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Galois functors and entwining structures

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Galois comodules over a coring can be characterised by properties of the relative injective comodules. They motivated the definition of *Galois functors* over some comonad (or monad) on any category and in the first section of the present paper we investigate the role of the relative injectives (projectives) in this context.

Then we generalise the notion of corings (derived from an entwining of an algebra and a coalgebra) to the entwining of a monad and a comonad. Hereby a key role is played by the notion of a *grouplike natural transformation* $g : I \to G$ generalising the grouplike elements in corings. We apply the evolving theory to Hopf monads on arbitrary categories, and to opmonoidal monads with antipode on autonomous monoidal categories (named *Hopf monads* by Bruguières and Virelizier) which can be understood as an entwining of two related functors.

As well known, for any set G the product $G \times -$ defines an endofunctor on the category of sets and this is a Hopf monad if and only if G allows for a group structure. In the final section the elements of this case are generalised to arbitrary categories with finite products leading to *Galois objects* in the sense of Chase and Sweedler.

Contents

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Introduction

An *entwining* of an algebra A and a coalgebra C over a commutative ring R is given by an R-linear map $\lambda : A \otimes_R C \to C \otimes_R A$ satisfying certain conditions (Brzeziński and Majid in [8]). The corresponding *entwined modules* are defined as R-modules M which allow for an A-module structure $\varrho_M : A \otimes_R M \to M$ and a C-comodule structure $\varrho^M : M \to C \otimes_R M$ with a compatibility condition expressed by the commutativity of the diagram (e.g. [9, 32.4])

$$\begin{tikzcd} A \otimes_R M \arrow[r, "\varrho_M"] \arrow[d, "I_A \otimes \varrho^M"'] & M \arrow[r, "\varrho^M"] & C \otimes M \\ A \otimes C \otimes M \arrow[rr, "\lambda \otimes I_M"] & & C \otimes A \otimes M. \arrow[u, "I_C \otimes \varrho_M"'] \end{tikzcd}$$

An entwining structure (A, C, λ) makes $\mathcal{C} := C \otimes_R A$ to an A-coring and the entwined modules are just the left comodules for the coring $\mathcal{C}$.

In [27], a left $\mathcal{C}$-comodule P with $S = \mathrm{End}^{\mathcal{C}}(P)$ is called a *Galois comodule* provided the natural transformation $\mathrm{Hom}_A(P, -) \otimes_S P \to \mathcal{C} \otimes_A -$ is an isomorphism. Such modules can be characterised by properties of the $(\mathcal{C}, A)$-injective comodules [27, 4.1].

If A itself is a $\mathcal{C}$-comodule, that is, it allows for a grouplike element, then $\mathcal{C}$ is called a *Galois coring* provided A is a Galois comodule.

As a special case, if an R-module B has entwined algebra and coalgebra structures, then $B \otimes_R B$ is a B-coring. In this situation, B is a $B \otimes_R B$-Galois comodule, that is, $B \otimes_R B$ is a Galois coring, if and only if B is a Hopf algebra over R.

Since the tensor product is fundamental for these notions, generalisations to monoidal categories were investigated, e.g. in McCrudden [17], Bruguières, and Virelizier [7], Loday [16], Mesablishvili [18], and others.

In this paper we are concerned with the extension of these formalisms to endofunctors on arbitrary categories. The key to this is the following observation. The R-algebra A induces a monad $A \otimes_R -$, and the R-coalgebra C yields a comonad $C \otimes_R -$ on the (monoidal) category of R-modules. Thus the entwining (A, C, λ) becomes a special case of the entwining of a monad with a comonad on any category which is known as *mixed distributive law* from early papers of Barr [1], Beck [2], van Osdol [26], Burroni [10], and others (see [28] for more references). The theory of the related entwined modules is well understood in this context.

Yet, the additional constructions and notions for (A, C, λ) mentioned above are only partly transferred to monads and comonads on arbitrary categories, e.g. in Gómez-Torrecillas [14], Böhm and Menini [5], Böhm, Brzeziński and Wisbauer [4], and [21]. The purpose of this article is to continue these investigations.

A basic notion for this approach is the following (e.g. [14,21]). Given a comonad $\mathbf{G} = (G, \delta, \varepsilon)$ on a category $\mathbb{A}$, a functor $F : \mathbb{B} \to \mathbb{A}$ is called a **G**-*comodule* if there is a natural transformation $\overline{\alpha} : F \to GF$ making F a G-comodule in an obvious sense. Such a functor is said to be **G**-*Galois* provided F has a right adjoint $R : \mathbb{A} \to \mathbb{B}$, and the induced comonad morphism $FR \to G$ is an isomorphism (Definition 1.3). In Section 1 we continue the investigation of these functors, in particular of their behaviour towards relative injective modules. Dually, given a monad $\mathbf{T} = (T, m, e)$ on $\mathbb{A}$, a functor $R : \mathbb{B} \to \mathbb{A}$ is said to be a $\mathbf{T}$-*module* if there is a natural transformation $\alpha : TR \to R$ making R a $\mathbf{T}$-module in an obvious way. Such a functor is called $\mathbf{T}$-*Galois* if the induced monad morphism $T \to RF$, where $F : \mathbb{A} \to \mathbb{B}$ is a left adjoint to R, is an isomorphism (Definition 1.16). These functors show a special behaviour toward relative projectives and this is outlined in the last part of Section 1.

Section 2 is concerned with $\mathbf{G}$-comodule and $\mathbf{T}$-module functors considered in the context of mixed distributive laws $\lambda : TG \to GT$. In particular, the relevance of the Galois property for making the comparison functor $K : \mathbb{A} \to (\mathbb{A}^G)_{\widehat{T}}$ an equivalence is of interest.

As mentioned before, in an entwining structure (A, C, λ), the grouplike elements allow for a $C \otimes_R A$-comodule structure on A. In Section 3 we introduce, for a comonad (G, δ, ε) on $\mathbb{A}$, grouplike morphisms $g : I \to G$ requiring suitable properties. For a monad $\mathbf{F}$ on $\mathbb{A}$ with a mixed distributive law $\lambda : FG \to GF$, the grouplike element g induces two $\mathbf{G}$-comodule structures on the functor F, namely $gF : F \to GF$ and $\tilde{g} : F \xrightarrow{\lambda \circ Fg} GF$. The equaliser $F^g \xrightarrow{i_F} F$ of these two structure maps can be seen as monad morphism. Properties of the resulting functors are investigated and eventually conditions are given to obtain an equivalence between $\mathbb{A}_{F^g}$ and the category $(\mathbb{A}_F)^{\widetilde{G}}$ (see 3.14). This generalises the characterisation of Galois corings in module categories (e.g. [9, 28.18]).

In Section 4, the preceding results are applied to the case of an endofunctor H, which is a monad (H, m, e) as well as a comonad (H, δ, ε) subject to some compatibility conditions. Such functors are called *bimonads* in [21]. Under mild conditions on the base category $\mathbb{A}$, it follows that H is a Hopf monad (has an antipode) if and only if (H, m, e) is an (H, δ, ε)-Galois comodule or – equivalently – (H, δ, ε) is an (H, m, e)-Galois module.

In Section 5 we consider *opmonoidal monads* $\mathbf{T} = (T, m, e)$ on a strict monoidal category $(\mathbb{V}, \otimes, \mathbb{I})$ (see [17]), called *bimonads* in [7]. Hereby $T(\mathbb{I})$ has the structure of a coalgebra in $\mathbb{V}$, and, as pointed out in [21, 2.2], their theory can be understood as an entwining between the monad T and the comonad $- \otimes T(\mathbb{I})$ on $\mathbb{V}$. Thus our theory applies and results from [7] are reconsidered from this point of view. This leads to an improvement of [7, Theorem 4.6] which may be seen as an extended version of the Fundamental Theorem of Hopf algebras for right autonomous strict monoidal categories.

In the final section we generalise known properties of the endofunctors $G \times -$ on the category of sets, G any set, to categories with finite products. This relates our notions with *Galois objects* in the sense of Chase and Sweedler [11] (in the category opposite to commutative algebras) and we obtain a more general form of their Theorem 12.5 by replacing the condition on the Hopf algebra to be finitely generated and projective over the base ring by flatness without finiteness condition.

1. Galois comodule and module functors

Let $\mathbb{A}$ and $\mathbb{B}$ denote any categories. By I_a, $I_{\mathbb{A}}$ or just by I we denote the identity morphism of an object $a \in \mathbb{A}$, respectively the identity functor of a category $\mathbb{A}$.

Recall (e.g. from [13]) that a *monad* $\mathbf{T}$ on $\mathbb{A}$ is a triple (T, m, e) where $T : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $m : TT \to T$, $e : I \to T$ satisfying associativity and unitality conditions. A T-*module* is an object $a \in \mathbb{A}$ with a morphism $h_a : T(a) \to a$ subject to associativity and unitality conditions. The (Eilenberg–Moore) category of $\mathbf{T}$-modules is denoted by $\mathbb{A}_T$ and there is a free functor $\phi_T : \mathbb{A} \to \mathbb{A}_T$, $a \mapsto (T(a), m_a)$ which is left adjoint to the forgetful functor $U_T : \mathbb{A}_T \to \mathbb{A}$.

Dually, a *comonad* $\mathbf{G}$ on $\mathbb{A}$ is a triple (G, δ, ε) where $G : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $\delta : G \to GG$, $\varepsilon : G \to I$, and G-*comodules* are objects $a \in \mathbb{A}$ with morphisms $\rho_a : a \to G(a)$. Both notions are subject to coassociativity and counitality conditions. The (Eilenberg–Moore) category of $\mathbf{G}$-comodules is denoted by $\mathbb{A}^G$ and there is a cofree functor $\phi^G : \mathbb{A} \to \mathbb{A}^G$, $a \mapsto (G(a), \delta_a)$ which is right adjoint to the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$.

For convenience we recall some notions from [21, Section 3].

1.1. G-comodule functors. Given a comonad $\mathbf{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$, a functor $F : \mathbb{B} \to \mathbb{A}$ is a *left* $\mathbf{G}$-*comodule* if there exists a natural transformation $\beta : F \to GF$ with commutative diagrams

$$\begin{tikzcd} F \arrow[r, "\beta"] \arrow[dr, equal] & GF \arrow[d, "\varepsilon F"] \\ & F, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\beta"] \arrow[d, "\beta"'] & GF \arrow[d, "\delta F"] \\ GF \arrow[r, "G\beta"'] & GGF. \end{tikzcd} \tag{1.1}$$

Obviously (G, δ) and $(GG, \delta G)$ both are left **G**-comodules.

A **G**-comodule structure on $F : \mathbb{B} \to \mathbb{A}$ is equivalent to the existence of a functor $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ (dual to [12, Proposition II.1.1]) leading to a commutative diagram

$$\begin{tikzcd} \mathbb{B} \arrow[r, "\overline{F}"] \arrow[dr, "F"'] & \mathbb{A}^G \arrow[d, "U^G"] \\ & \mathbb{A}. \end{tikzcd}$$

Indeed, if $\overline{F}$ is such a functor, then $\overline{F}(b) = (F(b), \beta_b)$ for some morphism $\beta_b : F(b) \to GF(b)$ and the collection $\{\beta_b,\ b \in \mathbb{B}\}$ constitutes a natural transformation $\beta : F \to GF$ making F a **G**-comodule. Conversely, if $(F, \beta : F \to GF)$ is a **G**-module, then $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ is defined by $\overline{F}(b) = (F(b), \beta_b)$.

If a **G**-comodule (F, β) admits a right adjoint $R : \mathbb{A} \to \mathbb{B}$, with counit $\sigma : FR \to I$, then the composite

$$t_{\overline{F}} : FR \xrightarrow{\beta R} GFR \xrightarrow{G\sigma} G$$

is a comonad morphism from the comonad generated by the adjunction $F \dashv R$ to the comonad **G**.

1.2. Proposition. *(See [18, Theorem 4.4].) The functor $\overline{F}$ is an equivalence of categories if and only if the functor F is comonadic and $t_{\overline{F}}$ is an isomorphism of comonads.*

1.3. Definition. (See [21, Definition 3.5].) A left **G**-comodule $F : \mathbb{B} \to \mathbb{A}$ with a right adjoint $R : \mathbb{A} \to \mathbb{B}$ is said to be **G**-*Galois* if the corresponding morphism $t_{\overline{F}} : FR \to G$ of comonads on $\mathbb{A}$ is an isomorphism.

Thus, $\overline{F}$ is an equivalence if and only if F is **G**-Galois and comonadic.

1.4. Right adjoint for $\overline{F}$. If the category $\mathbb{B}$ has equalisers of coreflexive pairs, the functor $\overline{F}$ has a right adjoint.

Proof. This can be described as follows (see [12]): With the composite

$$\gamma : R \xrightarrow{\eta R} RFR \xrightarrow{Rt_{\overline{F}}} RG,$$

a right adjoint to $\overline{F}$ is the equaliser $(\overline{R}, \bar{e})$ of the diagram

$$RU^G \underset{\gamma U^G}{\overset{RU^G\eta^G}{\rightrightarrows}} RGU^G = RU^G\phi^G U^G,$$

with $\eta^G : I \to \phi^G U^G$ the unit of $U^G \dashv \phi^G$.

An easy inspection shows that for any $(a, \theta_a) \in \mathbb{A}^G$, the (a, θ_a)-component of the above diagram is

$$R(a) \underset{\gamma_a}{\overset{R(\theta_a)}{\rightrightarrows}} RG(a).$$

Now, for any $a \in \mathbb{A}$, $(\overline{R}(\overline{F}))(a)$ can be seen as the equaliser

$$\big(\overline{R}(\overline{F})\big)(a) \xrightarrow{\bar{e}_{\overline{F}(a)}} RF(a) \underset{\gamma_{F(a)}}{\overset{R(\beta_a)}{\rightrightarrows}} RGF(a).$$

Thus, writing P for the monad on $\mathbb{A}$ generated by the adjunction $\overline{F} \dashv \overline{R}$, the diagram

$$P \xrightarrow{\bar{e}} RF \underset{\gamma F}{\overset{R\beta}{\rightrightarrows}} RGF$$

is an equaliser diagram. $\square$

In view of the characterisation of Galois functors we have a closer look at some related classes of relative injective objects.

Let $F : \mathbb{B} \to \mathbb{A}$ be any functor. Recall (from [25]) that an object $b \in \mathbb{B}$ is said to be *F-injective* if for any diagram in $\mathbb{B}$,

$$\begin{tikzcd} b_1 \arrow[r, "f"] \arrow[d, "g"'] & b_2 \arrow[dl, dotted, "h"] \\ b & \end{tikzcd}$$

with $F(f)$ a split monomorphism in $\mathbb{A}$, there exists a morphism $h : b_2 \to b$ such that $hf = g$. We write $\mathrm{Inj}(F, \mathbb{B})$ for the full subcategory of $\mathbb{B}$ with objects all F-injectives.

The following result from [25] will be needed.

1.5. Proposition. *Let $\eta, \varepsilon : F \dashv R : \mathbb{A} \to \mathbb{B}$ be an adjunction. For any object $b \in \mathbb{B}$, the following assertions are equivalent*:

(a) *b is F-injective*;
(b) *b is a coretract for some $R(a)$, with $a \in \mathbb{A}$*;
(c) *the b-component $\eta_b : b \to RF(b)$ of η is a split monomorphism.*

1.6. Remark. For any $a \in \mathbb{A}$, $R(\varepsilon_a) \cdot \eta_{R(a)} = I$ by one of the triangular identities for the adjunction $F \dashv R$. Thus, $R(a) \in \mathrm{Inj}(F, \mathbb{B})$ for all $a \in \mathbb{A}$. Moreover, since the composite of coretracts is again a coretract, it follows from (b) that $\mathrm{Inj}(F, \mathbb{B})$ is closed under coretracts.

1.7. Functor between injectives. Let $F : \mathbb{B} \to \mathbb{A}$ be a $\mathbf{G}$-module with a right adjoint $R : \mathbb{A} \to \mathbb{B}$ and unit $\eta : I \to RF$. Write $\mathbf{G}'$ for the comonad on $\mathbb{A}$ generated by the adjunction $F \dashv R$ and consider the comparison functor $K_{G'} : \mathbb{B} \to \mathbb{A}^{G'}$. If $b \in \mathbb{B}$ is F-injective, then $K_{G'}(b) = (F(b), F(\eta_b))$ is $U_{G'}$-injective, since by the fact that η_b is a split monomorphism in $\mathbb{B}$, $(\eta_{G'})_{\phi^{G'}(b)} = F(\eta_b)$ is a split monomorphism in $\mathbb{A}^{G'}$. Thus the functor $K_{G'} : \mathbb{B} \to \mathbb{A}_{G'}$ yields a functor

$$\mathrm{Inj}(K_{G'}) : \mathrm{Inj}(F, \mathbb{B}) \longrightarrow \mathrm{Inj}\big(U^{G'}, \mathbb{A}^{G'}\big).$$

When $\mathbb{B}$ has equalisers, this functor is an equivalence of categories (see [25]).

We shall henceforth assume that $\mathbb{B}$ has equalisers.

1.8. Proposition. *With the data given in* 1.7, *the functor* $\overline{R} : \mathbb{A}^G \to \mathbb{B}$ *restricts to a functor*

$$\overline{R}' : \mathrm{Inj}\big(U^G, \mathbb{A}^G\big) \longrightarrow \mathrm{Inj}(F, \mathbb{B}).$$

Proof. Let (a, θ_a) be an arbitrary object of $\mathrm{Inj}(U^G, \mathbb{A}^G)$. Then, by Proposition 1.5, there exists an object $a_0 \in \mathbb{A}$ such that (a, θ_a) is a coretraction of $\phi^G(a_0) = (G(a_0), \delta_{a_0})$ in $\mathbb{A}^G$, i.e., there exist morphisms

$$f : (a, \theta_a) \longrightarrow \big(G(a_0), \delta_{a_0}\big) \quad \text{and} \quad g : \big(G(a_0), \delta_{a_0}\big) \longrightarrow (a, \theta_a)$$

in $\mathbb{A}^G$ with $gf = I$. Since f and g are morphisms in $\mathbb{A}^G$, the diagram

$$\begin{tikzcd}
G(a_0) \arrow[r, "(\delta_G)_{a_0}"] \arrow[d, shift left, "g"] & GG(a_0) \arrow[d, shift left, "G(g)"] \\
a \arrow[u, shift left, "f"] \arrow[r, "\theta_a"'] & G(a) \arrow[u, shift left, "G(f)"]
\end{tikzcd}$$

commutes. By naturality of γ (see 1.4), the diagram

$$\begin{tikzcd}
RG(a_0) \arrow[r, "\gamma_{G(a_0)}"] \arrow[d, shift left, "R(g)"] & RGG(a_0) \arrow[d, shift left, "RG(g)"] \\
R(a) \arrow[u, shift left, "R(f)"] \arrow[r, "\gamma_a"'] & RG(a) \arrow[u, shift left, "RG(f)"]
\end{tikzcd}$$

also commutes. Consider now the following commutative diagram

$$\begin{tikzcd}
R(a_0) \arrow[r, "\gamma_{a_0}"] \arrow[d, dotted, shift left] & RG(a_0) \arrow[r, shift left, "\gamma_{G(a_0)}"] \arrow[r, shift right, "R((\delta_G)_{a_0})"'] \arrow[d, shift left, "R(g)"] & RGG(a_0) \arrow[d, shift left, "RG(g)"] \\
\overline{R}(a, \theta_a) \arrow[u, dotted, shift left] \arrow[r, "\bar{e}_{(a,\theta_a)}"'] & R(a) \arrow[u, shift left, "R(f)"] \arrow[r, shift left, "\gamma_a"] \arrow[r, shift right, "R(\theta_a)"'] & RG(a). \arrow[u, shift left, "RG(f)"]
\end{tikzcd} \tag{1.2}$$

It is not hard to see that the top row of this diagram is a (split) equaliser (see [14]), and since the bottom row is an equaliser by the very definition of $\bar{e}$, it follows from the commutativity of the diagram that $\overline{R}(a,\theta_a)$ is a coretract of $R(a_0)$, and thus is an object of $\mathrm{Inj}(F,\mathbb{B})$ (see Remark 1.6). It means that the functor $\overline{R}:\mathbb{A}^G\to\mathbb{B}$ can be restricted to a functor $\overline{R}':\mathrm{Inj}(U^G,\mathbb{A}^G)\to\mathrm{Inj}(F,\mathbb{B})$. □

1.9. Proposition. *With the data given in* 1.7*, suppose that for any $b\in\mathbb{B}$, $(t_{\overline{F}})_{F(b)}$ is an isomorphism. Then the functor $\overline{F}:\mathbb{B}\to\mathbb{A}^G$ can be restricted to a functor*

$$\overline{F}':\mathrm{Inj}(F,\mathbb{B})\longrightarrow\mathrm{Inj}\big(U^G,\mathbb{A}^G\big).$$

Proof. Let δ' denote the comultiplication in the comonad $\mathbf{G}'$ (see 1.7). Recall from [18] that $\overline{F}=\mathbb{A}_{t_{\overline{F}}}\cdot K_{G'}$, where $\mathbb{A}_{t_{\overline{F}}}$ is the functor $\mathbb{A}^{G'}\to\mathbb{A}^G$ induced by the comonad morphism $t_{\overline{F}}:G'\to G$. Then for any $b\in\mathbb{B}$,

$$\begin{aligned}\overline{F}\big(RF(b)\big)&=\mathbb{A}_{t_{\overline{F}}}\big(K_{G'}\big(UF(b)\big)\big)=\mathbb{A}_{t_{\overline{F}}}\big(FRF(b),F\eta_{RF(b)}\big)\\&=\mathbb{A}_{t_{\overline{F}}}\big(G'F(b),\delta'_{F(b)}\big)=\big(G'F(b),(t_{\overline{F}})_{G'F(b)}\cdot\delta'_{F(b)}\big).\end{aligned}$$

Consider now the diagram

$$\begin{tikzcd}
G'F(b) \arrow[rr, "(t_{\overline{F}})_{F(b)}"] \arrow[d, "\delta'_{F(b)}"'] & & GF(b) \arrow[dd, "\delta_{F(b)}"] \\
G'G'F(b) \arrow[d, "(t_{\overline{F}})_{G'F(b)}"'] \arrow[drr, "(t_{\overline{F}})_{F(b)}.(t_{\overline{F}})_{F(b)}"] & (1) & \\
GG'F(b) \arrow[rr, "G((t_{\overline{F}})_{F(b)})"'] & & GGF(b),
\end{tikzcd}$$

in which the triangle commutes by the definition of the composite $(t_{\overline{F}})_{F(b)}.(t_{\overline{F}})_{F(b)}$, while the diagram (1) commutes since $t_{\overline{F}}$ is a morphism of comonads. The commutativity of the outer diagram shows that $(t_{\overline{F}})_{F(b)}$ is a morphism from the G-coalgebra $\overline{F}(RF(b))=(G'F(b),(t_{\overline{F}})_{G'F(b)}\cdot\delta'_{F(b)})$ to the G-coalgebra $(GF(b),\delta_{F(b)})$. Moreover, $(t_{\overline{F}})_{F(b)}$ is an isomorphism by our assumption. Thus, for any $b\in\mathbb{B}$, $\overline{F}(RF(b))$ is isomorphic to the G-coalgebra $(GF(b),\delta_{F(b)})$, which is of course an object of the category $\mathrm{Inj}(U^G,\mathbb{A}^G)$. Now, since any $b\in\mathrm{Inj}(F,\mathbb{B})$ is a coretract of $RF(b)$ (see Remark 1.6), and since any functor takes coretracts to coretracts, it follows that, for any $b\in\mathrm{Inj}(F,\mathbb{B})$, $\overline{F}(b)$ is a coretract of the G-coalgebra $(GF(b),\delta_{F(b)})\in\mathrm{Inj}(U^G,\mathbb{A}^G)$, and thus is an object of the category $\mathrm{Inj}(U^G,\mathbb{A}^G)$, again by Remark 1.6. This completes the proof. □

The following technical observation is needed for the next proposition.

1.10. Lemma. *Let $\iota,\kappa:W\dashv W':\mathbb{Y}\to\mathbb{X}$ be an adjunction of any categories. If $i:x'\to x$ and $j:x\to x'$ are morphisms in $\mathbb{X}$ such that $ji=I$ and if ι_x is an isomorphism, then $\iota_{x'}$ is also an isomorphism.*

Proof. Since $ji=I$, the diagram

$$\begin{tikzcd} x' \arrow[r, "i"] & x \arrow[r, shift left, "I"] \arrow[r, shift right, "ij"'] & x \end{tikzcd}$$

is a split equaliser. Then the diagram

$$W'W\big(x'\big) \xrightarrow{W'W(i)} W'W(x) \underset{W'W(ij)}{\overset{I}{\rightrightarrows}} W'W(x)$$

is also a split equaliser. Now considering the following commutative diagram

$$\begin{tikzcd}
x' \arrow[r, "i"] \arrow[d, dotted, "\iota_{x'}"'] & x \arrow[r, shift left, "I"] \arrow[r, shift right, "ij"'] \arrow[d, "\kappa_x"'] & x \arrow[d, "\kappa_x"] \\
W'W(x') \arrow[r, "W'W(i)"'] & W'W(x) \arrow[r, shift left, "I"] \arrow[r, shift right, "W'W(ij)"'] & W'W(x)
\end{tikzcd}$$

and recalling that the vertical two morphisms are both isomorphisms by assumption, we get that the morphism $\iota_{x'}$ is also an isomorphism. $\square$

1.11. Proposition. *In the situation of Proposition* 1.9, $\mathrm{Inj}(F,\mathbb{B})$ *is* (*isomorphic to*) *a coreflective subcategory of the category* $\mathrm{Inj}(U^G,\mathbb{A}^G)$.

Proof. By Proposition 1.8, the functor $\overline{R}$ restricts to a functor

$$\overline{R}' : \mathrm{Inj}\big(U^G,\mathbb{A}^G\big) \longrightarrow \mathrm{Inj}(F,\mathbb{B}),$$

while according to Proposition 1.9, the functor $\overline{F}$ restricts to a functor

$$\overline{F}' : \mathrm{Inj}(F,\mathbb{B}) \longrightarrow \mathrm{Inj}\big(U^G,\mathbb{A}^G\big).$$

Since

- $\overline{F}$ is a left adjoint to $\overline{R}$,
- $\mathrm{Inj}(F,\mathbb{B})$ is a full subcategory of $\mathbb{B}$, and
- $\mathrm{Inj}(U^G,\mathbb{A}^G)$ is a full subcategory of $\mathbb{A}^G$,

the functor $\overline{F}'$ is left adjoint to the functor $\overline{R}'$, and the unit $\overline{\eta}' : I \to \overline{R}'\overline{F}'$ of the adjunction $\overline{F}' \dashv \overline{R}'$ is the restriction of $\overline{\eta} : \overline{F} \dashv \overline{R}$ to the subcategory $\mathrm{Inj}(F,\mathbb{B})$, while the counit $\overline{\varepsilon}' : \overline{F}'\overline{R}' \to I$ of this adjunction is the restriction of $\overline{\varepsilon} : \overline{F}\overline{R} \to I$ to the subcategory $\mathrm{Inj}(U^G,\mathbb{A}^G)$.

Next, since the top of the diagram (1.2) is a (split) equaliser, $\overline{R}(G(a_0),\delta_{a_0}) \simeq R(a_0)$. In particular, taking $(GF(b),\delta_{F(b)})$, we see that

$$RF(b) \simeq \overline{R}\big(GF(b),\delta_{F(b)}\big) = \overline{R}\,\overline{F}\big(UF(b)\big).$$

Thus, the $RF(b)$-component $\overline{\eta}'_{RF(b)}$ of the unit $\overline{\eta}' : I \to \overline{R}'\overline{F}'$ of the adjunction $\overline{F}' \dashv \overline{R}'$ is an isomorphism. It now follows from Lemma 1.10 – since any $b \in \mathrm{Inj}(F,\mathbb{B})$ is a coretraction of $RF(b)$ – that $\overline{\eta}'_b$ is an isomorphism for all $b \in \mathrm{Inj}(F,\mathbb{B})$, proving that the unit $\overline{\eta}'$ of the adjunction $\overline{F}' \dashv \overline{R}'$ is an isomorphism. Thus $\mathrm{Inj}(F,\mathbb{B})$ is (isomorphic to) a coreflective subcategory of the category $\mathrm{Inj}(U^G,\mathbb{A}^G)$. $\square$

1.12. Corollary. *In the situation of Proposition* 1.9, *suppose that each component of the unit* $\eta : I \to RF$ *is a split monomorphism. Then the category* $\mathbb{B}$ *is* (*isomorphic to*) *a coreflective subcategory of* $\mathrm{Inj}(U^G,\mathbb{A}^G)$.

Proof. When each component of the unit $\eta : I \to RF$ is a split monomorphism, it follows from Proposition 1.5 that every $b \in \mathbb{B}$ is F-injective; i.e. $\mathbb{B} = \mathrm{Inj}(F, \mathbb{B})$. The assertion now follows from Proposition 1.11. □

1.13. Characterisation of G-Galois comodules. *Assume $\mathbb{B}$ to admit equalisers, let $\mathbf{G}$ be a comonad on $\mathbb{A}$, and $F : \mathbb{B} \to \mathbb{A}$ a functor with right adjoint $R : \mathbb{A} \to \mathbb{B}$. If there exists a functor $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ with $U^G \overline{F} = F$, then the following are equivalent*:

(a) *F is $\mathbf{G}$-Galois, i.e. $t_{\overline{F}} : \mathbf{G}' \to \mathbf{G}$ is an isomorphism*;
(b) *the following composite is an isomorphism,*

$$\overline{F}R \xrightarrow{\eta_G \overline{F} R} \phi^G U^G \overline{F} R = \phi^G F R \xrightarrow{\phi^G \varepsilon} \phi^G;$$

(c) *the functor $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ restricts to an equivalence of categories*

$$\mathrm{Inj}(F, \mathbb{B}) \longrightarrow \mathrm{Inj}\big(U^G, \mathbb{A}^G\big);$$

(d) *for any $(a, \theta_a) \in \mathrm{Inj}(U^G, \mathbb{A}^G)$, the (a, θ_a)-component $\bar{\varepsilon}_{(a,\theta_a)}$ of the counit $\bar{\varepsilon}$ of the adjunction $\overline{F} \dashv \overline{R}$, is an isomorphism*;
(e) *for any $a \in \mathbb{A}$, $\bar{\varepsilon}_{\phi_G(a)} = \bar{\varepsilon}_{(G(a),\delta_a)}$ is an isomorphism.*

Proof. That (a) and (b) are equivalent is proved in [12]. By the proof of [14, Theorem of 2.6], for any $a \in \mathbb{A}$, $\bar{\varepsilon}_{\phi^G(a)} = \bar{\varepsilon}_{(G(a),\delta_a)} = (t_{\overline{F}})_a$, thus (a) and (e) are equivalent.

By Remark 1.6, (d) implies (e).

Since $\mathbb{B}$ admits equalisers by our assumptions, it follows from Proposition 1.7 that the functor $\mathrm{Inj}(K_{G'})$ is an equivalence of categories. Now, if $t_{\overline{F}} : \mathbf{G}' \to \mathbf{G}$ is an isomorphism of comonads, then the functor $\mathbb{A}_{t_{\overline{F}}}$ is an isomorphism of categories, and thus $\overline{F}$ is isomorphic to the comparison functor $K_{G'}$. It now follows from Proposition 1.7 that $\overline{F}$ restricts to the functor $\mathrm{Inj}(F, \mathbb{B}) \to \mathrm{Inj}(U^G, \mathbb{A}^G)$ which is an equivalence of categories. Thus (a) $\Rightarrow$ (c).

If the functor $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ restricts to a functor

$$\overline{F}' : \mathrm{Inj}(F, \mathbb{B}) \longrightarrow \mathrm{Inj}\big(U^G, \mathbb{A}^G\big),$$

then one can prove, as in the proof of Proposition 1.11, that $\overline{F}'$ is left adjoint to $\overline{R}'$ and that the counit $\bar{\varepsilon}' : \overline{F}'\overline{R}' \to I$ of this adjunction is the restriction of the counit $\bar{\varepsilon} : \overline{F}\overline{R} \to I$ of the adjunction $\overline{F} \dashv \overline{R}$ to the subcategory $\mathrm{Inj}(U^G, \mathbb{A}^G)$. Now, if $\overline{F}'$ is an equivalence of categories, then $\bar{\varepsilon}'$ is an isomorphism. Thus, for any $(a, \theta_a) \in \mathrm{Inj}(U^G, \mathbb{A}^G)$, $\bar{\varepsilon}'_{(a,\theta_a)}$ is an isomorphism proving that (c) $\Rightarrow$ (d). □

1.14. T-module functors. Given a monad $\mathbf{T} = (T, m, e)$ on $\mathbb{A}$, a functor $R : \mathbb{B} \to \mathbb{A}$ is said to be a (*left*) $\mathbf{T}$-*module* if there exists a natural transformation $\alpha : TR \to R$ with commuting diagrams

$$\begin{tikzcd} R \arrow[r, "eR"] \arrow[dr, equal] & TR \arrow[d, "\alpha"] \\ & R, \end{tikzcd} \qquad \begin{tikzcd} TTR \arrow[r, "mR"] \arrow[d, "T\alpha"'] & TR \arrow[d, "\alpha"] \\ TR \arrow[r, "\alpha"'] & R. \end{tikzcd} \tag{1.3}$$

It is easy to see that (T, m) and (TT, mT) both are left $\mathbf{T}$-modules.

A **T**-module structure on R is equivalent to the existence of a functor $\overline{R} : \mathbb{B} \to \mathbb{A}_T$ inducing a commutative diagram (see [12, Proposition II.1.1])

$$\begin{tikzcd} \mathbb{B} \arrow[r, "\overline{R}"] \arrow[dr, "R"'] & \mathbb{A}_T \arrow[d, "U_T"] \\ & \mathbb{A}. \end{tikzcd}$$

Indeed (compare [12]), if $\overline{R}$ is such a functor, then $\overline{R}(b) = (R(b), \alpha_b)$ for some morphism $\alpha_b : TR(b) \to R(b)$ and the collection $\{\alpha_b,\ b \in \mathbb{B}\}$ constitutes a natural transformation $\alpha : TR \to R$ making R a **T**-module. Conversely, if $(R, \alpha : TR \to R)$ is a **T**-module, then $\overline{R} : \mathbb{B} \to \mathbb{A}_T$ is defined by $\overline{R}(b) = (R(b), \alpha_b)$.

For any **T**-module $(R : \mathbb{B} \to \mathbb{A}, \alpha)$ admitting a left adjoint functor $F : \mathbb{A} \to \mathbb{B}$, the composite

$$t_{\overline{R}} : T \xrightarrow{T\eta} TRF \xrightarrow{\alpha F} RF,$$

where $\eta : I \to RF$ is the unit of the adjunction $F \dashv R$, is a monad morphism from **T** to the monad on $\mathbb{A}$ generated by the adjunction $F \dashv R$. Dual to [18, Lemma 4.3], we have a commutative diagram

$$\begin{tikzcd} \mathbb{B} \arrow[r, "K_R"] \arrow[dr, "\overline{R}"'] & \mathbb{A}_{RF} \arrow[d, "\mathbb{A}_{t_{\overline{R}}}"] \\ & \mathbb{A}_T, \end{tikzcd}$$

with the comparison functor $K_R : \mathbb{B} \to \mathbb{A}_{RF}$, $b \mapsto (R(b), R(\varepsilon_b))$, where ε is the counit of the adjunction $F \dashv R$. As the dual of [18, Theorem 4.4], we have

1.15. Proposition. *The functor $\overline{R}$ is an equivalence of categories if and only if the functor R is monadic (i.e. K_R is an equivalence) and $t_{\overline{R}}$ is an isomorphism of monads.*

Similar to 1.3 one defines ([21, Definition 3.5], [4, 2.19])

1.16. Definition. A left **T**-module $R : \mathbb{B} \to \mathbb{A}$ with a left adjoint $F : \mathbb{A} \to \mathbb{B}$ is said to be **T**-*Galois* if the corresponding morphism $t_{\overline{R}} : T \to RF$ of monads on $\mathbb{A}$ is an isomorphism.

Given a functor $R : \mathbb{B} \to \mathbb{A}$, we write $\mathrm{Proj}(R, \mathbb{B})$ for the full subcategory of $\mathbb{B}$ given by R-projective objects. The following is dual to 1.13.

1.17. Characterisation of T-Galois modules. *Assume the category $\mathbb{B}$ to have equalisers. Let $\mathbf{T} = (T, m, e)$ be a monad on $\mathbb{A}$, and $R : \mathbb{B} \to \mathbb{A}$ a left* **T***-module functor with left adjoint $F : \mathbb{A} \to \mathbb{B}$ (and unit η, counit ε). If there exists a functor $\overline{R} : \mathbb{B} \to \mathbb{A}_T$ with $U_T \overline{R} = R$, then the following are equivalent:*

(a) *R is* **T**-*Galois;*
(b) *the following composition is an isomorphism:*

$$\phi_T \xrightarrow{\phi_T \eta} \phi_T RF = \phi_T U_T \overline{R} F \xrightarrow{\varepsilon_T \overline{R} F} \overline{R} F;$$

(c) *the functor* $\overline{R} : \mathbb{B} \to \mathbb{A}_T$ *restricts to an equivalence between the categories* Proj$(R, \mathbb{B})$ *and* Proj$(U_T, \mathbb{A}_T)$;
(d) *for any* $(a, h_a) \in$ Proj$(U_T, \mathbb{A}_T)$, *the* (a, h_a)*-component of the unit* $\overline{\eta}$ *of the adjunction* $\overline{L} \dashv \overline{R}$, *is an isomorphism*;
(e) *for any* $a \in \mathbb{A}$, $\overline{\eta}_{\phi_T(a)} = \overline{\eta}_{(T(a), m_a)}$ *is an isomorphism.*

Dual to 1.4 we observe:

1.18. Left adjoint for $\overline{R}$. *If* $\mathbb{B}$ *admits coequalisers of reflexive pairs, then the functor* $\overline{R}$ *admits a left adjoint.*

Proof. Let $(R, \alpha : TR \to R)$ be a left **T**-module with a left adjoint $F : \mathbb{B} \to \mathbb{A}$. Consider the composite

$$\beta : FT \xrightarrow{Ft_{\overline{R}}} FRF \xrightarrow{\varepsilon F} F,$$

where $\varepsilon : FR \to I$ is the counit of $F \dashv R$. It is easy to check that (F, β) is a right **T**-module. According to [12, Theorem A.1], when a coequaliser (R, i) exists for the diagram of functors

$$FU_T\phi_T U_T = FTU_T \underset{\beta U_T}{\overset{FU_T\varepsilon_T}{\rightrightarrows}} FU_T, \tag{1.4}$$

where $\varepsilon_T : \phi_T U_T \to I$ is the counit of $\phi_T \dashv U_T$, then R is left adjoint to $\overline{R} : \mathbb{B} \to \mathbb{A}_T$. It is easy to see that for any $(a, h_a) \in \mathbb{A}_T$, the (a, h_a)-component in the diagram (1.4) is the pair

$$FT(a) \underset{\beta_a}{\overset{F(h_a)}{\rightrightarrows}} F(a) \tag{1.5}$$

which is a reflexive pair since $\beta_a \cdot F(e_a) = F(h_a) \cdot F(e_a) = I$. This proves our claim. $\square$

So far we have dealt with (co)module structures on functors. It is also of interest to consider the corresponding relations between monads and comonads.

1.19. Definitions. Let $\mathbf{T} = (T, m, e)$ be a monad and $\mathbf{G} = (G, \delta, \varepsilon)$ a comonad on $\mathbb{A}$. We say that **G** is **T**-*Galois*, if there exists a left **T**-module structure $\alpha : TG \to G$ on the functor G such that the composite

$$\gamma^G : TG \xrightarrow{T\delta} TGG \xrightarrow{\alpha G} GG$$

is an isomorphism.

Dually, **T** is **G**-*Galois*, if there is a left **G**-comodule structure $\beta : T \to GT$ on the functor T such that the composite

$$\gamma_T : TT \xrightarrow{\beta T} GTT \xrightarrow{Gm} GT$$

is an isomorphism.

We need the following (dual of [24, Lemma 21.1.5])

1.20. Proposition. *Let $\eta, \varepsilon : F \dashv R : \mathbb{C} \to \mathbb{A}$ and $\eta', \varepsilon' : F' \dashv R' : \mathbb{C} \to \mathbb{B}$ be adjunctions and let*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "X"] \arrow[dr, "F"'] & \mathbb{B} \arrow[d, "F'"] \\ & \mathbb{C} \end{tikzcd}$$

be a diagram of categories and functors with $F'X = F$. Write α for the composition

$$XR \xrightarrow{\eta' XR} R'F'XR = R'FR \xrightarrow{R'\varepsilon} R'.$$

Then the natural transformation $S_X = F'\alpha : FR = F'XR \to F'R'$ is a morphism of comonads.

Note that for the commutative diagram (see 1.1)

$$\begin{tikzcd} \mathbb{B} \arrow[r, "\overline{F}"] \arrow[dr, "F"'] & \mathbb{A}^G \arrow[d, "U^G"] \\ & \mathbb{A}, \end{tikzcd}$$

where F has a right adjoint R, the related comonad morphism $S_{\overline{F}} : FR \to G$ is just the comonad morphism $t_{\overline{F}} : FR \to G$.

From the proof of [24, Theorem 21.1.10(b)] we obtain:

1.21. Proposition. *Let $F \dashv R : \mathbb{D} \to \mathbb{A}$, $F' \dashv R' : \mathbb{D} \to \mathbb{B}$ and $F'' \dashv R'' : \mathbb{D} \to \mathbb{C}$ be adjunctions and let*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "X"] \arrow[dr, "F"'] & \mathbb{B} \arrow[r, "Y"] \arrow[d, "F'"] & \mathbb{C} \arrow[dl, "F''"] \\ & \mathbb{D} & \end{tikzcd}$$

be a commutative diagram of categories and functors. Write S_X for the comonad morphism $FR \to F'R'$, S_Y for the comonad morphism $F'R' \to F''R''$ and S_{YX} for the comonad morphism $FR \to F''R''$ that exist according to the previous proposition. Then $S_{YX} = S_Y S_X$.

2. Entwinings

2.1. Entwinings. We fix a mixed distributive law, also called an *entwining*, $\lambda : TG \to GT$ from the monad $\mathbf{T} = (T, m, e)$ to the comonad $\mathbf{G} = (G, \delta, \varepsilon)$, and write $\widehat{\mathbf{T}} = (\widehat{T}, \widehat{m}, \widehat{e})$ for a monad on $\mathbb{A}^G$ lifting $\mathbf{T}$, and $\widehat{\mathbf{G}} = (\widehat{G}, \widehat{\delta}, \widehat{\varepsilon})$ for a comonad on $\mathbb{A}_T$ lifting $\mathbf{G}$ (e.g. [28, Section 5]).

It is well known that for any object (a, h_a) of $\mathbb{A}_T$,

- $\widehat{G}(a, h_a) = \big(G(a), G(h_a) \cdot \lambda_a\big)$,
- $(\widehat{\delta})_{(a,h_a)} = \delta_a$,
- $(\widehat{\varepsilon})_{(a,h_a)} = \varepsilon_a$,

while for any object (a, θ_a) of the category $\mathbb{A}^G$,

- $\widehat{T}(a, \theta_a) = \big(T(a), \lambda_a \cdot T(\theta_a)\big)$;
- $(\widehat{m})_{(a,\theta_a)} = m_a$,
- $(\widehat{e})_{(a,\theta_a)} = e_a$,

and that there is an isomorphism of categories

$$\big(\mathbb{A}^G\big)_{\widehat{T}} \simeq (\mathbb{A}_T)^{\widehat{G}}.$$

We write $\mathbb{A}^G_T(\lambda)$ (or just $\mathbb{A}^G_T$, when the mixed distributive law λ is understood) for the category whose objects are triples (a, h_a, θ_a), where $(a, h_a) \in \mathbb{A}_T$ and $(a, \theta_a) \in \mathbb{A}^G$ with commuting diagram

$$\begin{tikzcd} T(a) \arrow[r, "h_a"] \arrow[d, "T(\theta_a)"'] & a \arrow[r, "\theta_a"] & G(a) \\ TG(a) \arrow[rr, "\lambda_a"'] & & GT(a). \arrow[u, "G(h_a)"'] \end{tikzcd} \tag{2.1}$$

Let $K : \mathbb{A} \to (\mathbb{A}^G)_{\widehat{T}}$ be a functor inducing a commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K"] \arrow[dr, "\phi^G"'] & (\mathbb{A}^G)_{\widehat{T}} \arrow[d, "U_{\widehat{T}}"] \\ & \mathbb{A}^G. \end{tikzcd} \tag{2.2}$$

Write $\alpha_K : \widehat{T}\phi^G \to \phi^G$ for the corresponding $\widehat{\mathbf{T}}$-module structure on ϕ^G (see 1.14). Since $\widehat{\mathbf{T}}$ is the lifting of $\mathbf{T}$ corresponding to λ, $U^G\widehat{T} = TU^G$ and one has the natural transformation

$$\alpha = U^G(\alpha_K) : U^G\widehat{T}\phi^G = TU^G\phi^G = TG \longrightarrow U^G\phi^G = G.$$

It is easy to see that α provides a left $\mathbf{T}$-module structure on G with commutative diagram

$$\begin{tikzcd} TG \arrow[r, "\alpha"] \arrow[d, "T\delta"'] & G \arrow[r, "\delta"] & GG \\ TGG \arrow[rr, "\lambda G"'] & & GTG. \arrow[u, "G\alpha"'] \end{tikzcd} \tag{2.3}$$

Conversely, a natural transformation

$$\alpha : U^G\widehat{T}\phi^G = TU^G\phi^G = TG \longrightarrow U^G\phi^G = G$$

making G a left $\mathbf{T}$-module, can be lifted to a left $\widehat{\mathbf{T}}$-module structure on ϕ^G if and only if for every $a \in \mathbb{A}$, $\alpha_a : TG(a) \to G(a)$ is a morphism in $\mathbb{A}^G$ from the $\mathbf{G}$-coalgebra $(TG(a), \lambda_{G(a)} \cdot T(\delta_a))$ to the $\mathbf{G}$-coalgebra $(G(a), \delta_a)$, which is just to say that the a-component of the diagram (2.3) commutes. Thus we have proved:

2.2. Proposition. *With the data given in* 2.1, *the assignment*

$$K : \mathbb{A} \longrightarrow \left(\mathbb{A}^G\right)_{\widehat{T}} \longmapsto U^G(\alpha_K) : TG \longrightarrow G,$$

yields a bijection between functors K making the diagram (2.2) *commute and left* **T***-module structures $\alpha : TG \to G$ on G for which the diagram* (2.3) *commutes.*

Now let $K' : \mathbb{A} \to (\mathbb{A}_T)^{\widehat{G}}$ be a functor inducing a commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K'"] \arrow[dr, "\phi_T"'] & (\mathbb{A}_T)^{\widehat{G}} \arrow[d, "U^{\widehat{G}}"] \\ & A_T . \end{tikzcd} \tag{2.4}$$

Write $\beta_{K'} : \phi_T \to \widehat{G}\phi_T$ for the corresponding $\widehat{\mathbf{G}}$-comodule structure on ϕ_T (see 1.1). One has the natural transformation

$$\beta = U_T(\beta_{K'}) : U_T\phi_T = T \longrightarrow U_T\widehat{G}\phi_T = GU_T\phi_T = GT$$

which induces a **G**-comodule structure on T with commutative diagram

$$\begin{tikzcd} TT \arrow[r, "m"] \arrow[d, "T\beta"'] & T \arrow[r, "\beta"] & GT \\ TGT \arrow[rr, "\lambda T"'] & & GTT . \arrow[u, "Gm"'] \end{tikzcd} \tag{2.5}$$

From this we obtain:

2.3. Proposition. *In the situation described above, the assignment*

$$K' : A \longrightarrow (\mathbb{A}_T)^{\widehat{G}} \quad \longmapsto \quad U_T(\beta_{K'}) : T \longrightarrow GT,$$

yields a bijection between functors K' making the diagram (2.4) *commute and left* **G***-comodule structures $\beta : T \to GT$ on the functor T for which the diagram* (2.5) *commutes.*

To give a functor $K' : \mathbb{A} \to (\mathbb{A}_T)^{\widehat{G}}$ making the diagram (2.4) commute is to give a natural transformation $\alpha : U_T \to U_T\widehat{G}$ making U_T a right $\widehat{G}$-comodule (see [14, Proposition 2.1]). For any $(a, h_a) \in \mathbb{A}_T$, $\widehat{G}(a, h_a) = (G(a), G(h_a) \cdot \lambda_a)$, the (a, h_a)-component $\alpha_{(a,h_a)}$ is a morphism $a \to G(a)$ in $\mathbb{A}$ with commutative diagrams

$$\begin{tikzcd} a \arrow[r, "\alpha_{(a,h_a)}"] \arrow[dr, equal] & G(a) \arrow[d, "\varepsilon_a"] \\ & a, \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha_{(a,h_a)}"] \arrow[d, "\alpha_{(a,h_a)}"'] & G(a) \arrow[d, "\delta_{G(a)}"] \\ G(a) \arrow[r, "G(\alpha_{(a,h_a)})"'] & GG(a), \end{tikzcd}$$

and the corresponding comonad morphism $t_{K'} : \phi_T U_T \to \widehat{G}$ is the composite

$$\phi_T U_T \xrightarrow{\phi_T \alpha} \phi_T U_T \widehat{G} \xrightarrow{\varepsilon_T \widehat{G}} \widehat{G}.$$

Then, since for any $(a, h_a) \in \mathbb{A}_T$, $(\varepsilon_T)_{(a,h_a)} = h_a$, the component $(t_{K'})_{(a,h_a)}$ is the composite

$$T(a) \xrightarrow{T(\alpha_{(a,h_a)})} TG(a) \xrightarrow{\lambda_a} GT(a) \xrightarrow{G(h_a)} G(a).$$

Now it follows from Proposition 1.2:

2.4. Theorem. *In the situation described above, the functor K' is an equivalence of categories if and only if for any $(a, h_a) \in \mathbb{A}_T$, the composite $G(h_a) \cdot \lambda_a \cdot T(\alpha_{(a,h_a)})$ is an isomorphism and the functor ϕ_T is comonadic.*

For the dual situation, let $K : \mathbb{A} \to (\mathbb{A}^G)_{\widehat{T}}$ be a functor inducing commutativity of the diagram (2.2). Since the functor ϕ^G has a left adjoint $U^G : \mathbb{A}^G \to \mathbb{A}$, it follows from [14] that to give such a functor is to give a right $\widehat{T}$-module structure $\alpha : U^G \widehat{T} \to U^G$ on U^G.

For any $(a, \theta_a) \in \mathbb{A}^G$, $\widehat{T}(a, \theta_a) = (T(a), \lambda_a \cdot T(\theta_a))$, the (a, θ_a)-component $\alpha_{(a,\theta_a)}$ is a morphism $T(a) \to a$ in $\mathbb{A}$ with commutative diagrams

$$\begin{tikzcd} a \arrow[r, "e_a"] \arrow[dr, equal] & T(a) \arrow[d, "\alpha_{(a,\theta_a)}"] \\ & a, \end{tikzcd} \qquad \begin{tikzcd} TT(a) \arrow[r, "T(\alpha_{(a,\theta_a)})"] \arrow[d, "m_a"'] & T(a) \arrow[d, "\alpha_{(a,\theta_a)}"] \\ T(a) \arrow[r, "\alpha_{(a,h_a)}"'] & a, \end{tikzcd}$$

and the corresponding monad morphism $t_K : \phi^G U^G \to \widehat{T}$ is the composite

$$\widehat{T} \xrightarrow{\eta^G \widehat{T}} \phi^G U^G \widehat{T} \xrightarrow{\phi^G \alpha} \phi^G U^G.$$

Now, since for any $(a, \theta_a) \in \mathbb{A}^G$, $(\eta^G)_{(a,\theta_a)} = \theta_a$, the component $(t_K)_{(a,\theta_a)}$ is the composite

$$T(a) \xrightarrow{T(\theta_a)} TG(a) \xrightarrow{\lambda_a} GT(a) \xrightarrow{G(\alpha_{(a,h_a)})} G(a).$$

As a consequence we get from Proposition 1.15:

2.5. Theorem. *In the situation described above, the functor K is an equivalence of categories if and only if for any $(a, \theta_a) \in \mathbb{A}^G$, the composite $G(\alpha_{(a,h_a)}) \cdot \lambda_a \cdot T(\theta_a)$ is an isomorphism and the functor ϕ^G is monadic.*

The following observation is probably known but we are not aware of a suitable reference. Recall that a functor $i : \mathbb{C} \to \mathbb{A}$ with $\mathbb{C}$ a small category is *dense*, if the functor

$$\widetilde{i} : \mathbb{A}^{op} \longrightarrow [\mathbb{C}, \mathrm{Set}], \quad a \longmapsto \mathrm{Mor}_{\mathbb{A}}\big(i(-), a\big),$$

is full and faithful.

2.6. Lemma. *Let $i : \mathbb{C} \to \mathbb{A}$ be a dense functor. Given two adjunctions*

$$F \dashv U,\ F' \dashv U' : \mathbb{A} \longrightarrow \mathbb{B},$$

and a natural transformation $\tau : F \to F'$, then τ is an isomorphism of functors if and only if $\tau i : Fi \to F'i$ is so.

Proof. Write $\tau' : U' \to U$ for the natural transformation corresponding to τ, that is τ and τ' are mates, denoted by $\tau \dashv \tau'$ (e.g. [21, 7.1], [4, 2.2]). Then τ is an isomorphism if and only if τ' is so. So it is enough to show that τ' is an isomorphism. Since $\tau \dashv \tau'$, the diagram

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{B}}(F'i(a), b) \arrow[r, "\alpha'_{i(a),b}"] \arrow[d, "\mathrm{Mor}_{\mathbb{B}}(\tau_{i(a)}, b)"'] & \mathrm{Mor}_{\mathbb{A}}(i(a), U'(b)) \arrow[d, "\mathrm{Mor}_{\mathbb{A}}(i(a), \tau'_b)"] \\
\mathrm{Mor}_{\mathbb{B}}(Fi(a), b) \arrow[r, "\alpha_{i(a),b}"'] & \mathrm{Mor}_{\mathbb{A}}(i(a), U(b)),
\end{tikzcd}$$

where α (resp. α') is the bijection corresponding to the adjunction $F \dashv U$ (resp. $F' \dashv U'$), commutes for all $a, b \in \mathbb{A}$. Since $\tau_{i(a)}$ is an isomorphism by our assumption on τ, it follows that the natural transformation $\mathrm{Mor}_{\mathbb{A}}(i(a), \tau'_b)$ is an isomorphism, implying – since i is dense – that $\tau' : U' \to U$ is an isomorphism. $\square$

2.7. Proposition. *With the data given in* 2.1, *let $K' : \mathbb{A} \to (\mathbb{A}_T)^{\widehat{G}}$ be a functor with $U^{\widehat{G}} K' = \phi_T$ and $\beta_{K'} : \phi_T \to \widehat{G}\phi_T$ the corresponding $\widehat{\mathbf{G}}$-comodule structure on ϕ_T (see* 1.1*). Suppose that*

(i) *$\mathbb{A}$ admits equalisers of coreflexive pairs and both T and G have right adjoints, or*
(ii) *$\mathbb{A}$ admits small colimits and both T and G preserve them.*

Then $(\phi_T, \beta_{K'})$ is $\widehat{\mathbf{G}}$-Galois if and only if $(T, U_T(\beta_{K'}))$ is $\mathbf{G}$-Galois.

Proof. For any $a \in \mathbb{A}$, the $\phi_T(a) = (T(a), m_a)$-component of $t_{K'} : \phi_T U_T \to \widehat{G}$ is just $(\gamma_T)_a$ (see 1.19). Thus it is enough to show that $t_{K'}$ is an isomorphism if and only if its restriction to free $\mathbf{T}$-modules is.

(i) If T has a right adjoint, there exists a comonad $\mathbf{H}$ inducing an isomorphism of categories $\mathbb{A}_T \simeq \mathbb{A}^H$; this implies that the functor U_T is comonadic and hence has a right adjoint. It follows that the composite GU_T also has a right adjoint. Next, since $\widehat{\mathbf{G}}$ is the lifting of $\mathbf{G}$, we have the commutative diagram

$$\begin{tikzcd}
\mathbb{A}_T \arrow[r, "\widehat{G}"] \arrow[d, "U_T"'] & \mathbb{A}_T \arrow[d, "U_T"] \\
\mathbb{A} \arrow[r, "G"'] & \mathbb{A}.
\end{tikzcd}$$

Since

- GU_T has a right adjoint,
- the functor U_T is comonadic, and
- $\mathbb{A}_T$ admits equalisers of coreflexive pairs (since $\mathbb{A}$ does so),

it follows from the dual of [12, Theorem A.1] that the functor $\widehat{G}$ has a right adjoint.

Now, since the full subcategory of $\mathbb{A}_T$ given by free $\mathbf{T}$-modules is dense in $\mathbb{A}_T$, it follows from Lemma 2.6 that $t_{K'} : \phi_T U_T \to \widehat{G}$ is an isomorphism if and only if its restriction to free $\mathbf{T}$-modules is.

(ii) Since T preserves colimits, the category $\mathbb{A}_T$ admits colimits and the functor $U_T : \mathbb{A}_T \to \mathbb{A}$ creates them. Thus

- the functor $\phi_T U_T$ preserves colimits;
- any functor $L : \mathbb{B} \to \mathbb{A}_T$ preserves colimits if and only if the composite $U_T L$ does; so, in particular, the functor $\widehat{G}$ preserves colimits, since $U_T \widehat{G} = G U_T$ and $G U_T$ is the composite of two colimit-preserving functors.

The full subcategory of $\mathbb{A}_T$ given by the free $\mathbf{T}$-modules is dense and since the functors $\phi_T U_T$ and $\widehat{G}$ both preserve colimits, it follows from [24, Theorem 17.2.7] that the natural transformation

$$t_{K'} : \phi_T U_T \longrightarrow \widehat{G}$$

is an isomorphism if and only if its restriction to the free $\mathbf{T}$-modules is so; i.e. if $(t_{K'})_{\phi_T(a)}$ is an isomorphism for all $a \in \mathbb{A}$. This completes the proof. $\square$

Dually, one has

2.8. Proposition. *With the data given in 2.1, let $K : \mathbb{A} \to (\mathbb{A}^G)_{\widehat{T}}$ be a functor with $U_{\widehat{T}} K = \phi^G$ and $\alpha_K : \widehat{T}\phi^G \to \phi^G$ the corresponding $\widehat{\mathbf{T}}$-module structure on ϕ^G. Suppose that*

(i) *$\mathbb{A}$ admits coequalisers of reflexive pairs and both T and G have left adjoints, or*
(ii) *$\mathbb{A}$ admits all small limits and both T and G preserve them.*

Then (ϕ^G, α_K) is $\widehat{\mathbf{T}}$-Galois if and only if $(G, U^G(\alpha_K))$ is $\mathbf{T}$-Galois.

The results of the preceding two propositions may be compared with Böhm and Menini's [5, Theorem 3.3].

3. Grouplike morphisms

In this section we extend the theory of Galois corings $\mathcal{C}$ over a ring A to entwinings of a monad F and a comonad G on general categories. For this we extend the notion of a grouplike element in $\mathcal{C}$ (e.g. [9, 28.1]) to the notion of a grouplike natural transformation $I \to G$.

3.1. Definition. Let $\mathbf{G} = (G, \delta, \varepsilon)$ be a comonad on a category $\mathbb{A}$. A natural transformation $g : I \to G$ is called a *grouplike morphism* provided it induces commutative diagrams

$$\begin{tikzcd} I \arrow[r, "g"] \arrow[dr, "="'] & G \arrow[d, "\varepsilon"] \\ & I, \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "g"] \arrow[dr, "gg"'] & G \arrow[d, "\delta"] \\ & GG. \end{tikzcd}$$

Comonads with grouplike morphisms are called *computational* in [6] (see also [23]). The next result transfers Proposition 5.1 in [18].

3.2. Grouplike morphisms and comodule structure. *Let $\mathbf{F} = (F, m, e)$ be a monad and $\mathbf{G} = (G, \delta, \varepsilon)$ a comonad on a category $\mathbb{A}$ with an entwining $\lambda : FG \to GF$. If G has a grouplike morphism $g : I \to G$, then F has two left G-comodule structures* (*see* 1.1) *given by*

$$(1) \quad \tilde{g} : F \xrightarrow{Fg} FG \xrightarrow{\lambda} GF \quad \text{and} \quad (2) \quad gF : F \longrightarrow GF.$$

Proof. (1) In the diagram

$$\begin{tikzcd}
F \arrow[r, "Fg"] \arrow[dr, "="'] & FG \arrow[r, "\lambda"] \arrow[d, "F\varepsilon"] & GF \arrow[d, "\varepsilon F"] \\
& F \arrow[r, "="'] & F,
\end{tikzcd}$$

the triangle is commutative by the grouplike properties of g and the square is commutative by the properties of the entwining λ. In the diagram

$$\begin{tikzcd}
F \arrow[r, "Fg"] \arrow[d, "Fg"'] \arrow[dr, "Fgg"] & FG \arrow[r, "\lambda"] \arrow[d, "F\delta"] & GF \arrow[dd, "\delta F"] \\
FG \arrow[d, "\lambda"'] & FGG \arrow[d, "\lambda G"] & \\
GF \arrow[r, "GFg"] & GFG \arrow[r, "G\lambda"] & GGF,
\end{tikzcd}$$

the right rectangle is commutative by properties of entwinings, the triangle is commutative by properties of the grouplike morphism g, and the pentagon is commutative by naturality of composition. This shows that $\tilde{g}$ makes F a left G-comodule.

(2) To say that $(F, gF : F \to GF)$ is a left $\mathbf{G}$-comodule is to say that the diagrams

$$\begin{tikzcd}
F \arrow[r, "gF"] \arrow[dr, "="'] & GF \arrow[d, "\varepsilon F"] \\
& F,
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "gF"] \arrow[d, "gF"'] & GF \arrow[d, "\delta F"] \\
GF \arrow[r, "GgF"'] & GGF
\end{tikzcd}$$

are commutative. Using the fact that

$$GgF \cdot gF = ggF,$$

the commutativity of these diagrams follows from the definition of a grouplike morphism. $\square$

The pattern of the proof of [18, Proposition 5.3] also yields:

3.3. *F* as mixed bimodule. *With the data given in* 3.2, $(F, m, \tilde{g})$ *is a mixed* (F, G)*-bimodule.*

Proof. We need to show commutativity of the diagram

$$\begin{tikzcd} FF \arrow[r, "m"] \arrow[d, "F\tilde{g}"'] & F \arrow[r, "\tilde{g}"] & GF \\ FGF \arrow[rr, "\lambda F"] & & GFF. \arrow[u, "Gm"'] \end{tikzcd}$$

However, by the definition of $\tilde{g}$, we get the diagram

$$\begin{tikzcd} FF \arrow[r, "m"] \arrow[d, "FFg"'] & F \arrow[r, "Fg"] & FG \arrow[r, "\lambda"] & GF \\ FFG \arrow[urr, "mG"] \arrow[r, "F\lambda"'] & FGF \arrow[rr, "\lambda F"] & & GFF, \arrow[u, "Gm"'] \end{tikzcd}$$

in which the right pentagon is commutative since λ is an entwining and the triangle is commutative by naturality of composition. This proves our claim. □

Combining 2.3, 3.2 and 3.3 yields the existence of a functor $K_g : \mathbb{A} \to (\mathbb{A}_F)^{\widehat{G}}$ making the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K_g"] \arrow[dr, "\phi_F"'] & (\mathbb{A}_F)^{\widehat{G}} \arrow[d, "U^{\widehat{G}}"] \\ & \mathbb{A}_F \end{tikzcd}$$

commute. Note that $K_g(a) = ((F(a), m_a), \tilde{g}_a)$.

Now assume that $\mathbb{A}$ admits equalisers. Then the category of endofunctors of $\mathbb{A}$ also has equalisers and we have the

3.4. Equaliser functor. *With the data given in* 3.2, *define a functor* F^g *as an equaliser of functors*

$$\begin{tikzcd} F^g \arrow[r, "i_F"] & F \arrow[rr, "gF"] \arrow[dr, "Fg"'] & & GF \\ & & FG. \arrow[ur, "\lambda"'] & \end{tikzcd}$$

Then F^g *is a monad on* $\mathbb{A}$ *and* $i_F : F^g \to F$ *is a monad morphism.*

Proof. We adapt the proof of [18, 5.2]. The following two diagrams are commutative by naturality of composition,

$$\begin{tikzcd} I \arrow[r, "e"] \arrow[d, "g"'] & F \arrow[d, "gF"] \\ G \arrow[r, "Ge"'] & GF, \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "e"] \arrow[d, "g"'] & F \arrow[d, "Fg"] \\ G \arrow[r, "eG"'] & FG. \end{tikzcd}$$

Since $\lambda \cdot eG = Ge$, it follows that

$$\lambda \cdot Fg \cdot e = \lambda \cdot eG \cdot g = Ge \cdot g = gF \cdot e.$$

Thus there exists a unique morphism $e' : I \to F^g$ yielding a commutative diagram

$$\begin{tikzcd} F^g \arrow[r, "i_F"] & F \\ I. \arrow[u, "e'"] \arrow[ur, "e"'] & \end{tikzcd}$$

Observe that

(α) the diagrams

$$\begin{tikzcd} FF \arrow[r, "FFg"] \arrow[d, "m"'] & FFG \arrow[d, "mG"] \\ F \arrow[r, "Fg"'] & FG, \end{tikzcd} \qquad \begin{tikzcd} FF \arrow[r, "gFF"] \arrow[d, "m"'] & GFF \arrow[d, "Gm"] \\ F \arrow[r, "gF"'] & GF \end{tikzcd}$$

commute by naturality of composition,
(β) $\lambda \cdot mG = Gm \cdot \lambda F \cdot F\lambda$, since λ is an entwining,
(γ) $\lambda \cdot Fg \cdot i_F = gF \cdot i_F$, since i_F is an equaliser of gF and $\lambda \cdot Fg$,
(δ) $i_F i_F = i_F F \cdot F^g i_F = Fi_F \cdot i_F F^g$, by naturality of composition.

Hence we have

$$\begin{aligned} \lambda \cdot Fg \cdot m \cdot i_F i_F &=_{(\alpha)} \lambda \cdot mG \cdot FFg \cdot i_F i_F \\ &=_{(\beta)} Gm \cdot \lambda F \cdot F\lambda \cdot FFg \cdot i_F i_F \\ &=_{(\delta)} Gm \cdot \lambda F \cdot F\lambda \cdot FFg \cdot Fi_F \cdot i_F F^g \\ &=_{(\gamma)} Gm \cdot \lambda F \cdot FgF \cdot Fi_F \cdot i_F F^g \\ &=_{(\delta)} Gm \cdot \lambda F \cdot FgF \cdot i_F F \cdot F^g i_F \\ &=_{(\gamma)} Gm \cdot gFF \cdot i_F F \cdot F^g i_F \\ &=_{(\delta)} Gm \cdot gFF \cdot i_F i_F \\ &=_{(\alpha)} gF \cdot m \cdot i_F i_F. \end{aligned}$$

Considering now the diagram

$$\begin{tikzcd}
F^gF^g \arrow[r, "i_Fi_F"] \arrow[d, dashed, "m'"'] & FF \arrow[r, "FgF"] \arrow[rr, bend left, "gFF"] \arrow[d, "m"'] & FGF \arrow[r, "\lambda F"] & GFF \arrow[d, "Gm"] \\
F^g \arrow[r, "i_F"'] & F \arrow[r, "Fg"] \arrow[rr, bend left, "gF"] & FG \arrow[r, "\lambda"] & GF,
\end{tikzcd}$$

one sees that there exists a unique morphism $m' : F^gF^g \to F^g$ making the left square of the diagram commute. The result now follows from [3, Lemma 3.2]. $\square$

As we have seen, the morphism $\beta = \tilde{g} : F \to GF$ makes F a left G-comodule. Consider the related functor $K_g : \mathbb{A} \to (\mathbb{A}_F)^{\widehat{G}}$ and write $t : \phi_F U_F \to \widehat{G}$ for the corresponding morphism of comonads on $\mathbb{A}_F$. It is easy to see that for any $(a, h_a) \in \mathbb{A}_F$, $t_{(a,h_a)}$ is the composite

$$F(a) \xrightarrow{F(g_a)} FG(a) \xrightarrow{\lambda_a} GF(a) \xrightarrow{G(h_a)} G(a). \tag{3.1}$$

Since $Fg \cdot e = eG \cdot g$ by naturality of composition and $\lambda \cdot eG = Ge$, the $(a, h_a) \in \mathbb{A}_F$-component of the morphism

$$\gamma : U_F \xrightarrow{\eta^F U_F} U_F \phi_F U_F \xrightarrow{U_F t} U_F \widehat{G}$$

is just the morphism $g_a : a \to G(a)$. It follows that the monad generated by the functor K_g and its right adjoint R_g is given by the equaliser of the diagram

$$F \underset{\tilde{g} = \lambda \cdot Fg}{\overset{gF}{\rightrightarrows}} GF.$$

Thus F^g is just the monad on $\mathbb{A}$ generated by the adjunction $K_g \dashv R_g$.

Since any functor with a right adjoint is full and faithful if and only if the unit of the adjunction is an isomorphism, we have

3.5. Proposition. *Let $g : I \to G$ be a grouplike morphism. Then the corresponding functor $K_g : \mathbb{A} \to (\mathbb{A}_F)^{\widehat{G}}$ is full and faithful if and only if the functor F^g is (isomorphic to) the identity monad on $\mathbb{A}$.*

For an entwining $\lambda : TG \to GT$ and a grouplike morphism $g : I \to G$, for any $(a, h_a) \in \mathbb{A}_F$, the (a, h_a)-component $t_{(a,h_a)}$ of the comonad morphism $t : \phi_F U_F \to \widehat{G}$, corresponding to the functor $K_g : \mathbb{A} \to (\mathbb{A}_F)^{\widehat{G}}$, is given in (3.1). Consider the diagram

$$
\begin{tikzcd}
F(a) \arrow[r, "F(e_a)"] \arrow[d, "F(g_a)"'] \arrow[dr, phantom, "(1)"] & FF(a) \arrow[d, "F(g_{F(a)})"] \arrow[dr, phantom, "(2)"] & F(a) \arrow[l, "e_{F(a)}"'] \arrow[d, "g_{F(a)}"] \\
FG(a) \arrow[r, "FG(e_a)"] \arrow[d, "\lambda_a"'] \arrow[dr, phantom, "(3)"] & FGF(a) \arrow[d, "\lambda_{F(a)}"'] \arrow[r, phantom, "(4)"] & GF(a) \arrow[l, "e_{GF(a)}"'] \arrow[dl, "G(e_{F(a)})"] \arrow[dd, equal] \\
GF(a) \arrow[r, "GF(e_a)"] \arrow[dr, equal] \arrow[r, phantom, shift right=4ex, "(5)"] & GFF(a) \arrow[d, "G(m_a)"] \arrow[r, phantom, "(6)"] & \\
 & GF(a) \arrow[r, equal] & GF(a),
\end{tikzcd}
$$

in which

- diagram (1) is commutative by naturality of $g : I \to G$;
- diagram (2) is commutative by naturality of composition;
- diagram (3) is commutative by naturality of $\lambda : FG \to GF$;
- diagram (4) is commutative since λ is an entwining, and
- diagrams (5) and (6) are commutative since F is a monad.

It follows from the commutativity of this diagram that the diagram

$$
\begin{tikzcd}
F(a) \arrow[r, shift left, "e_{F(a)}"] \arrow[r, shift right, "F(e_a)"'] \arrow[dddd, equal] & FF(a) \arrow[d, "F(g_{F(a)})"] \\
 & FGF(a) \arrow[d, "\lambda_{F(a)}"] \\
 & GFF(a) \arrow[d, "G(m_a)"] \\
 & \\
F(a) \arrow[r, shift left, "g_{F(a)}"] \arrow[r, shift right, "\lambda_a \cdot F(g_a)"'] & GF(a)
\end{tikzcd}
\tag{3.2}
$$

is serially commutative.

3.6. Proposition. *Let $\lambda : TG \to GT$ be an entwining and $g : I \to G$ be a grouplike morphism. If the monad F is of descent type (that is, the free F-algebra functor $\phi_F : \mathbb{A} \to \mathbb{A}_F$ is precomonadic) and if the monad F is G-Galois w.r.t. the G-coaction $\tilde{g} : F \to GF$ (see 3.2), then the monad F^g is (isomorphic to) the identity monad.*

Proof. To say that F is of descent type is to say that the diagram

$$
\begin{tikzcd}
a \arrow[r, "e_a"] & F(a) \arrow[r, shift left, "e_{F(a)}"] \arrow[r, shift right, "F(e_a)"'] & FF(a)
\end{tikzcd}
$$

is a coequaliser diagram for all $a \in \mathbb{A}$ (see [9]), while to say that the monad F is G-Galois w.r.t. G-coaction $\tilde{g} : F \to GF$ is to say that, for any $a \in \mathbb{A}$, the composite $G(m_a) \cdot \lambda_{F(a)} \cdot F(g_{F(a)})$ is an isomorphism. The result now follows from the commutativity of the diagram (3.2). □

3.7. Left adjoint of $(i_F)^*$. Since $i_F : F^g \to F$ is a morphism of monads, it induces a functor

$$(i_F)^* : \mathbb{A}_F \longrightarrow \mathbb{A}_{F^g}, \quad (a, h_a) \longmapsto \big(a, h_a \cdot (i_F)_a\big).$$

Moreover, when the category $\mathbb{A}_F$ has coequalisers of reflexive pairs (which is certainly the case if $\mathbb{A}$ has coequalisers of reflexive pairs and F preserves them), $(i_F)^*$ has a left adjoint $(i_F)_! : \mathbb{A}_{F^g} \to \mathbb{A}_F$ which is defined as follows: For notational reasons, write

$$\eta, \sigma : V \dashv U : \mathbb{A}_F \longrightarrow \mathbb{A} \quad \big(\text{resp. } \eta', \sigma' : V' \dashv U' : \mathbb{A}_{F^g} \longrightarrow \mathbb{A}\big)$$

for the forgetful-free adjunction (ϕ_F, U_F) (resp. (ϕ_{F^g}, U_{F^g})). Then $(i_F)_!$ is the coequaliser of the diagram of functors and natural transformations

$$VU'V'U' \underset{\beta}{\overset{VU'\sigma'}{\rightrightarrows}} VU' \xrightarrow{\quad q \quad} (i_F)_!, \tag{3.3}$$

where β is the composite

$$VU'V'U' \xrightarrow{VU'V'\eta U'} VU'V'UVU' = VU'V'U'(i_F)^*VU'$$
$$\xrightarrow{VU'\sigma'(i_F)^*VU'} VU'(i_F)^*VU' = VUVU' \xrightarrow{\quad \sigma VU' \quad} VU'.$$

It is not hard to see that for any $(a, h_a) \in \mathbb{A}_{F^g}$, the (a, h_a)-component of the diagram (3.3) is the diagram

$$\begin{tikzcd} FF^g(a) \arrow[r, "F((i_F)_a)"'] \arrow[rr, bend left, "F(h_a)"] & FF(a) \arrow[r, "m_a"'] & F(a) \arrow[r, "q_a"] & (i_F)_!(a, h_a). \end{tikzcd}$$

Let $\widehat{G}$ be the comonad on $\mathbb{A}_F$ that is the lifting of the comonad **G** corresponding to the entwining λ. Then for any $(a, h_a) \in \mathbb{A}_F$, $\widehat{G}(a, h_a) = (G(a), G(h_a) \cdot \lambda_a)$.

3.8. Lemma. *With the data given in* 3.2, *we have the morphism*

$$\tilde{g}_a : F(a) \longrightarrow GF(a) \quad \text{for any } (a, h_a) \in \mathbb{A}_F.$$

(i) *Each $\tilde{g}_a$ can be seen as a morphism in $\mathbb{A}_F$ from the free F-module $V(a) = (F(a), m_a)$ to the F-module*

$$\widehat{G}\big(V(a)\big) = \widehat{G}\big(F(a), m_a\big) = \big(GF(a), G(m_a) \cdot \lambda_{F(a)}\big).$$

(ii) *The family $(\tilde{g}_{F(a)})_{a \in \mathbb{A}}$ induces a natural transformation $\alpha_V : V \to \widehat{G}V$ making V a left $\widehat{G}$-comodule.*

Proof. (i) Consider the diagram

$$\begin{tikzcd}
FF(a) \arrow[r, "FF(g_a)"] \arrow[dd, "m_a"'] \arrow[ddr, phantom, "(1)"] & FFG(a) \arrow[r, "F(\lambda_a)"] \arrow[dd, "m_{G(a)}"'] \arrow[ddr, phantom, "(2)"] & FGF(a) \arrow[d, "\lambda_{F(a)}"] \\
 & & GFF(a) \arrow[d, "G(m_a)"] \\
F(a) \arrow[r, "F(g_a)"'] & FG(a) \arrow[r, "\lambda_a"'] & GF(a),
\end{tikzcd}$$

in which part (1) commutes by naturality of m, while part (2) commutes since λ is an entwining. Thus the outer rectangle is commutative, which just means that $\tilde{g} : F(a) \to GF(a)$ is a morphism in $\mathbb{A}_F$ from the free F-module $V(a) = (F(a), m_a)$ to the F-module $(GF(a), G(m_a) \cdot \lambda_{F(a)})$.

(ii) Using that for any $(a, h_a) \in \mathbb{A}_F$,

- $(\varepsilon_{\widehat{G}})_{(a,h_a)} = (\varepsilon_G)_a$,
- $(\delta_{\widehat{G}})_{(a,h_a)} = (\delta_G)_a$,
- $(F, \tilde{g})$ is a left G-comodule,

it is not hard to prove that the pair (V, α_V) is a left $\widehat{G}$-comodule. □

3.9. Lemma. *With the notation above,*

(1) *the left rectangle in the diagram*

$$\begin{tikzcd}
VU'V'U' \arrow[r, shift left, "VU'\sigma'"] \arrow[r, shift right, "\beta"'] \arrow[d, "\alpha_V U'V'U'"'] & VU' \arrow[r, "q"] \arrow[d, "\alpha_V U'"] & (i_F)_! \arrow[d, dashed, "\alpha_{(i_F)_!}"] \\
\widehat{G}VU'V'U' \arrow[r, shift left, "\widehat{G}VU'\sigma'"] \arrow[r, shift right, "\widehat{G}\beta"'] & \widehat{G}VU' \arrow[r, "\widehat{G}q"] & \widehat{G}(i_F)_!
\end{tikzcd}$$

is serially commutative;

(2) *there exists a unique natural transformation $\alpha_{(i_F)_!} : (i_F)_! \to \widehat{G}(i_F)_!$ making the right square of the diagram commute.*

Proof. (2) follows from the fact that q is a coequaliser of $VU'\sigma'$ and β.

(1) To show that the left square is serially commutative, we have to show that for any $(a, h_a) \in \mathbb{A}_F$, the diagram

$$\begin{tikzcd}
FF^g(a) \arrow[r, "F((i_F)_a)"'] \arrow[rr, bend left, "F(h_a)"] \arrow[d, "(\tilde{g})_{F^g(a)}"'] & FF(a) \arrow[r, "m_a"'] & F(a) \arrow[d, "(\tilde{g})_a"] \\
GFF^g(a) \arrow[r, "GF((i_F)_a)"'] \arrow[rr, bend left, "GF(h_a)"] & GFF(a) \arrow[r, "G(m_a)"'] & GF(a)
\end{tikzcd}$$

is so. The left diagram below is commutative by naturality of $g : I \to G$,

$$\begin{tikzcd}
F^g(a) \arrow[r, "h_a"] \arrow[d, "g_{F^g(a)}"'] & a \arrow[d, "g_a"] \\
GF^g(a) \arrow[r, "G(h_a)"'] & G(a),
\end{tikzcd}
\qquad
\begin{tikzcd}
FGF^g(a) \arrow[r, "FG(h_a)"] \arrow[d, "\lambda_{F^g(a)}"'] & FG(a) \arrow[d, "\lambda_a"] \\
GFF^g(a) \arrow[r, "GF(h_a)"'] & GF(a)
\end{tikzcd}$$

while the right square is commutative by naturality of λ. From this we obtain the commutative diagram

$$\begin{tikzcd}
FF^g(a) \arrow[r, "F(h_a)"] \arrow[d, "F(g_{F^g(a)})"'] & F(a) \arrow[d, "F(g_a)"] \\
FGF^g(a) \arrow[r, dashed, "FG(h_a)"'] \arrow[d, "\lambda_{F^g(a)}"'] & FG(a) \arrow[d, "\lambda_a"] \\
GFF^g(a) \arrow[r, "GF(h_a)"'] & GF(a).
\end{tikzcd}$$

Next, consider the diagram

$$\begin{tikzcd}
FF^g(a) \arrow[rr, "F((i_F)_a)"] \arrow[dr, "F((i_F)_a)"] \arrow[dd, "F(g_{F^g(a)})"'] & \arrow[d, phantom, "(2)"] & FF(a) \arrow[r, "m_a"] \arrow[d, "FF(g_a)"] \arrow[dr, phantom, "(3)"] & F(a) \arrow[d, "F(g_a)"] \\
\arrow[r, phantom, "(1)"] & FF(a) \arrow[dr, "F(g_{F(a)})"] & FFG(a) \arrow[r, "m_{G(a)}"'] \arrow[d, "F(\lambda_a)"] \arrow[dr, phantom, "(4)"] & FG(a) \arrow[d, "\lambda_{F(a)}"] \\
FGF^g(a) \arrow[rr, "FG((i_F)_a)"'] \arrow[d, "\lambda_{F^g(a)}"'] & \arrow[d, phantom, "(5)"] & FGF(a) \arrow[d, "\lambda_{F(a)}"'] & GF(a) \\
GFF^g(a) \arrow[rr, "GF((i_F)_a)"'] & & GFF(a) \arrow[ur, "G(m_a)"'] &
\end{tikzcd}$$

in which

- diagram (1) commutes by naturality of composition;
- diagram (2) commutes by (γ) in proof of 3.4;
- diagram (3) commutes by naturality of m;
- diagram (4) commutes since λ is an entwining, and
- diagram (5) commutes by naturality of λ.

Thus the outer diagram is commutative and this completes the proof of the lemma. $\square$

3.10. Natural transformation $S_{\phi_{F^g}}$. In 3.8, a left $\widehat{G}$-comodule (V, α_V) is considered and by commutativity of the diagram in 3.9, the pair $((i_F)_!, \alpha_{(i_F)_!})$ is also a left $\widehat{G}$-comodule. Thus, as noted in 1.1, there exists a unique functor $\overline{i_F} : \mathbb{A}_{F^g} \to (\mathbb{A}_F)^{\widehat{G}}$ yielding commutativity in the right triangle of the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\phi_{F^g}"] \arrow[dr, "\phi_F"'] & \mathbb{A}_{F^g} \arrow[r, "\overline{i_F}"] \arrow[d, "(i_F)_!"] & (\mathbb{A}_F)^{\widehat{G}} \arrow[dl, "U^{\widehat{G}}"] \\ & \mathbb{A}_F & \end{tikzcd} \tag{3.4}$$

where $U^{\widehat{G}} : (\mathbb{A}_F)^{\widehat{G}} \to \mathbb{A}_F$ is the evident forgetful functor.

A direct inspection shows that the diagram

$$\begin{tikzcd} FF^gF^g \arrow[r, "Fi_FF^g"] \arrow[rr, bend left, "Fm'"] & FFF^g \arrow[r, "mF^g"] & FF^g \arrow[r, "Fi_F"] \arrow[ll, bend left, "FF^ge'"] & FF \arrow[r, "m"] & F \arrow[ll, bend left, "Fe'"] \end{tikzcd}$$

is a split coequaliser diagram. This means in particular that for any $a \in \mathbb{A}$,

$$(i_F)_!\big(\phi_{F^g}(a)\big) = (i_F)_!\big(F^g(a), m'_a\big) = \big(F(a), m_a\big) = \phi_F(a).$$

Thus the left triangle in the diagram is also commutative.

Consider the related comonad morphisms

- $S_{\phi_{F^g}} : \phi_F U_F \to (i_F)_!(i_F)^*$ corresponding to the left triangle in (3.4),
- $S_{\overline{i_F}} : (i_F)_!(i_F)^* \to \widehat{G}$ corresponding to the right triangle in (3.4),
- and $S_{\overline{i_F}\cdot\phi_{F^g}} = t : \phi_F U_F \to \widehat{G}$ corresponding to the outer diagram in (3.4).

Then it follows from Proposition 1.21 that $t = S_{\overline{i_F}} \cdot S_{\phi_{F^g}}$.

3.11. Lemma. *With the notation from* 3.10*, for any* $(a, h_a) \in \mathbb{A}_F$*, the* (a, h_a)*-component of the natural transformation* $S_{\phi_{F^g}}$ *is the morphism*

$$q_a : F(a) \longrightarrow (i_F)_!\big((i_F)^*(a, h_a)\big) = (i_F)_!\big(a, h_a \cdot (i_F)_a\big).$$

Proof. Consider the natural transformation $\alpha : \phi_{F^g} U_F \to (i_F)^*$ corresponding to the left triangle in (3.4) which is the composite

$$\phi_{F^g} U_F \xrightarrow{\ \overline{\eta}\phi_{F^g}U_F\ } (i_F)^*(i_F)_!\phi_{F^g}U_F = (i_F)^*\phi_F U_F \xrightarrow{\ (i_F)^*\varepsilon_F\ } (i_F)^*,$$

where $\overline{\eta} : I \to (i_F)^*(i_F)_!$ is the unit of the adjunction $(i_F)_! \dashv (i_F)^*$. A simple calculation shows that, for any $(a, h_a) \in \mathbb{A}_F$, $\alpha_{(a,h_a)}$ is the composite

$$F^g(a) \xrightarrow{(i_F)_a} F(a) \xrightarrow{h_a} a.$$

Thus, the (a, h_a)-component of $S_{\phi_{F^g}}$ is the morphism

$$(i_F)_!\big(h_a \cdot (i_F)_a\big) : (i_F)_!\big(\phi_{F^g} U_F(a, h_a)\big) \longrightarrow (i_F)_!\big((i_F)^*(a, h_a)\big).$$

Since $\phi_{F^g} U_F(a, h_a) = \phi_{F^g}(a) = (F^g(a), m'_a)$ and $(i_F)^*(a, h_a) = (a, h_a \cdot (i_F)_a)$, it follows from the definition of $(i_F)_!$ that the diagram

$$\begin{tikzcd}
FF^gF^g(a) \arrow[rr, "F((i_F)_{F^g(a)})"'] \arrow[rrr, bend left, "F(m'_a)"] \arrow[d, "FF^g(h_a \cdot (i_F)_a)"'] & & FFF^g(a) \arrow[r, "m_{F^g(a)}"'] \arrow[dl, "FF(h_a \cdot (i_F)_a)"'] & FF^g(a) \arrow[r, "q_{F^g(a)}"] \arrow[d, "F(h_a \cdot (i_F)_a)"'] & (i_F)_!(\phi_{F^g} U_F(a, h_a)) \arrow[d, "(S_{\phi_{F^g}})_{(a,h_a)}"] \\
FF^g(a) \arrow[r, "F((i_F)_a)"] & FF(a) \arrow[rr, shift left, "F(h_a)"] \arrow[rr, shift right, "m_a"'] & & F(a) \arrow[r, "q_a"] & (i_F)_!((i_F)^*(a, h_a)),
\end{tikzcd}$$

whose rows are coequaliser diagrams, is commutative. Note now that the diagram

$$\begin{tikzcd}
FF^gF^g \arrow[r, "Fi_F F^g"] \arrow[rr, bend left, "Fm'"] & FFF^g \arrow[r, "mF^g"] & FF^g \arrow[r, "Fi_F"] \arrow[ll, bend left, "FF^g e'"] & FF \arrow[r, "m"] & F \arrow[ll, bend left, "Fe'"]
\end{tikzcd}$$

is a split coequaliser diagram. It follows that the diagram

$$\begin{tikzcd}
FF^g(a) \arrow[r, "m_a \cdot F((i_F)_a)"] \arrow[d, "F((i_F)_a)"'] & F(a) \arrow[d, "(S_{\phi_{F^g}})_{(a,h_a)}"] \\
F(a) \arrow[r, "q_a"'] & (i_F)_!((i_F)^*(a, h_a))
\end{tikzcd}$$

is commutative. Now, since $q_a \cdot F(h_a) \cdot F((i_F)_a) = q_a \cdot m_a \cdot F((i_F)_a)$ and since $(S_{\phi_{F^g}})_{(a,h_a)}$ is the unique morphism making the square commute, we see that $(S_{\phi_{F^g}})_{(a,h_a)} = q_a$. $\square$

3.12. Proposition. *With the notation from* 3.10, *suppose the natural transformation $t : \phi_F U_F \to \widehat{G}$ to be componentwise a monomorphism. Then $S_{\phi_{F^g}} : \phi_F U_F \to (i_F)_!(i_F)^*$ is an isomorphism. Thus, $S_{\overline{i_F}} : (i_F)_!(i_F)^* \to \widehat{G}$ is an isomorphism if and only if t is so.*

Proof. First note that, by the previous lemma, $S_{\phi_{F^g}}$ is a componentwise regular epimorphism. Now, since any regular epimorphism that is a monomorphism is an isomorphism and since $t = S_{\overline{i_F}} \cdot S_{\phi_{F^g}}$ (see 3.10), the result follows. $\square$

3.13. Galois entwinings. Write $\widetilde{G}$ for the comonad on the category $\mathbb{A}_F$ generated by the adjunction $(i_F)_! \dashv (i_F)^*$ and let $t_g : \widetilde{G} \to \widehat{G}$ be the related comonad morphism (see [18, Theorem 4.1]). This leads to a commutative diagram with the canonical comparison functor $K_{\widetilde{G}}$ (e.g. [18, Lemma 4.3])

$$\begin{tikzcd} \mathbb{A}_{F^g} \arrow[r, "K_{\widetilde{G}}"] \arrow[dr, "\overline{i_F}"'] & (\mathbb{A}_F)^{\widetilde{G}} \arrow[d, "(\mathbb{A}_F)_{t_g}"] \\ & (\mathbb{A}_F)^{\widehat{G}}. \end{tikzcd}$$

By Definition 1.3, the functor $(i_F)_!$ is $\widehat{G}$-Galois provided $t_g : \widetilde{G} \to \widehat{G}$ is an isomorphism. If this is the case we call $(\mathbf{F}, \mathbf{G}, \lambda, g)$ a *Galois entwining* and $g : I \to G$ a *Galois* (*grouplike*) *morphism* and we have:

3.14. Theorem. *Let $\lambda : FG \to GF$ be an entwining from a monad $\mathbf{F}$ to a comonad $\mathbf{G}$ on a category $\mathbb{A}$. Suppose that $g : I \to G$ is a grouplike morphism such that the corresponding functor $(i_F)^* : \mathbb{A}_F \to \mathbb{A}_{F^g}$ admits a left adjoint functor $(i_F)_! : \mathbb{A}_{F^g} \to \mathbb{A}_F$ (see 3.7). Then the comparison functor $\overline{i_F} : \mathbb{A}_{F^g} \to (\mathbb{A}_F)^{\widehat{G}}$ is an equivalence of categories if and only if $(\mathbf{F}, \mathbf{G}, \lambda, g)$ is a Galois entwining and the functor $(i_F)_!$ is comonadic.*

In the situation of the preceding theorem, if g is such that the corresponding comparison functor $K_g : \mathbb{A} \to (\mathbb{A}_F)^{\widehat{G}}$ is full and faithful, it follows from Proposition 3.5 that the functor $\overline{i_F}$ reduces to the functor K_g.

4. Bimonads

4.1. Properties of bimonads. Recall from [21, Definition 4.1] that a bimonad $\mathbf{H}$ on a category $\mathbb{A}$ is an endofunctor $H : \mathbb{A} \to \mathbb{A}$ which has a monad structure $\underline{H} = (H, m, e)$ and a comonad structure $\overline{H} = (H, \delta, \varepsilon)$ with an entwining $\lambda : HH \to HH$ inducing commutativity of the diagrams

$$\begin{tikzcd} HH \arrow[r, shift left, "\varepsilon H"] \arrow[r, shift right, "H\varepsilon"'] \arrow[d, "m"'] & H \arrow[d, "\varepsilon"] \\ H \arrow[r, "\varepsilon"] & I, \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "e"] \arrow[d, "e"'] & H \arrow[d, "\delta"] \\ H \arrow[r, shift left, "eH"] \arrow[r, shift right, "He"'] & HH, \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "e"] \arrow[dr, "="'] & H \arrow[d, "\varepsilon"] \\ & I, \end{tikzcd} \tag{4.1}$$

$$\begin{tikzcd} HH \arrow[r, "m"] \arrow[d, "H\delta"'] & H \arrow[r, "\delta"] & HH \\ HHH \arrow[rr, "\lambda H"'] & & HHH. \arrow[u, "Hm"'] \end{tikzcd} \tag{4.2}$$

Joining H from the left to the central diagram in (4.1) and attaching the resulting square on the left-hand side of (4.2), one derives the relation

$$\lambda \cdot He = \delta. \tag{4.3}$$

For the bimonad $\mathbf{H}$ we obtain the comparison functor

$$K_H : \mathbb{A} \longrightarrow \mathbb{A}_H^H, \quad a \longrightarrow \big(H(a), m_a, \delta_a\big),$$

where $\mathbb{A}_H^H = \mathbb{A}_{\underline{H}}^{\overline{H}}(\lambda)$, with commutative diagrams

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "K_H"] \arrow[dr, "\phi_{\underline{H}}"'] & (\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}} \simeq \mathbb{A}_H^H \arrow[d, "U^{\widehat{\overline{H}}}"] \\
 & \mathbb{A}_{\underline{H}},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[r, "K_H"] \arrow[dr, "\phi^{\overline{H}}"'] & (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}} \simeq \mathbb{A}_H^H \arrow[d, "U_{\widehat{\underline{H}}}"] \\
 & \mathbb{A}^{\overline{H}}.
\end{tikzcd}
\tag{4.4}
$$

As noticed in [28, 5.13], the comparison functor K_H is full and faithful by the isomorphism

$$\mathrm{Mor}_H^H\big(H(a), H(b)\big) \longrightarrow \mathrm{Mor}_{\mathbb{A}}(a, b), \quad f \longmapsto \varepsilon_b \circ f \circ e_a.$$

We now reconsider bimonads and Hopf monads in view of the notions introduced in the preceding sections. It is clear from (4.1) that the unit $e : I \to H$ is a grouplike morphisms (as defined in 3.1). Write γ for the composite $Hm \cdot \delta H$. Then since $\gamma \cdot He = \delta$ (see [21, (5.2)]), it is easy to see that the functor K_H is just the functor K_e corresponding to the grouplike morphisms $e : I \to H$. Then, since the functor K_H is full and faithful, it follows from Proposition 3.5 that the diagram

$$
\begin{tikzcd}
I \arrow[r, "e"] & H \arrow[r, shift left, "eH"] \arrow[r, shift right, "\delta=\lambda\cdot He"'] & HH
\end{tikzcd}
$$

is an equaliser diagram. Therefore the functor F^e from 3.4, that is $\underline{H}^{\overline{H}}$, is just the identity on $\mathbb{A}$. Thus $(i_{\underline{H}})^*$ turns out to be the forgetful functor $U_{\underline{H}} : \mathbb{A}_{\underline{H}} \to \mathbb{A}$ and its left adjoint $(i_{\underline{H}})_!$ is the free functor $\phi_{\underline{H}} : \mathbb{A} \to \mathbb{A}_{\underline{H}}$. Now, since the unit of the adjunction $\phi_H \dashv U_H$ is a split monomorphism, the functor $\phi_{\underline{H}} : \mathbb{A} \to \mathbb{A}_{\underline{H}}$ is always comonadic, provided the category $\mathbb{A}$ is Cauchy complete (see Corollary 3.19 in [19]), it follows from 3.14:

4.2. $\phi_{\underline{H}}$ as $\widehat{\overline{H}}$-Galois functor. *For a bimonad* **H** *on a Cauchy complete category $\mathbb{A}$, the following are equivalent*:

(a) *$\phi_{\underline{H}}$ is an $\widehat{\overline{H}}$-Galois functor*;
(b) *the unit $e : I \to H$ is a Galois grouplike morphism*;
(c) *the functor $K_H : \mathbb{A} \to \mathbb{A}_H^H$ is an equivalence of categories.*

4.3. Proposition. *Assume that $\mathbb{A}$ admits equalisers and that H has a right adjoint. Then the following are equivalent*:

(a) *the functor $K_H : \mathbb{A} \to \mathbb{A}_H^H$ is an equivalence of categories*;
(b) *(H, m) is $\overline{H}$-Galois*;
(c) **H** *has an antipode.*

Proof. Clearly (a) implies (b), while the equivalence of (a) and (c) is proved in [21, 5.6]. So suppose that (H, m) is $\overline{H}$-Galois. Then it follows from Proposition 2.7 that $\phi_{\underline{H}}$ is $\widehat{\overline{H}}$-Galois, i.e. the comonad morphism $t_{\phi_{\underline{H}}} : \phi_{\underline{H}} U_{\underline{H}} \to \widehat{\overline{H}}$ is an isomorphism. Now, since the category $\mathbb{A}$ admits equalisers, it is Cauchy complete, and as it was noted above, the functor $\phi_{\underline{H}}$ is always comonadic, it follows from Proposition 1.2 that K_H is an equivalence of categories. This completes the proof. $\square$

Dually, one has

4.4. Proposition. *Assume that $\mathbb{A}$ admits coequalisers and that H has a left adjoint. Then the following are equivalent*:

(a) *the functor $K_H : \mathbb{A} \to \mathbb{A}_H^H$ is an equivalence of categories*;
(b) *(H, δ) is $\underline{H}$-Galois*;
(c) **H** *has an antipode.*

Summarising the previous observations yields the main result of this section:

4.5. Theorem. *Let* **H** *be a bimonad on the category $\mathbb{A}$ and assume that*

(i) *$\mathbb{A}$ has small limits or colimits and H preserves them, or*
(ii) *$\mathbb{A}$ admits equalisers and H has a right adjoint, or*
(iii) *$\mathbb{A}$ admits coequalisers and H has a left adjoint.*

Then the functor $K_H : \mathbb{A} \to \mathbb{A}_H^H$ is an equivalence of categories if and only if **H** *has an antipode.*

Proof. The assertions follow by Propositions 2.7, 2.8, 4.3 and 4.4. □

5. Opmonoidal monads

Let $(\mathbb{V}, \otimes, \mathbb{I})$ be a strict monoidal category.

5.1. Opmonoidal monads. Let $\mathbf{T} = (T, m, e)$ be a monad on $\mathbb{V}$, such that the functor T and the natural transformations m and e are opmonoidal, that is, there are natural transformations

$$\chi_{X,Y} : T(X \otimes Y) \longrightarrow T(X) \otimes T(Y) \quad \text{for } X, Y \in \mathbb{V}$$

and a morphism $\theta_{\mathbb{I}} : T(\mathbb{I}) \to \mathbb{I}$ satisfying certain compatibility axioms. Following McCrudden [17] we call such monads *opmonoidal monads*. They were introduced in Moerdijk [22] under the name *Hopf monads* and are named *bimonads* by Bruguières and Virelizier in [7, Section 2.3].

It follows from the definition of an opmonoidal monad **T** that the triple

$$\big(T(\mathbb{I}), \chi_{\mathbb{I},\mathbb{I}} : T(\mathbb{I}) \longrightarrow T(\mathbb{I}) \otimes T(\mathbb{I}), \theta_{\mathbb{I}} : T(\mathbb{I}) \longrightarrow \mathbb{I}\big)$$

is a coalgebra in $\mathbb{V}$ (see [7, p. 704]), and thus one has a comonad **G** on $\mathbb{V}$ whose functor part is $G = - \otimes T(\mathbb{I})$. Then the compatibility axioms ensure that the natural transformation

$$\lambda := \big(T(-) \otimes m_{\mathbb{I}}\big) \cdot \chi_{-,T(\mathbb{I})} : TG \longrightarrow GT,$$

is a mixed distributive law (entwining) from the monad **T** to the comonad **G**.

5.2. Entwined modules. For an opmonoidal monad **T** on $\mathbb{V}$, the *entwined modules* are objects $M \in \mathbb{V}$ with a T-module structure $h : T(M) \to M$ and a comodule structure $\rho : M \to M \otimes T(\mathbb{I})$ inducing commutativity of the diagram

$$\begin{tikzcd}
T(M) \arrow[r, "h"] \arrow[d, "T(\rho)"'] & M \arrow[r, "\rho"] & M \otimes T(\mathbb{I}) \\
T(M \otimes T(\mathbb{I})) \arrow[r, "\chi_{M,T(\mathbb{I})}"] & T(M) \otimes TT(\mathbb{I}) \arrow[r, "T(M)\otimes m_{\mathbb{I}}"] & T(M) \otimes T(\mathbb{I}). \arrow[u, "h\otimes T(\mathbb{I})"']
\end{tikzcd}$$

(In [7, Section 4.2] these are named *right Hopf T-modules*.) They form a category in an obvious way which we denote by $\mathbb{V}_T^G$.

From the ingredients of the definition one obtains the commutative diagram

$$\begin{tikzcd}
TT \arrow[r, "m"] \arrow[d, "T\chi_{-,\mathbb{I}}"'] & T \arrow[r, "\chi_{-,\mathbb{I}}"] & T(-)\otimes T(\mathbb{I}) \\
T(T(-)\otimes T(\mathbb{I})) \arrow[r, "\chi_{T(-),T(\mathbb{I})}"] & TT(-)\otimes TT(\mathbb{I}) \arrow[r, "TT(-)\otimes m_{\mathbb{I}}"] & TT(-)\otimes T(\mathbb{I}) \arrow[u, "m\otimes T(\mathbb{I})"']
\end{tikzcd}$$

which shows that for any $X \in \mathbb{V}$, $T(X)$ is an entwined **T**-module leading to the commutative diagram

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "K"] \arrow[dr, "\phi_T"'] & (\mathbb{V}_T)^{\widehat{G}} = \mathbb{V}_T^G \arrow[d, "U^{\widehat{G}}"] \\
& \mathbb{V}_T,
\end{tikzcd}$$

with a comparison functor $K(X) = (T(X), m_X, \chi_{X,\mathbb{I}})$.

For the corresponding comonad morphism $t_K : \phi_T U_T \to \widehat{G}$, it is easy to see that for any $(X, h_X) \in V_T$, the (X, h_X)-component of t_K is the composite

$$T(X) \xrightarrow{\chi_{X,\mathbb{I}}} T(X)\otimes T(\mathbb{I}) \xrightarrow{h_X \otimes T(\mathbb{I})} X \otimes T(\mathbb{I}).$$

Since t_K is a comonad morphism, we have the commutative diagram

$$\begin{tikzcd}
\phi_T U_T \arrow[r, "t_K"] \arrow[dr, "\varepsilon_T"'] & \widehat{G} \arrow[d, "\varepsilon_{\widehat{G}}"] \\
& I,
\end{tikzcd}$$

and since, for any $(X, h_X) \in \mathbb{V}$, $(\varepsilon_T)_{(X,h_X)} = h_X$ and $(\varepsilon_{\widehat{G}})_{(X,h_X)} = X \otimes \theta_{\mathbb{I}}$, we have

5.3. Lemma. *For any* $(X, h_X) \in \mathbb{V}_T$,

$$(X \otimes \theta_{\mathbb{I}}) \cdot (t_K)_{(X,h_X)} = h_X.$$

5.4. Remark. Note that there are also functors

$$\mathbb{V}_T \longrightarrow \mathbb{V}_T^G, \qquad (M, h) \longmapsto \big(M \otimes T(\mathbb{I}), \tilde{h} := (h \otimes m_{\mathbb{I}}) \circ \chi_{M,T(\mathbb{I})}, M \otimes \chi_{\mathbb{I},\mathbb{I}}\big),$$
$$\mathbb{V}^G \longrightarrow \mathbb{V}_T^G, \qquad (N, \rho) \longmapsto \big(T(N), m_N, \widehat{\rho} := \big(T(N) \otimes m_{\mathbb{I}}\big) \circ \chi_{N,T\mathbb{I}} \circ T(\rho)\big).$$

The second functor corresponds to [7, Lemma 4.3].

5.5. Grouplike morphism. Since $\mathbf{T}$ is an opmonoidal monad on $\mathbb{V}$, the following two diagrams

$$\begin{tikzcd} \mathbb{I} \arrow[r, "e_{\mathbb{I}}"] \arrow[dr, equal] & T(\mathbb{I}) \arrow[d, "\theta_{\mathbb{I}}"] \\ & \mathbb{I} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{I} \arrow[r, "e_{\mathbb{I}}"] \arrow[dr, "e_{\mathbb{I}}\otimes e_{\mathbb{I}}"'] & T(\mathbb{I}) \arrow[d, "\chi_{\mathbb{I},\mathbb{I}}"] \\ & T(\mathbb{I})\otimes T(\mathbb{I}) \end{tikzcd}$$

both are commutative, implying that the natural transformation

$$g := -\otimes e_{\mathbb{I}} : 1 \longrightarrow -\otimes T(\mathbb{I})$$

is a grouplike morphism. Note that $gT : T \to T \otimes T(\mathbb{I})$ is the natural transformation given by $T(X) \otimes e_{\mathbb{I}} : T(X) \to T(X) \otimes T(\mathbb{I})$, while $\lambda \cdot Tg : T \to T \otimes T(\mathbb{I})$ is given by the composite

$$T(X) \xrightarrow{T(X\otimes e_{\mathbb{I}})} T\big(X \otimes T(\mathbb{I})\big) \xrightarrow{\chi_{X,T(\mathbb{I})}} T(X) \otimes TT(\mathbb{I}) \xrightarrow{T(X)\otimes m_{\mathbb{I}}} T(X) \otimes T(\mathbb{I}).$$

Since T is an opmonoidal monad, the diagram

$$\begin{tikzcd} T(X) \arrow[r, "T(X\otimes e_{\mathbb{I}})"] \arrow[d, "\chi_{X,\mathbb{I}}"'] & T(X \otimes T(\mathbb{I})) \arrow[d, "\chi_{X,T(\mathbb{I})}"] \\ T(X) \otimes T(\mathbb{I}) \arrow[r, "T(X)\otimes T(e_{\mathbb{I}})"'] & T(X) \otimes TT(\mathbb{I}) \end{tikzcd}$$

is commutative. But since $m_{\mathbb{I}} \cdot e_{\mathbb{I}} = I$, we see that $\lambda \cdot Tg$ is just the natural transformation $\chi_{-,\mathbb{I}}$. Thus, for any $X \in \mathbb{V}$, $T^g(X)$ is the equaliser

$$\begin{tikzcd} T^g(X) \arrow[r] & T(X) \arrow[r, shift left, "T(X)\otimes e_{\mathbb{I}}"] \arrow[r, shift right, "\chi_{X,\mathbb{I}}"'] & T(X) \otimes T(\mathbb{I}). \end{tikzcd}$$

Note that the functor $K : \mathbb{V} \to \mathbb{V}_T^G$ is just the functor $K_g : \mathbb{V} \to \mathbb{V}_T^G = (\mathbb{V}_T)^{\widehat{G}}$.

5.6. Antipodes. For opmonoidal monads on a right autonomous category $\mathbb{V}$, a *right antipode* is defined in [7, Section 3.3] and its existence is equivalent to the fact that the category of $\mathbf{T}$-modules is right autonomous [7, Theorem 3.8].

From now on we suppose that $\mathbf{T}$ is an opmonoidal monad, on a right autonomous category, with a right antipode (a *right Hopf monad* in the sense of [7, Section 3.6]).

Consider the natural transformation $\Gamma : G \to TT$ defined in [7, Section 4.5]. We shall need the following simple properties of this functor (see [7, Lemma 4.9]):

$$m \cdot \Gamma = e \otimes \theta_{\mathbb{I}}, \tag{5.1}$$

$$Tm \cdot \Gamma T \cdot \chi_{-,\mathbb{I}} = Te. \tag{5.2}$$

Using these, one can calculate (see [7]) that for any $(X, h_X, \vartheta_X) \in \mathbb{V}_T^G$,

$$h_X \cdot T(h_X) \cdot \Gamma_X \cdot \vartheta_X = I_X. \tag{5.3}$$

5.7. Lemma. *For any* $(X, h_X, \vartheta_X) \in \mathbb{V}_T^G$*, the morphism*

$$(t_K)_{(X,h_X)} = \big(h_X \otimes T(\mathbb{I})\big) \cdot \chi_{X,\mathbb{I}} : T(X) \longrightarrow X \otimes T(\mathbb{I})$$

is a split monomorphism.

Proof. For any $(X, h_X, \vartheta_X) \in \mathbb{V}_T^G$, consider the composite

$$q_{(X,h_X)} = T(h_X) \cdot \Gamma_X : X \otimes T(\mathbb{I}) \longrightarrow T(X).$$

We claim that $q_{(X,h_X)} \cdot (t_K)_{(X,h_X)} = I$. Indeed, consider the diagram

$$\begin{tikzcd}
T(X) \arrow[r, "\chi_{X,\mathbb{I}}"] \arrow[ddr, "T(e_X)"'] & T(X) \otimes T(\mathbb{I}) \arrow[r, "h_X \otimes T(\mathbb{I})"] \arrow[d, "\Gamma_{T(X)}"] \arrow[dr, phantom, "(1)"] & X \otimes T(\mathbb{I}) \arrow[d, "\Gamma_X"] \\
 & TTT(X) \arrow[r, "TT(h_X)"'] \arrow[d, "T(m_X)"] \arrow[dr, phantom, "(2)"] & TT(X) \arrow[d, "T(h_X)"] \\
 & TT(X) \arrow[r, "T(h_X)"'] & T(X).
\end{tikzcd}$$

In this diagram

- square (1) commutes because Γ is a functor,
- square (2) commutes because (X, h_X) is a **T**-algebra, and
- the triangle commutes because of (5.2).

It follows that

$$q_{(X,h_X)} \cdot (t_K)_{(X,h_X)} = T(h_X) \cdot T(e_X) = T(h_X \cdot e_X) = I_X.$$

Thus

$$q_{(X,h_X)} \cdot (t_K)_{(X,h_X)} = I_X. \qquad \square \tag{5.4}$$

Since, by Lemma 5.7, t_K is a componentwise (split) monomorphism, Proposition 3.12 yields the

5.8. Corollary. $t_K : \phi_T U_T \to \widehat{G}$ *is an isomorphism if and only if* $e_\mathbb{I} : \mathbb{I} \to T(\mathbb{I})$ *is a Galois grouplike morphism.*

5.9. Proposition. *With the data given in* 5.6*, for any* $(X, h_X, \vartheta_X) \in \mathbb{V}_T^G$*, the diagram*

$$\begin{tikzcd}
X \arrow[r, "q_{(X,h_X)} \cdot \vartheta_X"] & T(X) \arrow[r, shift left, "T(\vartheta_X)"] \arrow[r, shift right, "T(X \otimes e_\mathbb{I})"'] & T\big(X \otimes T(\mathbb{I})\big),
\end{tikzcd} \tag{5.5}$$

is a split equaliser diagram.

Proof. Note first that, by [7, Lemma 4.11], the composite $q_{(X,h_X)} \cdot \vartheta_X$ equalises the pair $(T(\vartheta_X), T(X \otimes e_{\mathbb{I}}))$. Next the following diagram is serially commutative (see [14])

$$\begin{tikzcd}
 & T(X) \arrow[r, shift left, "T(\vartheta_X)"] \arrow[r, shift right, "T(X\otimes e_{\mathbb{I}})"'] \arrow[d, "s_1"'] & T(X \otimes T(\mathbb{I})) \arrow[d, "s_2"] \\
X \arrow[r, "\vartheta_X"'] & X \otimes T(\mathbb{I}) \arrow[r, shift left, "\vartheta_X \otimes T(\mathbb{I})"] \arrow[r, shift right, "X\otimes \chi_{\mathbb{I},\mathbb{I}}"'] & X \otimes T(\mathbb{I}) \otimes T(\mathbb{I})
\end{tikzcd} \tag{5.6}$$

where $s_1 = (t_K)_{(X,h_X)}$ and $s_2 = (t_K)_{(X\otimes T(\mathbb{I}), h_{X\otimes T(\mathbb{I})})}$. Note that the bottom row of this diagram is split by the morphisms $X \otimes \theta_{\mathbb{I}}$ and $X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}$. Recall that this means

$$(X \otimes \theta_{\mathbb{I}}) \cdot \vartheta_X = I, \tag{5.7}$$

$$\big(X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}\big) \cdot (X \otimes \chi_{\mathbb{I},\mathbb{I}}) = I, \quad \text{and} \tag{5.8}$$

$$\big(X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}\big) \cdot \big(\vartheta_X \otimes T(\mathbb{I})\big) = \vartheta_X \cdot (X \otimes \theta_{\mathbb{I}}). \tag{5.9}$$

By 5.3, we now have

$$h_X \cdot q_{(X,h_X)} \cdot \vartheta_X = h_X \cdot T(h_X) \cdot \Gamma_X \cdot \vartheta_X = I_X.$$

Furthermore, since $s_2 \cdot T(X \otimes e_{\mathbb{I}}) = (X \otimes \chi_{\mathbb{I},\mathbb{I}}) \cdot s_1$,

$$\begin{aligned}
&q_{(X,h_X)} \cdot \big(X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}\big) \cdot s_2 \cdot T(X \otimes e_{\mathbb{I}}) \\
&\quad = q_{(X,h_X)} \cdot \big(X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}\big) \cdot (X \otimes \chi_{\mathbb{I},\mathbb{I}}) \cdot s_1 \\
&\quad =_{(5.8)} q_{(X,h_X)} \cdot s_1 = q_{(X,h_X)} \cdot (t_K)_{(X,h_X)} =_{(5.4)} I_X,
\end{aligned}$$

and since $s_2 \cdot T(\vartheta_X) = (\vartheta_X \otimes T(\mathbb{I})) \cdot s_1$,

$$\begin{aligned}
&q_{(X,h_X)} \cdot \big(X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}\big) \cdot s_2 \cdot T(\vartheta_X) \\
&\quad = q_{(X,h_X)} \cdot \big(X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}\big) \cdot \big(\vartheta_X \otimes T(\mathbb{I})\big) \cdot s_1 \\
&\quad =_{(5.9)} q_{(X,h_X)} \cdot \vartheta_X \cdot (X \otimes \theta_{\mathbb{I}}) \cdot s_1 \\
&\quad = q_{(X,h_X)} \cdot \vartheta_X \cdot (X \otimes \theta_{\mathbb{I}}) \cdot (t_K)_{(X,h_X)} \\
&\quad =_{L.5.3} q_{(X,h_X)} \cdot \vartheta_X \cdot h_X.
\end{aligned}$$

We have proved that

$$h_X \cdot q_{(X,h_X)} \cdot \vartheta_X = I_X, \tag{5.10}$$

$$q_{(X,h_X)} \cdot \big(X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}\big) \cdot s_2 \cdot T(X \otimes e_{\mathbb{I}}) = I_X, \quad \text{and} \tag{5.11}$$

$$q_{(X,h_X)} \cdot \big(X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}\big) \cdot s_2 \cdot T(\vartheta_X) = q_{(X,h_X)} \cdot \vartheta_X \cdot h_X, \tag{5.12}$$

which just means that (5.5) is a split equaliser: a splitting is given by h_X and by $q_{(X,h_X)} \cdot (X \otimes T(\mathbb{I}) \otimes \theta_{\mathbb{I}}) \cdot s_2$. $\square$

5.10. Proposition. *Given the data from* 5.6, *the functor* $K : \mathbb{V} \to \mathbb{V}_T^G$ *has a fully faithful right adjoint if and only if for any* $(X, h_X, \vartheta_X) \in \mathbb{V}_T^G$, *the pair of morphisms*

$$\begin{tikzcd} X \arrow[r, shift left, "\vartheta_X"] \arrow[r, shift right, "X\otimes e_{\mathbb{I}}"'] & X \otimes T(\mathbb{I}) \end{tikzcd} \tag{5.13}$$

has an equaliser and this equaliser is preserved by T.

Proof. By 1.4, K has a right adjoint if and only if (5.13) has an equaliser for all $(X, h_X, \vartheta_X) \in \mathbb{V}_T^G$. We write $(\overline{X}, i_X : \overline{X} \to X)$ for this equaliser. Thus $R(X, h_X, \vartheta_X) = (\overline{X}, i_X)$. Since the diagram (5.6) is commutative and since $T(i_X)$ equalises $T(\vartheta_X)$ and $T(X \otimes e_{\mathbb{I}})$, there exists a unique morphism $k_X = k_{(X,h_X,\vartheta_X)} : T(\overline{X}) \to X$ making the diagram

$$\begin{tikzcd} T(\overline{X}) \arrow[r, "T(i_X)"] \arrow[d, "k_X"'] & T(X) \arrow[r, shift left, "T(\vartheta_X)"] \arrow[r, shift right, "T(X\otimes e_{\mathbb{I}})"'] \arrow[d, "s_1"] & T(X \otimes T(\mathbb{I})) \arrow[d, "s_2"] \\ X \arrow[r, "\vartheta_X"'] & X \otimes T(\mathbb{I}) \arrow[r, shift left, "\vartheta_X\otimes T(\mathbb{I})"] \arrow[r, shift right, "X\otimes\delta_{\mathbb{I}}"'] & X \otimes T(\mathbb{I}) \otimes T(\mathbb{I}) \end{tikzcd} \tag{5.14}$$

commute. Since $q_{(X,h_X)} \cdot \vartheta_X \cdot k_X = q_{(X,h_X)} \cdot s_1 \cdot T(i_X) = T(i_X)$ and since $(X, q_{(X,h_X)} \cdot \vartheta_X)$ is an equaliser of the pair $(T(\vartheta_X), T(X \otimes e_{\mathbb{I}}))$ by Proposition 5.9, it follows from the universal property of equalisers that k_X is an isomorphism if and only if the top row of diagram (5.14) is an equaliser diagram, i.e. if T preserves the equaliser of (5.13). Since according to [14], $k_X = k_{(X,h_X,\vartheta_X)}$ is the (X, h_X, ϑ_X)-component of the counit $\bar{\varepsilon}$ of the adjunction $K \dashv R$ and since R is full and faithful if and only if $\bar{\varepsilon}$ is an isomorphism, it follows that R is a fully faithful functor if and only if for any $(X, h_X, \vartheta_X) \in \mathbb{V}_T^G$, the pair of morphisms $(\vartheta_X, X \otimes e_{\mathbb{I}})$ has an equaliser and this equaliser is preserved by T. $\square$

Recall that any functor is called *conservative* provided it reflects isomorphisms. The preceding propositions allow a refinement of [7, Theorem 4.6]:

5.11. Theorem. *Let* **T** *be an opmonoidal monad on a right autonomous category with a right antipode. Then the functor* $K : \mathbb{V} \to \mathbb{V}_T^G$ *is an equivalence of categories if and only if the functor* T *is conservative and for any* $(X, h_X, \vartheta_X) \in \mathbb{V}_T^G$, *the pair of morphisms* $(\vartheta_X, X \otimes e_{\mathbb{I}})$ *has an equaliser and this equaliser is preserved by* T.

Proof. According to the previous proposition it is enough to show that the fully faithful functor R is an equivalence of categories if and only if T is conservative. But since any fully faithful functor with a left adjoint is an equivalence of categories if and only if the left adjoint is conservative, it is sufficient to prove that T is conservative if and only if the functor K is, which is indeed the case since $T = U_T \phi_T = U_T U^{\widehat{G}} K$ and the functors U_T and $U^{\widehat{G}}$ are both conservative. $\square$

Recall (e.g. [19]) that a monad **T** on an arbitrary category $\mathbb{A}$ is of *effective descent type* if the functor $\phi_T : \mathbb{A} \to \mathbb{A}_T$ is comonadic.

5.12. Theorem. *Let* $\mathbf{T} = (T, m, e)$ *be an opmonoidal monad with right antipode on a right autonomous Cauchy complete monoidal category* $\mathbb{V}$.

(1) $K : \mathbb{V} \to \mathbb{V}_T^G$ *is an equivalence if and only if* **T** *is of effective descent type. In this case,*
 (i) *the natural transformation* $t_K : \phi_T U_T \to \widehat{G}$ *is an isomorphism of comonads*;

(ii) $e_{\mathbb{I}} : \mathbb{I} \to T(\mathbb{I})$ *is a Galois grouplike morphism*;
(iii) *the monad* T^g *is* (*isomorphic to*) *the identity monad.*

(2) *If* $e : I \to T$ *is a split monomorphism, the functor* $K : \mathbb{V} \to \mathbb{V}_T^G$ *is an equivalence.*

Proof. (1) If K is an equivalence of categories, then the functor ϕ_T is comonadic by Proposition 1.2.

Conversely, suppose that $\mathbf{T}$ is of effective descent type. Since $\mathbb{V}$ is Cauchy complete, it follows from [19, Proposition 3.11] that $\mathbf{T}$ is of effective descent type if and only if T is conservative and $\mathbb{V}$ has equalisers of T-split pairs and these equalisers are preserved by T. Now, if $(X, h_X, \vartheta_X) \in \mathbb{V}_T^G$, then the pair of morphisms $(T(\vartheta_X), T(X \otimes e_{\mathbb{I}}))$ is split by Proposition 5.9 and thus there exists an equaliser $(\overline{X}, i_X)$ of the pair $(\vartheta_X, X \otimes e_{\mathbb{I}})$ and this equaliser is preserved by T. The preceding theorem completes the proof.

(1)(i) and (ii) follow by Proposition 1.2 and Corollary 5.8.

(1)(iii) is a consequence of Proposition 3.5.

(2) Any monad on a Cauchy complete category whose unit is a split monomorphism is of effective descent type (see [19]). Thus the assertion follows from (1). $\square$

5.13. Bimonads in braided categories. As before, let $(\mathbb{V}, \otimes, \mathbb{I})$ be a strict monoidal category and $\mathbf{T} = (T, m, e)$ an opmonoidal monad on $\mathbb{V}$, and consider the corresponding mixed distributive law (entwining)

$$\lambda := \big(T(-) \otimes m_{\mathbb{I}}\big) \cdot \chi_{-,T(\mathbb{I})} : TG \longrightarrow GT,$$

from the monad $\mathbf{T}$ to the comonad $\mathbf{G} = - \otimes T(\mathbb{I})$. It is pointed out in [7] that, when $\mathbb{V}$ is a *braided monoidal category* with braiding $\tau_{X,Y} : X \otimes Y \to Y \otimes X$, then for any bialgebra $\mathbf{A} = (A, e, m, \varepsilon, \delta)$ in $\mathbb{V}$, the monad $A \otimes -$ is a comonoidal monad, where the natural transformation $\chi_{X,Y} : A \otimes X \otimes Y \to A \otimes X \otimes A \otimes Y$ is the composite

$$A \otimes X \otimes Y \xrightarrow{\delta \otimes X \otimes Y} A \otimes A \otimes X \otimes Y \xrightarrow{A \otimes \tau_{A,X} \otimes Y} A \otimes X \otimes A \otimes Y.$$

Then, for any $X \in \mathbb{V}$, λ_X is the composite

$$A \otimes X \otimes A \xrightarrow{\delta \otimes X \otimes A} A \otimes A \otimes X \otimes A \xrightarrow{A \otimes \tau_{A,X} \otimes X} A \otimes X \otimes A \otimes A \xrightarrow{A \otimes X \otimes m} A \otimes X \otimes A.$$

Consider now the diagram

$$\begin{tikzcd}
A \otimes X \otimes A \arrow[r, "\delta \otimes X \otimes A"] \arrow[d, "\tau_{A,X} \otimes A"'] \arrow[dr, phantom, "(1)"] & A \otimes A \otimes X \otimes A \arrow[r, "A \otimes \tau_{A,X} \otimes X"] \arrow[d, "\tau_{A \otimes A, X} \otimes A"'] \arrow[dr, phantom, "(2)"] & A \otimes X \otimes A \otimes A \arrow[r, "A \otimes X \otimes m"] \arrow[dl, "\tau^{-1}_{A,X} \otimes A \otimes A"] \arrow[d, phantom, "(3)"] & A \otimes X \otimes A \arrow[dl, "\tau^{-1}_{A,X} \otimes A"] \\
X \otimes A \otimes A \arrow[r, "X \otimes \delta \otimes A"'] & X \otimes A \otimes A \otimes A \arrow[r, "X \otimes A \otimes m"'] & X \otimes A \otimes A, &
\end{tikzcd}$$

in which the diagrams (1) and (2) commute by naturality of τ, while diagram (3) commutes by naturality of composition. Since each component of τ is an isomorphism, λ_X is an isomorphism if and only if the composite $(X \otimes A \otimes m)(X \otimes \delta \otimes A)$ is so. Since $(X \otimes A \otimes m)(X \otimes \delta \otimes A) = X \otimes ((A \otimes m)(\delta \otimes A))$ and since $(A \otimes m)(\delta \otimes A)$ is an isomorphism if and only if $\mathbf{A}$ has an antipode, it follows that the composite $(X \otimes A \otimes m)(X \otimes \delta \otimes A)$ – and hence λ_X – is an isomorphism for all $X \in \mathbb{V}$ if and only if $\mathbf{A}$ has an antipode.

6. Categories with finite products and Galois objects

In the category Set of sets, for any object G, the product $G \times -$ defines an endofunctor. This is always a comonad with the coproduct given by the diagonal map, and it is a monad provided G is a semigroup. In this case $G \times -$ is a (mixed) bimonad and it is a Hopf monad if and only if G is a group. We refer to [28, 5.19] for more details.

In this final section we study similar operations in more general categories and this leads eventually to the *Galois objects* in such categories as studied in Chase and Sweedler [11].

Let $\mathbb{A}$ be a category with finite products. In particular, $\mathbb{A}$ has a terminal object, which is the product over the empty set. Then $(\mathbb{A}, \times, \mathbb{1})$ is a symmetric monoidal category, where $a \times b$ is some chosen product of a and b, and $\mathbb{1}$ is a chosen terminal object in $\mathbb{A}$, while the symmetry $\tau_{a,b} : a \times b \to b \times a$ is the unique morphism for which the diagram

$$\begin{tikzcd} & a & \\ a \times b \arrow[ur, "p_1"] \arrow[rr, "\tau_{a,b}"] \arrow[dr, "p_2"'] & & b \times a \arrow[ul, "p_2"'] \arrow[dl, "p_1"] \\ & b & \end{tikzcd}$$

commutes. The associativity and unit constraints are defined via the universal property for products. Such a category is called a *cartesian monoidal category*.

Similarly, a *cocartesian monoidal category* is a monoidal category whose monoidal structure is given by the categorical coproduct and whose unit object is the initial object. Any category with finite coproducts can be considered as a cocartesian monoidal category.

Given morphisms $f : a \to x$ and $g : a \to y$ in $\mathbb{A}$, we write $\langle f, g\rangle : a \to x \times y$ for the unique morphism making the diagram

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[d, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ x & x \times y \arrow[l, "p_1"] \arrow[r, "p_2"'] & y \end{tikzcd}$$

commute. In particular, $\Delta_a = \langle I_a, I_a\rangle : a \to a \times a$ is the diagonal morphism.

It is well known that every object c of $\mathbb{A}$ has a unique (cocommutative) comonoid structure in the monoidal category $(\mathbb{A}, \times, \mathbb{1})$. Indeed, the counit $\varepsilon : c \to \mathbb{1}$ is the unique morphism $!_c$ to the terminal object $\mathbb{1}$, and the comultiplication $\delta : c \to c \times c$ is the diagonal morphism Δ_c. This yields an isomorphism of categories $\mathrm{Comon}(\mathbb{A}) \simeq \mathbb{A}$. Given an arbitrary object $c \in \mathbb{A}$, we write $\bar{\mathbf{c}}$ for the corresponding comonoid in $(\mathbb{A}, \times, \mathbb{1})$.

6.1. Proposition. *The assignment*

$$(a, \theta_a : a \longrightarrow a \times c) \longmapsto (p_2 \cdot \theta_a : a \longrightarrow c)$$

yields an isomorphism of categories

$${}^{\bar{\mathbf{c}}}\mathbb{A} \simeq \mathbb{A} \downarrow c,$$

where ${}^{\bar{\mathbf{c}}}\mathbb{A} = \mathbb{A}_{c\times -}$, *while* $\mathbb{A} \downarrow c$ *is the comma-category of objects over* c, *that is,* objects *are morphisms* $f : a \to c$ *with codomain* c *and* morphisms *are commutative diagrams*

$$\begin{tikzcd} a \arrow[rr, "h"] \arrow[dr, "f"'] & & a' \arrow[dl, "f'"] \\ & c. & \end{tikzcd}$$

If the category $\mathbb{A}$ has pullbacks, then for any morphism $f : c \to d$ in $\mathbb{A}$, the functor $f_* : \mathbb{A} \downarrow c \to \mathbb{A} \downarrow d$ given by the composition with f has the right adjoint $f^* : \mathbb{A} \downarrow d \to \mathbb{A} \downarrow c$ given by pulling back along the morphism f. Now, identifying $f : c \to d$ with the morphism $f : \bar{\mathbf{c}} \to \bar{\mathbf{d}}$ of the corresponding comonoids in $\mathbb{A}$, one can see the functors f^* and f_* as the induction functor ${}^{\bar{\mathbf{c}}}\mathbb{A} \to {}^{\bar{\mathbf{d}}}\mathbb{A}$ and the coinduction functor ${}^{\bar{\mathbf{d}}}\mathbb{A} \to {}^{\bar{\mathbf{c}}}\mathbb{A}$, respectively. Given an object $c \in \mathbb{A}$, we write P_c and U_c for the functors $(!_c)^*$ and $(!_c)_*$.

Given a symmetric monoidal category $\mathbb{V} = (V, \otimes, \mathbb{I})$, the category Mon($\mathbb{V}$) of monoids in $\mathbb{V}$ is again a monoidal category. For two $\mathbb{V}$-monoids $\mathbf{A} = (A, m_A, e_A)$ and $\mathbf{B} = (B, m_B, e_B)$, their tensor product is defined as

$$\mathbf{A} \otimes \mathbf{B} = \big(A \otimes B, (m_A \otimes m_B)(1 \otimes \tau_{A,B} \otimes 1), e_A \otimes e_B\big),$$

where τ is the symmetry in $\mathbb{V}$. The unit object for this tensor product is the trivial $\mathbb{V}$-monoid $\mathbf{I} = (\mathbb{I}, I_{\mathbb{I}}, I_{\mathbb{I}})$. Similarly, the category Comon($\mathbb{V}$) of $\mathbb{V}$-comonoids inherits, in a canonical way, the monoidal structure from $\mathbb{V}$ making it a monoidal category.

It is well known that one can describe bimonoids in any symmetric monoidal category $\mathbb{V}$ as monoids in the monoidal category of comonoids in $\mathbb{V}$. Thus, writing Bimon($\mathbb{V}$) for the category of bimonoids in $\mathbb{V}$, then Bimon($\mathbb{V}$) = Mon(Comon($\mathbb{V}$)). In particular, since Comon($\mathbb{A}$) $\simeq \mathbb{A}$ for any cartesian monoidal category $\mathbb{A}$, one has Bimon($\mathbb{A}$) = Mon(Comon($\mathbb{A}$)) $\simeq$ Mon($\mathbb{A}$). Thus, for any monoid $\mathbf{b} = (b, m_b, e_b)$ in $(\mathbb{A}, \times, 1)$, the 6-tuple

$$\widehat{\mathbf{b}} = \big((b, m_b, e_b), (b, \Delta_b, !_b)\big)$$

is a bimonoid in $(\mathbb{A}, \times, 1)$. In particular, then the functor $b \times - : \mathbb{A} \to \mathbb{A}$ is a $(\tau_{b,b} \times -)$-bimonad (in the sense of [21]).

Fix now a monoid $\mathbf{b} = (b, m_b, e_b)$ in $(\mathbb{A}, \times, 1)$. Since $\widehat{\mathbf{b}}$ is a bimonoid in $(\mathbb{A}, \times, 1)$, the category ${}_{\mathbf{b}}\mathbb{A} := \mathbb{A}_{b\times -}$ of $\mathbf{b}$-modules is monoidal. More precisely, if $(x, \alpha_x), (y, \alpha_y) \in {}_{\mathbf{b}}\mathbb{A}$, then their tensor product is the pair $(x \times y, \alpha_{x\times y})$, where $\alpha_{x\times y}$ is the composite

$$b \times x \times y \xrightarrow{\Delta_b \times x \times y} b \times b \times x \times y \xrightarrow{b \times \tau_{b,x} \times y} b \times x \times b \times y \xrightarrow{\alpha_x \times \alpha_y} x \times y.$$

It is easy to see that this monoidal structure is cartesian and coincides with the cartesian structure on ${}_{\mathbf{b}}\mathbb{A}$ which can be lifted from $\mathbb{A}$ along the forgetful functor ${}_{\mathbf{b}}\mathbb{A} \to \mathbb{A}$.

Suppose now that $(c, \alpha_c : b \times c \to c) \in {}_{\mathbf{b}}\mathbb{A}$. Applying the previous proposition to the comonoid $\overline{(c, \alpha_c)}$ in the cartesian monoidal category ${}_{\mathbf{b}}\mathbb{A}$ gives

6.2. Proposition. *If* $(c, \alpha_c) \in {}_{\mathbf{b}}\mathbb{A}$, *then the assignment*

$$\big((x, \alpha_x), \theta_{(x,\alpha_x)}\big) \longrightarrow \big((x, \alpha_x), p_2 \cdot \theta_{(x,\alpha_x)}\big)$$

yields an isomorphism of categories

$${}^{\overline{(c,\alpha_c)}}({}_{\mathbf{b}}\mathbb{A}) \simeq {}_{\mathbf{b}}\mathbb{A} \downarrow (c, \alpha_c).$$

We have seen that the data

$$\widetilde{\mathbf{b}} = \big(\underline{\mathbf{b}} = (b \times -, m_b \times -, e_b \times -), \bar{\mathbf{b}} = (b \times -, \Delta_b \times -, !_b \times -), \tau_{b,b} \times -\big)$$

define a $(\tau_{b,b} \times -)$-bimonad on $\mathbb{A}$ and, considering b as an object of ${}_{\mathbf{b}}\mathbb{A}$ via the multiplication $m_b : b \times b \to b$, one obtains easily that the categories $\mathbb{A}_b^b := \mathbb{A}_{\underline{\mathbf{b}}}^{\bar{\mathbf{b}}}(\tau_{b,b} \times -)$ (compare 4.1) and ${}^{\overline{(b,m_b)}}({}_{\mathbf{b}}\mathbb{A})$ are isomorphic. Thus, by the previous proposition, the categories $\mathbb{A}_b^b$ and ${}_{\mathbf{b}}\mathbb{A} \downarrow (b, m_b)$ are also isomorphic.

6.3. Theorem. *Assume that*

(i) $\mathbb{A}$ *has small limits, or*
(ii) $\mathbb{A}$ *has colimits and the functor* $b \times -$ *preserves them, or*
(iii) $\mathbb{A}$ *admits equalisers and* $b \times -$ *has a right adjoint, or*
(iv) $\mathbb{A}$ *admits coequalisers and* $b \times -$ *has a left adjoint.*

Then the functor

$$K : \mathbb{A} \longrightarrow {}_{\mathbf{b}}\mathbb{A} \downarrow (b, m_b), \quad a \longmapsto (b \times a, p_1 : b \times a \longrightarrow b),$$

is an equivalence of categories if and only if $\mathbf{b}$ *is a group.*

Proof. It is easy to see that, modulo the isomorphism $\mathbb{A}_b^b \simeq {}_{\mathbf{b}}\mathbb{A} \downarrow (b, m_b)$, the functor $K : \mathbb{A} \to {}_{\mathbf{b}}\mathbb{A} \downarrow (b, m_b)$ can be identified with the comparison functor $K : \mathbb{A} \to \mathbb{A}_b^b$, which by 4.5 is an equivalence of categories if and only if the bimonad $\widetilde{\mathbf{b}}$ has an antipode, which is the case if and only if the $\mathbb{A}$-bimonoid $\widehat{\mathbf{b}}$ has one, i.e., $\widehat{\mathbf{b}}$ is a Hopf monoid in $(\mathbb{A}, \times, 1)$. Now the result follows from the fact that in any cartesian monoidal category, a Hopf algebra is nothing but a group (see, for example, [28, 5.20]). $\square$

Consider now an object $(c, \alpha_c) \in {}_{\mathbf{b}}\mathbb{A}$. Since (c, α_c) is a comonoid in the cartesian monoidal category $({}_{\mathbf{b}}\mathbb{A}, \times, 1)$, the composite

$$b \times c \times - \xrightarrow{\Delta_b \times c \times -} b \times b \times c \times - \xrightarrow{b \times \tau_{b,c} \times -} b \times c \times b \times - \xrightarrow{\alpha_c \times b \times -} c \times b \times -$$

is an entwining from the monad $\mathbf{T_b} = b \times -$ to the comonad $\mathbf{G}_{\bar{\mathbf{c}}} = c \times -$. Then one has a lifting $\widetilde{\mathbf{T}_{\mathbf{b}}}$ of the monad $\mathbf{T_b}$ along the forgetful functor ${}^{\bar{\mathbf{c}}}\mathbb{A} = \mathbb{A} \downarrow c \to \mathbb{A}$. It is easy to see that if $(x, f : x \to c) \in \mathbb{A} \downarrow c$, then

$$\widetilde{T_{\mathbf{b}}}(x, f) = \big(b \times x, \alpha_c \cdot (b \times f) : b \times x \longrightarrow c\big).$$

We write ${}_{\mathbf{b}}(\mathbb{A} \downarrow c)$ for the category $(\mathbb{A} \downarrow c)_{\widetilde{\mathbf{T}_{\mathbf{b}}}}$. It is also easy to see that the functor

$$K : \mathbb{A} \longrightarrow {}_{\mathbf{b}}(\mathbb{A} \downarrow c)$$

that takes an object $a \in \mathbb{A}$ to the object

$$(c \times a, \alpha_c \times a : b \times c \times a \longrightarrow c \times a),$$

makes the diagram

$$\begin{tikzcd} & {}_{\mathbf{b}}(\mathbb{A} \downarrow c) \arrow[d, "U"] \\ \mathbb{A} \arrow[ur, "K"] \arrow[r, "P_c"'] & \mathbb{A} \downarrow c \end{tikzcd}$$

commute, where U is the evident forgetful functor. Then the corresponding $\widetilde{\mathbf{T}_{\mathbf{b}}}$-module structure on P_c is given by the morphism $\alpha_c \times - : b \times c \times - \to c \times -$. Since the forgetful functor $U_c : \mathbb{A} \downarrow c \to \mathbb{A}$ that takes $f : x \to c$ to x is left adjoint to the functor P_c and since the $(f : x \to c)$-component of the unit of the adjunction $U_c \dashv P_c$ is the morphism $\langle f, I_x \rangle : x \to c \times x$, the $(f : x \to c)$-component t_f of the monad morphism $t : \widetilde{\mathbf{T}_{\mathbf{b}}} \to P_c U_c$ is the composite

$$b \times x \xrightarrow{b \times \langle f, I_x \rangle} b \times c \times x \xrightarrow{\alpha_c \times I_x} c \times x.$$

We write γ_c for the morphism $t_{I_c} : b \times c \to c \times c$.

One says that a morphism $f : a \to b$ in $\mathbb{A}$ is an (*effective*) *descent morphism* if the corresponding functor $f^* : \mathbb{A} \downarrow b \to \mathbb{A} \downarrow a$ is precomonadic (resp. monadic).

6.4. Theorem. *Let $\mathbf{b} = (b, m_b, e_b)$ be a monoid in $\mathbb{A}$ and let $(c, \alpha_c) \in {}_{\mathbf{b}}\mathbb{A}$. Suppose that*

(i) *$\mathbb{A}$ admits all small limits, or*
(ii) *$\mathbb{A}$ admits coequalisers of reflexive pairs and the functors $b \times - : \mathbb{A} \to \mathbb{A}$ and $c \times - : \mathbb{A} \to \mathbb{A}$ both have left adjoints.*

Then the functor $K : \mathbb{A} \to {}_{\mathbf{b}}(\mathbb{A} \downarrow c)$ is an equivalence of categories if and only if $\gamma_c : b \times c \to c \times c$ is an isomorphism and $!_c : c \to 1$ is an effective descent morphism.

Proof. According to Proposition 1.15, the functor K is an equivalence of categories if and only if the functor P_c is comonadic (i.e. if the morphism $!_c : c \to 1$ is an effective descent morphism) and $t : \widetilde{\mathbf{T}_{\mathbf{b}}} \to P_c U_c$ is an isomorphism of monads. Since the functors $b \times - : \mathbb{A} \to \mathbb{A}$ and $c \times - : \mathbb{A} \to \mathbb{A}$ both preserve those limits that exist in $\mathbb{A}$, it follows from 2.8 that if $\mathbb{A}$ satisfies (i) or (ii), t is an isomorphism if and only if its restriction on free $P_c U_c$-algebras is so. But any free $P_c U_c$-algebra has the form $(c \times x, p_1)$ for some $x \in \mathbb{A}$ and it is not hard to see that the $(c \times x, p_1)$-component $t_{(c \times x, p_1)}$ of t is the morphism $\gamma_c \times x$. It follows that $t_{(c \times x, p_1)}$ is an isomorphism for all $x \in \mathbb{A}$ if and only if the morphism γ_c is an isomorphism. This completes the proof. $\square$

We say an object $a \in \mathbb{A}$ is *faithful* if the functor $a \times - : \mathbb{A} \to \mathbb{A}$ is conservative. Note that an arbitrary $a \in \mathbb{A}$ for which the unique morphism $!_a : a \to 1$ is a descent morphism is necessarily faithful.

Following Chase and Sweedler [11] we call an object $(c, \alpha_c) \in {}_{\mathbf{b}}\mathbb{A}$ a *Galois* $\mathbf{b}$-*object* if c is a faithful object in $\mathbb{A}$ such that the morphism $\gamma_c : b \times c \to c \times c$ is an isomorphism. Using this notion, we can rephrase the previous theorem as follows.

6.5. Theorem. *In the situation of the previous theorem, if $(c, \alpha_c) \in {}_{\mathbf{b}}\mathbb{A}$ is a Galois $\mathbf{b}$-object, then the functor $K : \mathbb{A} \to {}_{\mathbf{b}}(\mathbb{A} \downarrow c)$ is an equivalence of categories if and only if $!_c : c \to 1$ is an effective descent morphism.*

If any descent morphism in $\mathbb{A}$ is effective (as surely it is when $\mathbb{A}$ is an exact category in the sense of Barr, see [15]), then one has

6.6. Corollary. *If every descent morphism in $\mathbb{A}$ is effective, then for any Galois* $\mathbf{b}$*-object* (c, α_c)*, the functor* $K : \mathbb{A} \to {}_{\mathbf{b}}(\mathbb{A} \downarrow c)$ *is an equivalence of categories.*

Note that if $g : I \to G_{\bar{\mathbf{c}}}$ is a grouplike morphism for the comonad $G_{\bar{\mathbf{c}}}$, then the composite $1 \xrightarrow{g_1} G_{\bar{\mathbf{c}}}(1) = c \times 1 \xrightarrow{p_2} 1$ is the identity morphism, implying that the morphism $!_c : c \to 1$ is a split epimorphism. It is then easy to see that the counit of the adjunction $U_c \dashv P_c$ is a split epimorphism, and it follows from the dual of [19, Proposition 3.16] that the functor P_c is monadic (i.e., $!_c : c \to 1$ is an effective descent morphism) provided that the category $\mathbb{A}$ is Cauchy complete. In the light of the previous theorem, we get:

6.7. Theorem. *In the situation of Theorem* 6.4*, if $\mathbb{A}$ is Cauchy complete and if there exists a grouplike morphism for the comonad $G_{\bar{\mathbf{c}}}$, then the functor $K : \mathbb{A} \to {}_{\mathbf{b}}(\mathbb{A} \downarrow c)$ is an equivalence of categories if and only if $(c, \alpha_c) \in {}_{\mathbf{b}}\mathbb{A}$ is a Galois* $\mathbf{b}$*-object.*

Recall from [11] that an object $a \in \mathbb{A}$ is (*faithfully*) *coflat* if the functor

$$a \times - : \mathbb{A} \longrightarrow \mathbb{A}$$

preserves coequalisers (resp. preserves and reflects coequalisers).

6.8. Theorem. *Let $\mathbb{A}$ be a category with finite products and coequalisers, and $\mathbf{b} = (b, m_b, e_b)$ a monoid in the cartesian monoidal category $\mathbb{A}$ with b coflat and let $(c, \alpha_c) \in {}_{\mathbf{b}}\mathbb{A}$ be a* $\mathbf{b}$*-Galois object with $!_c : c \to 1$ an effective descent morphism. Assume*

(i) *$\mathbb{A}$ admits all small limits, or*
(ii) *the functors $b \times - : \mathbb{A} \to \mathbb{A}$ and $c \times - : \mathbb{A} \to \mathbb{A}$ both have left adjoints.*

Then c is (*faithfully*) *coflat.*

Proof. Note first that since $\mathbb{A} \downarrow c \simeq {}^{\bar{\mathbf{c}}}\mathbb{A}$ and since the category $\mathbb{A}$ admits coequalisers, the category $\mathbb{A} \downarrow c$ also admits coequalisers and the forgetful functor $U_c : \mathbb{A} \downarrow c \to \mathbb{A}$ creates them. Now, if b is coflat, then the functor $b \times - : \mathbb{A} \to \mathbb{A}$ preserves coequalisers, and it follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{A} \downarrow c \arrow[r, "\widetilde{T_{\mathbf{b}}}"] \arrow[d, "U_c"'] & \mathbb{A} \downarrow c \arrow[d, "U_c"] \\
\mathbb{A} \arrow[r, "T_{\mathbf{b}} = b \times -"'] & \mathbb{A}
\end{tikzcd}$$

that the functor $\widetilde{T_{\mathbf{b}}}$ also preserves coequalisers. As in the proof of 6.4, one can show that the morphism $t : \mathbf{T_b} \to P_c U_c$ is an isomorphism of monads. Thus, in particular, the monad $P_c U_c$ preserves coequalisers. Since the morphism $!_c : c \to 1$ is an effective descent morphism by our assumption on c, the functor P_c is monadic. Applying now the dual of [19, Proposition 3.11], one gets that the functor $U_c P_c = c \times -$ also preserves coequalisers. Thus c is coflat. $\square$

As a consequence, we have:

6.9. Theorem. *Let $\mathbb{A}$ be a category with finite products and coequalisers in which all descent morphisms are effective. Suppose that $\mathbf{b} = (b, m_b, e_b)$ is a monoid in the cartesian monoidal category $\mathbb{A}$ with b coflat and that $(c, \alpha_c) \in {}_{\mathbf{b}}\mathbb{A}$ is a* $\mathbf{b}$*-Galois object. If*

(i) *$\mathbb{A}$ admits all small limits, or*
(ii) *the functors $b \times - : \mathbb{A} \to \mathbb{A}$ and $c \times - : \mathbb{A} \to \mathbb{A}$ both have left adjoints,*

then c is (faithfully) coflat.

6.10. Opposite category of commutative algebras. Let k be a commutative ring (with unit) and let $\mathbb{A}$ be the opposite of the category of commutative unital k-algebras.

It is well known that $\mathbb{A}$ has finite products and coequalisers. If $\mathbf{A} = (A, m_A, e_A)$ and $\mathbf{B} = (B, m_B, e_B)$ are objects of $\mathbb{A}$ (i.e. if A and B are commutative k-algebras), then $A \otimes_k B$ with the obvious k-algebra structure is the product of $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{A}$: the projections $p_1 : A \otimes_k B \to A$ and $p_2 : A \otimes_k B \to B$ are given by $I_A \otimes_k e_B : A \to A \otimes_k B$ and $e_A \otimes_k I_B : B \to A \otimes_k B$, respectively. Furthermore, if $f, g : A \to B$ are morphisms in $\mathbb{A}$, then the pair (C, i), where $C = \{b \in B \mid f(b) = g(b)\}$ and $i : C \to B$ is the canonical embedding of k-algebras, defines a coequaliser in $\mathbb{A}$. The terminal object in $\mathbb{A}$ is k.

An object A in $\mathbb{A}$ (i.e. a commutative k-algebra) is (faithfully) coflat if and only if A is a (faithfully) flat k-module (see, [11]). Moreover, a monoid in the cartesian monoidal category $\mathbb{A}$ is a commutative k-bialgebra, which is a group in $\mathbb{A}$ if and only if it has an antipode, and if $\mathbf{B}$ is a commutative k-bialgebra, then $(C, \alpha_C) \in {}_{\mathbf{B}}\mathbb{A}$ if and only if C is a commutative $\mathbf{B}$-comodule algebra.

Note that in the present context, $(C, \alpha_C) \in {}_{\mathbf{B}}\mathbb{A}$ is a Galois $\mathbf{B}$-object if C is a faithful k-module and the composite

$$\gamma_C : C \otimes_k C \xrightarrow{\alpha_C \otimes_k I_C} B \otimes_k C \otimes_k C \xrightarrow{B \otimes_k m_C} B \otimes_k C,$$

where $m_C : C \otimes_k C \to C$ is the multiplication in C, is an isomorphism.

Since the category $\mathbb{A}$ admits all small limits and since in $\mathbb{A}$ every descent morphism is effective (see [20]), one can apply Theorem 6.9 to deduce the following

6.11. Theorem. *Let $\mathbf{B}$ be a commutative k-bialgebra with B a flat k-module. Then any Galois $\mathbf{B}$-object in $\mathbb{A}$ is a faithfully flat k-module.*

Note finally that when $\mathbf{B}$ is a Hopf algebra which is finitely generated and projective as a k-module, the result was obtained by Chase and Sweedler, see [11, Theorem 12.5].

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