\| = 0$. In the same time, $\|y\| - \|T^n x\| \leq \|T^n x - y\| < \varepsilon \|y\|$ and so $\|T^n x\| \geq (1 - \varepsilon) \|y\|$ which implies that $\sup_n \|T^n x\| = \infty$. Therefore $x$ is a hypercyclic vector. It is not known at this point if $x$ is not in fact a hypercyclic vector (in the context of Hilbert spaces). Nevertheless, we can say the following:

### 3.11. Proposition

Every component of the spectrum of an $\varepsilon$ hypercyclic operator must intersect the unit circle.

#### Proof

Let $x$ be an $\varepsilon$ hypercyclic vector of the operator $T$. Suppose that $x$ is a component of $\sigma(T)$ which does not intersect the unit circle. Then we can write $\sigma(T) = \sigma_1 \cup \sigma_2$, with $\sigma_1, \sigma_2$, non-empty, disjoint, and compact, and such that $\sigma_2$ does not intersect the unit circle (see for example Corollary 1 in [13, p. 83]). Let $\mathcal{M}$ be the Riesz spectral subspace corresponding to $\sigma_1$. Then, with respect to $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, we have

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{with} \quad T^n = \begin{pmatrix} A^n & B^n \\ 0 & C^n \end{pmatrix}.$$ 

If, with respect to the same decomposition of $\mathcal{H}$ we have $x = x_1 \oplus x_2$, then it is easy to see that $x_2$ is an $\varepsilon$ hypercyclic vector of $C$. In particular, it is an irregular vector, and thus the spectrum of $C$ must intersect the circle. This is a contradiction because $\sigma(C) = \sigma_2$. □

The previous proof may seem unnecessary long, but at this point we do not know if $\varepsilon$-hypercyclicity is similarity invariant.

Combining the previous proposition with Lemma 2.3 in [1] we get the following:

### 3.12. Corollary

Every $\varepsilon$-hypercyclic operator is norm limit of hypercyclic operators.
3.13. Theorem. There are operators with all non-zero vectors irregular, none of them hypercyclic.

Proof. Let \( c_0 = 1 \). We will construct inductively several sequences.

Let \( \beta_1 \) be a positive integer such that \( 2^{\frac{1}{\beta_1}} \leq 1 + \frac{1}{\beta_1} \) and \( \beta_1 > c_0 \). Let \( \alpha_1 \) be a positive integer such that \( \beta_1 \) divides \( 1 + \alpha_1 \) and \( 2^{\frac{1}{\beta_1} + \alpha_1} \geq 1 \). Let

\[
a_1 = c_0 + \beta_1, \quad b_1 = a_1 + \alpha_1 \beta_1, \quad c_1 = b_1 + 1 + \frac{1 + \alpha_1}{\beta_1}.
\]

Suppose that we selected \( c_0 < a_1 < b_1 < c_1 < \cdots < a_{n-1} < b_{n-1} < c_{n-1}, \beta_1 < \beta_2 < \cdots < \beta_{n-1} \), and \( \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \) such that \( c_{j-1} < \beta_j, 2^{\frac{1}{\beta_j}} \leq 1 + \frac{1}{\beta_j}, \beta_j \) divides \( 1 + \alpha_j \) and \( 2^{\frac{1}{\beta_j} + \alpha_j} \geq j2^{j-1} \) for \( 1 \leq j \leq n - 1 \).

Let \( \beta_n \) be a positive integer such that \( 2^{\frac{1}{\beta_n}} \leq 1 + \frac{1}{\beta_n} \), \( \beta_n > \beta_{n-1} \) and \( \beta_n > c_{n-1} \).

Let \( \alpha_n \) be a positive integer such that \( \beta_n \) divides \( 1 + \alpha_n \) and \( 2^{\frac{1}{\beta_n} + \alpha_n} \geq n2^{n-1} \).

We define

\[
a_n = c_{n-1} + \beta_n, \quad b_n = a_n + \alpha_n \beta_n, \quad c_n = b_n + 1 + \frac{1 + \alpha_n}{\beta_n}.
\]

Let \( w_j = 2^{\frac{1}{\beta_j}} \) if \( c_{n-1} \leq j < b_n \) and \( w_j = \frac{1}{2} \) if \( b_n \leq j < c_n \).

Then

\[
\prod_{j=0}^{a_0-1} w_j = \left( 2^{\frac{1}{\beta_0}} \right)^{a_0-c_0} = \left( 2^{\frac{1}{\beta_0}} \right)^{c_0} = 2^{\frac{1}{2^0}} = 1 + \frac{1}{2^0},
\]

\[
\prod_{j=b_0}^{b_1-1} w_j = \left( 2^{\frac{1}{\beta_1}} \right)^{b_1-a_0} = \left( 2^{\frac{1}{\beta_1}} \right)^{a_0} = 2^{\frac{a_0}{\beta_1}},
\]

\[
\prod_{j=b_0}^{c_1-1} w_j = \left( \frac{1}{2} \right)^{c_0-b_0} = \left( \frac{1}{2} \right)^{1 + \frac{1 + \alpha_0}{\beta_0}},
\]

\[
\prod_{j=b_0}^{c_1-1} w_j = 2^{\frac{1}{\beta_1} + \frac{a_0}{\beta_1}} \left( \frac{1}{2} \right)^{1 + \frac{1 + \alpha_0}{\beta_0}} = 1,
\]

and

\[
\prod_{j=1}^{b_{n-1}} w_j = \prod_{k=1}^{n-1} \left( \prod_{j=c_{k-1}}^{c_k-1} w_j \right) = \prod_{j=c_{k-1}}^{b_{k-1}} w_j = \left( \frac{1}{2} \right)^{n-1} 2^{\frac{n}{\beta_0} + \frac{a_0}{\beta_0}} \geq n.
\]

Let \( T \) be the unilateral forward weighted shift of weights \( (w_j) \).

Therefore

\[
\left\| T^{b_{n-1}} e_1 \right\| = \left\| \prod_{j=1}^{b_{n-1}} w_j e_{b_0} \right\| = \prod_{j=1}^{b_{n-1}} w_j \geq n
\]

which implies that \( \sup_n \left\| T^n e_1 \right\| = \infty \) and so \( \sup_n \left\| T^n x \right\| = \infty \) for every \( x \neq 0 \).

If \( y \in \mathcal{C} \) we will write it as \( y = \sum_{k=1}^{\infty} y_k \), where \( y_k = \sum_{j=c_{k-1}}^{c_k-1} \gamma_j e_j \).

We will evaluate \( T^n y_k \) for arbitrary \( n \) and \( k \). We have

\[
T^n y_k = \sum_{j=c_{k-1}}^{c_k-1} \gamma_j w_j w_{j+1} \cdots w_{j+b_n-1} e_j e_{j+b_n}
\]

and so

\[
\left\| T^n y_k \right\|^2 = \sum_{j=c_{k-1}}^{c_k-1} |\gamma_j|^2 (w_j w_{j+1} \cdots w_{j+b_n-1})^2.
\]
Suppose that $k \geq n$ and $c_{k-1} \leq j \leq c_k - 1$. Then the indices of the weights involved in the computation of the norm above are in the range from $c_{k-1}$ to $c_k + 1$. In this range, the greatest possible value of a weight is $\frac{1}{2^n}$ and hence
\[
w_j w_{j+1} \cdots w_{j+n-1} \leq \left( \frac{1}{2^{n-k}} \right)^{\beta_j} \leq \left( \frac{1}{2^{n-k}} \right)^{\beta_k} \leq 1 + \frac{1}{2^k} < 2.
\]
Therefore
\[
\|T^\beta y_k\|^2 \leq 4 \sum_{j=c_{k-1}}^{c_k} |y_j|^2 = 4\|y_k\|^2
\]
and so
\[
\left\|T^\beta \left( \sum_{k \geq n} y_k \right) \right\|^2 \leq \sum_{k \geq n} \|T^\beta y_k\|^2 \leq 4 \sum_{k \geq n} \|y_k\|^2.
\]
Assume that $k < n$. Then
\[
j + \beta_n - 1 \leq c_k + \beta_n - 1 \leq c_n - 1 + \beta_n - 1 = a_n - 1 < c_n.
\]
Let $p_j$ be the last integer such that $c_{p_j} \leq j + \beta_n - 1$. By the inequality above, $p_j < n$. In the same time, since $\beta_n > c_{n-1}$, $c_{n-1} < j + \beta_n - 1$. This implies that $p_j > n - 1$. From the two inequalities we conclude that $p_j = n - 1$. In this case we write
\[
w_j w_{j+1} \cdots w_{j+p_n-1} = w_j \cdots w_{c_k-1} w_{c_k} \cdots w_{c_{k+1}-1} w_{c_{k+1}} \cdots w_{c_{k+2}-1} \cdots w_{c_n-1} \cdots w_{c_n-2} \cdots w_{c_k-1} \cdots w_{j+p_n-1}.
\]
Because of the way the weights were defined, each product $w_{c_k} \cdots w_{c_{k+1}-1} = \frac{1}{2}$. Thus
\[
w_j w_{j+1} \cdots w_{j+p_n-1} = w_j \cdots w_{c_k-1} w_{c_k} \cdots w_{j+p_n-1} \left( \frac{1}{2} \right)^{n-1-k}.
\]
If $j \geq b_k$, then in $w_j \cdots w_{c_k-1}$ each factor is $\frac{1}{2}$ and hence the product is $< \frac{1}{2}$. If $j < b_k$ then
\[
w_j \cdots w_{c_k-1} = w_{c_k-1} \cdots W_{j-1} \leq \frac{1}{2}
\]
because the numerator equals $\frac{1}{2}$ and the denominator is a product of numbers greater than 1. Therefore, either way, $w_j \cdots w_{c_k-1} \leq \frac{1}{2}$.

To evaluate the product $w_{c_{n-1}} \cdots w_{j+p_n-1}$ we use the fact noticed above, that $j + \beta_n - 1 < a_n - 1$. Therefore
\[
w_{c_{n-1}} \cdots w_{j+p_n-1} = \frac{w_{c_{n-1}} \cdots w_{a_n-1}}{w_j \cdots w_{a_n-1}} \leq 1 + \frac{1}{2^n-1} < 2
\]
because the numerator is $\leq 1 + \frac{1}{2^n-1}$ and the denominator is a product of numbers greater than 1. Thus
\[
w_j w_{j+1} \cdots w_{j+p_n-1} < \left( \frac{1}{2} \right)^{n-1-k}
\]
and so
\[
\|T^\beta y_k\|^2 \leq \left( \frac{1}{2} \right)^{n-1-k} \sum_{j=c_{k-1}}^{c_k} |y_j|^2 = \left( \frac{1}{4} \right)^{n-1-k} \|y_k\|^2.
\]
This implies that
\[
\left\|T^\beta \left( \sum_{k \leq n} y_k \right) \right\|^2 = \sum_{k \leq n} \|T^\beta y_k\|^2 \leq \sum_{k \leq n} \left( \frac{1}{4} \right)^{n-1-k} \|y_k\|^2,
\]
from where we conclude that
\[
\left\|T^\beta \left( \sum_{k=1}^{\infty} y_k \right) \right\|^2 = \left\|T^\beta \left( \sum_{k \leq n} y_k \right) \right\|^2 + \left\|T^\beta \left( \sum_{k>n} y_k \right) \right\|^2 \leq \sum_{k \leq n} \left( \frac{1}{4} \right)^{n-1-k} \|y_k\|^2 + 4 \sum_{k>n} \|y_k\|^2
\]
which implies that $\lim_n \|T^\beta y\| = 0$ and hence $\inf_n \|T^n y\| = 0$. \qed
3.14. Remark. The operator defined in the previous theorem is the adjoint of a hypercyclic operator. Indeed, if $S$ is the unilateral backward weighted shift of weights $(w_j)$, then, since $\lim_n \prod_{j=1}^{n-1} w_j = \infty$ we conclude that $\sup_n w_1 \cdots w_n = \infty$ and so, according to Theorem 2.8 in [15], $S$ is a hypercyclic operator.

A more powerful example, with several extra properties, was constructed in [16].

4. Completely irregular operators

In the previous section we saw that a backward weighted shift is hypercyclic if and only if every non-zero orbit of its adjoint is unbounded (part (i) of Proposition 3.8 and the result from [15] mentioned at the end of the previous section). This was not just a coincidence. It is known that if $T$ is a hypercyclic operator and $y \neq 0$ then $\sup_n \|T^* y\| = \infty$ (see the comment after Corollary 4.2 in [6]).

This result seems to imply that the best candidates for operators with irregular vectors are adjoints of hypercyclic operators. In this case half of the irregularity condition is true for every vector in the space. We just hope that for at least one vector the other half is also true. That this is not always the case can be easily seen by looking at the classic twice the unilateral backward shift. The operator is hypercyclic, but for its adjoint, twice the unilateral forward shift, all non-zero orbits are converging (in norm) to $\infty$.

Stronger forms of hypercyclicity imply more restrictions on the behavior of the orbits of the adjoint. Recall that an operator $T$ is called topologically mixing if for any two open sets $U$ and $V$ in $\mathbb{H}$ there is $k$ such that $T^k U \cap V \neq \emptyset$ for all $n \geq k$. The following result is from [12].

4.1. Theorem. If $T$ is topologically mixing then all non-zero orbits of $T^*$ are converging to $\infty$.

Although it looks like there is no real connection between existence of irregular vectors and adjoints of hypercyclic operators, stronger forms of irregularity are indeed connected (although in a different way) to adjoints of hypercyclic operators.

4.2. Definition. An operator is called completely irregular if all its non-zero orbits are irregular.

The operator from Theorem 3.8 and the operator constructed in [16] are the only known examples of a completely irregular operator in a Hilbert space. For an example in $l^1$ see [14]. Notice that if the invariant subset problem has a negative solution then the example will be a completely irregular operator. This is the case in [14]. So far, though, the three known examples of completely irregular operators are either with no hypercyclic vectors or only with hypercyclic vectors. We can reformulate the invariant subset problem in the following way: Does every completely irregular operator have a non-trivial closed set?

Asking that all orbits are irregular implies more constraints on the spectrum. In this case we get stronger conclusions. For example, if $A \oplus B$ is completely irregular then both $A$ and $B$ are. This implies that if $T$ is a completely irregular operator then all components of its spectrum must intersect the unit circle. The spectral radius of a completely irregular operator must be $1$. This is because we already know that is $\geq 1$ and if $r(T) > 1$ there are vectors with orbit going to $\infty$ (in norm). Moreover, it is easy to see that to a completely irregular operator cannot have eigenvalues. In particular this means that $\sigma_0(T) = \emptyset$ and that any Fredholm index for $T$ (if any) must be negative. Recall that $\sigma_0(T)$ denotes the set of normal eigenvalues; that is the set of all eigenvalues that are both isolated and have the property that the corresponding Riesz spectral invariant subspace is of finite dimension. This is the same as the set of all isolated points of the spectrum that do not belong to the essential spectrum. The Weyl spectrum of an operator $T$ is defined by $\sigma_W(T) = \sigma(T) \setminus \{ \lambda \in \mathbb{C}; T - \lambda$ has Fredholm index $0\}$.

4.3. Theorem. If $T$ is a completely irregular operator then there is $(T_n)$, a sequence of hypercyclic operators, such that $\lim_n T_n^+ = T$.

Proof. By the discussion above, any Fredholm index for $T^*$ (if any) must be positive. Moreover, since $\sigma_0(T^+) = \sigma_0(T)$, $\sigma_0(T^+) = \emptyset$. This implies that $\sigma_W(T) = \sigma(T)$.

Thus the following facts are true for $T^*$: Each component of its Weyl spectrum intersects the unit circle. It does not have normal eigenvalues. All Fredholm index (if any) is positive. Therefore Theorem 2.1 in [10] implies that there is $(T_n)$, a sequence of hypercyclic operators such that $T^+ = \lim_n T_n$, from where we obtain the conclusion. \qed

4.4. Proposition. If $T$ is completely irregular than $T^*$ does not have any eigenvalues of absolute value $\geq 1$.

Proof. Let $\lambda$ be an eigenvalue of $T^*$ and $x \neq 0$ be an eigenvector. Then $\|T^n x\| \geq 0$ is the norm. Therefore Theorem 2.1 in [10] implies that there is $(T_n)$, a sequence of hypercyclic operators such that $T^+ = \lim_n T_n$, from where we obtain the conclusion. \qed
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