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On a problem of D.H. Lehmer over short intervals ☆

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Abstract

Let p be an odd prime and a be an integer coprime to p. Denote by $N(a,p)$ the number of pairs of integers b, c with $bc \equiv a \pmod p$, $1 \leqslant b, c \leqslant (p-1)/2$ and with b, c having different parity. The main purpose of this paper is to study the sum $\sum_{a=1}^{p-1}(N(a,p)-(p-1)/8)^2$, and obtain a sharp asymptotic formula.

Keywords: D.H. Lehmer problem; Short intervals; The error term; Mean value

1. Introduction

Let p be an odd prime and a be an integer coprime to p. For each integer b with $1 \leqslant b < p$, there is a unique integer c with $1 \leqslant c < p$ such that $bc \equiv a \pmod p$. Let $M(a,p)$ denote the number of solutions of the congruence equation $bc \equiv a \pmod p$ with $1 \leqslant b, c < p$ such that b, c are of opposite parity. D.H. Lehmer posed the problem to find $M(1,p)$ or at least to say some thing nontrivial about it (see problem F12 of [1, p. 251]. The second author [2] proved that

$$M(1,p) = \frac{p-1}{2} + O\left(p^{\frac{1}{2}} \ln^2 p\right).$$

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For the further properties of $M(a, p)$, he studied the mean square value of the error term $M(a, p) - \frac{p-1}{2}$, and obtained

$$\sum_{a=1}^{p-1}\left(M(a, p) - \frac{p-1}{2}\right)^2 = \frac{3}{4}p^2 + O\left(p\exp\left(\frac{3\ln p}{\ln\ln p}\right)\right),$$

see [3]. In [4], the second author used the properties of Dedekind sums and Cochrane sums to study the D.H. Lehmer problem for the general case of an odd number $q \geqslant 3$, and obtained the following sharp asymptotic formula:

$$\sum_{a=1}^{q}{}'\left(M(a, q) - \frac{\phi(q)}{2}\right)^2 = \frac{3}{4}\phi^2(q)\prod_{p^\alpha \| q}\frac{\frac{(p+1)^3}{p^2(p^2+1)} - \frac{1}{p^{3\alpha}}}{1 + \frac{1}{p} + \frac{1}{p^2}} + O\left(q\exp\left(\frac{4\ln q}{\ln\ln q}\right)\right).$$

In this paper, we will study the D.H. Lehmer problem over a short interval $\left[1, \frac{p-1}{2}\right]$. Denote by $N(a, p)$ the number of pairs of integers b, c with $bc \equiv a \pmod p$, $1 \leqslant b, c \leqslant \frac{p-1}{2}$ and with b, c having different parity. The methods of [2] give also the formula:

$$N(1, p) = \frac{1}{8}(p-1) + O\left(p^{\frac{1}{2}}\ln^2 p\right).$$

For any fixed positive integer a with $(a, p) = 1$, let

$$E(a, p) = N(a, p) - \frac{1}{8}(p-1).$$

The main purpose of this paper is to study distribution properties of $E(a, p)$ by using the mean value theorems of Dirichlet L-functions. That is, we will prove the following result:

Theorem. *Let p be an odd prime. Then we have the asymptotic formula*

$$\sum_{a=1}^{p-1}E^2(a, p) = \frac{9}{64}p^2 + O\left(p^{1+\epsilon}\right),$$

where ϵ is any fixed positive number.

In fact, our method also works for the D.H. Lehmer problem over the shorter interval $[1, \frac{p}{4}]$. For the general case of an odd number $q \geqslant 3$, whether there exists an asymptotic formula for mean square value

$$\sum_{a=1}^{q}{}' E^2(a, q)$$

is an open problem, where $\sum'$ denotes the summation over all a such that $(a, q) = 1$.

2. Some lemmas

To prove the theorem, we need the following lemmas.

Lemma 1. *Let χ be a primitive character modulo m with $\chi(-1)=-1$. Then we have*

$$\frac{1}{m}\sum_{b=1}^{m} b\chi(b)=\frac{i}{\pi}\tau(\chi)L(1,\bar{\chi}),$$

where $\tau(\chi)=\sum_{a=1}^{m}\chi(a)e(\frac{a}{q})$ is the Gauss sum, $e(y)=e^{2\pi iy}$, and $L(s,\chi)$ denotes the Dirichlet L-function corresponding to χ.

Proof. This can be easily deduced from Theorems 12.11 and 12.20 of [5]. □

Lemma 2. *Let $q\geqslant 3$ be an odd number. For any nonprincipal character χ mod q, we have*

$$\sum_{a=1}^{q} a\chi(a)=\frac{\chi(2)q}{1-2\chi(2)}\sum_{a=1}^{\frac{q-1}{2}}\chi(a).$$

Proof. See [6]. □

Lemma 3. *Let $q\geqslant 5$ be an odd integer and χ be a primitive Dirichlet character modulo q such that $\chi(-1)=1$. Then we have*

$$\sum_{a=1}^{[\frac{q}{4}]}\chi(a)=-\frac{i\bar{\chi}(4)}{2\pi}\tau(\chi\chi_4)L(1,\bar{\chi}\chi_4),$$

where χ_4 is the primitive Dirichlet character modulo 4.

Proof. First, we suppose $q\equiv 1 \pmod 4$. Since χ_4 is the primitive Dirichlet character modulo 4, so we have $\chi_4(1)=\chi_4(-3)=1$ and $\chi_4(3)=\chi_4(-1)=-1$. Then the following identity is obvious:

$$\sum_{a=1}^{4q} a\chi(a)\chi_4(a)=\sum_{a=0}^{q-1}(4a+1)\chi(4a+1)-\sum_{a=0}^{q-1}(4a+3)\chi(4a+3). \tag{1}$$

Noting that $\sum_{a=0}^{q-1}\chi(a)=0$, we can write

$$\begin{aligned}
&\sum_{a=0}^{q-1}(4a+1)\chi(4a+1)\\
&\quad=4\chi(4)\sum_{a=0}^{q-1}a\chi(a+\bar{4})=4\chi(4)\sum_{a=0}^{q-1}(a+\bar{4})\chi(a+\bar{4})\\
&\quad=4\chi(4)\sum_{a=0}^{\frac{q-1}{4}}(a+\bar{4})\chi(a+\bar{4})+4\chi(4)\sum_{a=\frac{q-1}{4}+1}^{q-1}(a+\bar{4})\chi(a+\bar{4}),
\end{aligned} \tag{2}$$

where $4 \cdot \bar{4} \equiv 1 \pmod q$. We know that $\frac{3q+1}{4}$ is an integer and $\bar{4} = \frac{3q+1}{4}$, so

$$0 \leqslant a + \bar{4} \leqslant q, \quad \text{if } a \leqslant \frac{q-1}{4}$$

and

$$q < a + \bar{4} \leqslant 2q - 1, \quad \text{if } a > \frac{q-1}{4}.$$

Hence, we have

$$\begin{aligned}
&4\chi(4)\sum_{a=0}^{\frac{q-1}{4}}(a+\bar{4})\chi(a+\bar{4}) + 4\chi(4)\sum_{a=\frac{q-1}{4}+1}^{q-1}(a+\bar{4})\chi(a+\bar{4}) \\
&\quad = 4\chi(4)\sum_{a=0}^{\frac{a-1}{4}}(a+\bar{4})\chi(a+\bar{4}) + 4\chi(4)\sum_{a=\frac{q-1}{4}+1}^{q-1}(a+\bar{4}-q)\chi(a+\bar{4}-q) \\
&\qquad + 4\chi(4)\sum_{a=\frac{q-1}{4}+1}^{q-1} q\chi(a+\bar{4}) \\
&\quad = 4\chi(4)\sum_{a=1}^{q-1} a\chi(a) + 4\chi(4)q\sum_{a=\frac{q-1}{4}+1}^{q-1}\chi(a+\bar{4}).
\end{aligned} \tag{3}$$

Noting that $\chi(-1) = 1$, we can get (see [5, Theorem 12.20])

$$\sum_{a=1}^{q-1} a\chi(a) = 0.$$

Now combining (2) and (3), we have

$$\begin{aligned}
&\sum_{a=0}^{q-1}(4a+1)\chi(4a+1) \\
&\quad = 4\chi(4)q\sum_{a=\frac{q-1}{4}+1}^{q-1}\chi(a+\bar{4}) = -4\chi(4)q\sum_{a=0}^{\frac{q-1}{4}}\chi\left(a+\frac{3p+1}{4}\right) \\
&\quad = -4\chi(4)q\sum_{a=0}^{\frac{q-1}{4}}\chi\left(a-\frac{p-1}{4}\right) = -4\chi(4)q\sum_{a=0}^{\frac{q-1}{4}}\chi\left(\frac{q-1}{4}-s\right) \\
&\quad = -4\chi(4)q\sum_{a=1}^{\frac{q-1}{4}}\chi(a).
\end{aligned} \tag{4}$$

By using the same method, we can also get

$$\sum_{a=0}^{q-1}(4a+3)\chi(4a+3)=4\chi(4)q\sum_{a=1}^{\frac{q-1}{4}}\chi(a). \tag{5}$$

From (1), (4) and (5), we have

$$\sum_{a=1}^{4q}a\chi(a)\chi_4(a)=-8\chi(4)q\sum_{a=1}^{\frac{q-1}{4}}\chi(a). \tag{6}$$

Since $\chi(a)$ is a primitive character modulo q, χ_4 is a primitive character modulo 4 and $(q,2)=1$, so $\chi\chi_4$ is also a primitive character modulo $4q$. Noting that

$$\chi\chi_4(-1)=\chi(-1)\chi_4(-1)=-1,$$

combining (6) and Lemma 1, we can easily get

$$\sum_{a=1}^{\frac{q-1}{4}}\chi(a)=-\frac{i\bar{\chi}(4)}{2\pi}\tau(\chi\chi_4)L(1,\bar{\chi}\chi_4). \tag{7}$$

For the case of $q\equiv 3 \pmod 4$, by the same argue we can get

$$\sum_{a=1}^{\frac{q-3}{4}}\chi(a)=-\frac{i\bar{\chi}(4)}{2\pi}\tau(\chi\chi_4)L(1,\bar{\chi}\chi_4). \tag{8}$$

Combining (7) and (8), we have

$$\sum_{a=1}^{[\frac{q}{4}]}\chi(a)=-\frac{i\bar{\chi}(4)}{2\pi}\tau(\chi\chi_4)L(1,\bar{\chi}\chi_4).$$

This proves Lemma 3. □

Lemma 4. *Let p be an odd prime. Then for any positive integer a with $(a,p)=1$, we have the identities*

$$E(a,p)=\frac{1}{2\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\bar{\chi}(a)\big(\chi(4)-1\big)\big(\bar{\chi}(2)-2\big)^2\tau^2(\chi)L^2(1,\bar{\chi})$$
$$+\frac{1}{2\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=1\\ \chi\neq\chi_0}}\bar{\chi}(a)\bar{\chi}(4)\tau^2(\chi\chi_4)L^2(1,\bar{\chi}\chi_4)+r(p),$$

where

$$r(p)=\begin{cases}0 & \text{if } p\equiv 1 \pmod 4;\\ \frac{1}{2(p-1)} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Proof. From the orthogonality relation for character sums modulo p and the definition of $N(a,p)$ we have

$$
\begin{aligned}
N(a,p) &= \frac{1}{2}\sum_{\substack{b=1\\ bc\equiv a \pmod p}}^{\frac{p-1}{2}}\sum_{c=1}^{\frac{p-1}{2}}\left(1-(-1)^{b+c}\right)\\
&= \frac{1}{2(p-1)}\left(\sum_{\chi \bmod p}\bar{\chi}(a)\sum_{b=1}^{\frac{p-1}{2}}\sum_{c=1}^{\frac{p-1}{2}}\chi(bc)\right.\\
&\quad \left. -\sum_{\chi \bmod p}\bar{\chi}(a)\left(\sum_{b=1}^{\frac{p-1}{2}}(-1)^b\chi(b)\right)^2\right)\\
&= \frac{p-1}{8}+\frac{1}{2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}}\bar{\chi}(a)\left(\sum_{b=1}^{\frac{p-1}{2}}\chi(b)\right)^2\\
&\quad -\frac{1}{2(p-1)}\sum_{\chi \bmod p}\bar{\chi}(a)\left(\sum_{b=1}^{\frac{p-1}{2}}(-1)^b\chi(b)\right)^2. \qquad (9)
\end{aligned}
$$

Now if $p\equiv 1 \pmod 4$ and $\chi(-1)=1$, then we can deduce that

$$\sum_{b=1}^{\frac{p-1}{2}}\chi(b)=0 \qquad (10)$$

and

$$
\begin{aligned}
\sum_{b=1}^{\frac{p-1}{2}}(-1)^b\chi(b) &= \sum_{b=1}^{\frac{p-1}{4}}\chi(2b)-\sum_{b=1}^{\frac{p-1}{4}}\chi(p-2b+1)\\
&= \sum_{b=1}^{\frac{p-1}{4}}\chi(2b)-\sum_{b=\frac{p+3}{4}}^{\frac{p-1}{2}}\chi(2b)\\
&= \begin{cases} 0 & \text{if } \chi=\chi_0 \text{ is a principal character;}\\ 2\chi(2)\sum_{b=1}^{\frac{p-1}{4}}\chi(b) & \text{if } \chi(-1)=1 \text{ and } \chi\neq\chi_0, \end{cases} \qquad (11)
\end{aligned}
$$

while if $\chi(-1)=-1$, then we have

$$\sum_{b=1}^{\frac{p-1}{2}}(-1)^b\chi(b)=\sum_{b=1}^{\frac{p-1}{4}}\chi(2b)+\sum_{b=1}^{\frac{p-1}{4}}\chi(p-2b+1)$$

$$
\begin{aligned}
&= \sum_{b=1}^{\frac{p-1}{4}} \chi(2b) + \sum_{b=\frac{p+3}{4}}^{\frac{p-1}{2}} \chi(2b) \\
&= \chi(2) \sum_{b=1}^{\frac{p-1}{2}} \chi(b) \qquad (12)
\end{aligned}
$$

and (see Lemma 2)

$$
\sum_{b=1}^{\frac{p-1}{2}} \chi(b) = \frac{\bar{\chi}(2)-2}{p} \sum_{b=1}^{p-1} b\chi(b). \qquad (13)
$$

For the case of $p \equiv 3 \pmod 4$, we can also get

$$
\sum_{b=1}^{\frac{p-1}{2}} (-1)^b \chi(b) = \begin{cases} -1 & \text{if } \chi = \chi_0; \\ 2\chi(2) \sum_{b=1}^{\frac{p-3}{4}} \chi(b) & \text{if } \chi(-1) = 1 \text{ and } \chi \neq \chi_0; \\ \chi(2) \sum_{b=1}^{\frac{p-1}{2}} \chi(b) & \text{if } \chi(-1) = -1 \end{cases} \qquad (14)
$$

and

$$
\sum_{b=1}^{\frac{p-1}{2}} \chi(b) = \begin{cases} 0 & \text{if } \chi(-1) = 1; \\ \frac{\bar{\chi}(2)-2}{p} \sum_{b=1}^{p-1} b\chi(b) & \text{if } \chi(-1) = -1. \end{cases} \qquad (15)
$$

Now combining (9)–(15), Lemmas 1 and 3, we can write

$$
\begin{aligned}
N(a,p) &= \frac{p-1}{8} + \frac{1}{2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bar{\chi}(a)\big(1-\chi(4)\big) \left(\sum_{b=1}^{\frac{p-1}{2}} \chi(b) \right)^2 \\
&\quad - \frac{2}{(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \bar{\chi}(a)\chi(4) \left(\sum_{b=1}^{[\frac{p}{4}]} \chi(b) \right)^2 \\
&= \frac{p-1}{8} \\
&\quad + \frac{1}{2\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bar{\chi}(a)\big(\chi(4)-1\big)\big(\bar{\chi}(2)-2\big)^2 \tau^2(\chi) L^2(1,\bar{\chi}) \\
&\quad + \frac{1}{2\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \bar{\chi}(a)\bar{\chi}(4)\tau^2(\chi\chi_4) L^2(1,\bar{\chi}\chi_4),
\end{aligned}
$$

if $p \equiv 1 \pmod 4$. Similarly, we have

$$N(a,p)=\frac{p-1}{8}+\frac{1}{2\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\bar\chi(a)\big(\chi(4)-1\big)\big(\bar\chi(2)-2\big)^2\tau^2(\chi)L^2(1,\bar\chi)+\frac{1}{2\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=1\\ \chi\neq\chi_0}}\bar\chi(a)\bar\chi(4)\tau^2(\chi\chi_4)L^2(1,\bar\chi\chi_4)+\frac{1}{2(p-1)},$$

if $p\equiv 3 \pmod 4$. Now Lemma 4 can be easily obtained from the definition of $E(a,p)$. □

Lemma 5. *Let p be a prime and m be a fixed positive integer. Then we have the identity*

$$\sum_{\substack{n=1\\ (n,p)=1}}^{\infty}\frac{\tau(2^m n)\tau(n)}{n^2}=\frac{(3m+5)\pi^4}{72}\frac{(p^2-1)^3}{p^4(p^2+1)}.$$

Proof. Noting that $\tau(n)$ is a multiplicative function, we can write

$$\sum_{\substack{n=1\\ (n,p)=1}}^{\infty}\frac{\tau(2^m n)\tau(n)}{n^2}=\tau\big(2^m\big)\sum_{\substack{n=1\\ (n,p)=1\\ 2\nmid n}}^{\infty}\frac{\tau^2(n)}{n^2}+\sum_{\substack{n=1\\ (n,p)=1\\ 2\mid n}}^{\infty}\frac{\tau(2^m n)\tau(n)}{n^2}$$
$$=(m+1)\sum_{\substack{n=1\\ (n,p)=1\\ 2\nmid n}}^{\infty}\frac{\tau^2(n)}{n^2}+\sum_{\substack{r=1\\ (r,p)=1\\ 2\nmid r}}^{\infty}\sum_{j=1}^{\infty}\frac{\tau(2^{m+j})\tau(2^j)\tau^2(r)}{(r\cdot 2^j)^2}$$
$$=\left(m+1+\sum_{j=1}^{\infty}\frac{(m+j+1)(j+1)}{4^j}\right)\sum_{\substack{n=1\\ (n,p)=1\\ 2\nmid n}}^{\infty}\frac{\tau^2(n)}{n^2}. \tag{16}$$

After some simple calculation, we can get

$$\sum_{j=1}^{\infty}\frac{(m+j+1)(j+1)}{4^j}=\frac{21m+53}{27}. \tag{17}$$

For the summation

$$\sum_{\substack{n=1\\ (n,p)=1\\ 2\nmid n}}^{\infty}\frac{\tau^2(n)}{n^2},$$

by using the Euler product formula we can write

$$\sum_{\substack{n=1\\(n,p)=1\\2\nmid n}}^{\infty}\frac{\tau^2(n)}{n^2}=\prod_{\substack{p_1\neq p\\p_1\neq 2}}\left(1+\frac{\tau^2(p_1)}{p_1^2}+\frac{\tau^2(p_1^2)}{p_1^4}+\cdots\right)$$
$$=\prod_{\substack{p_1\neq p\\p_1\neq 2}}\left(1+\frac{2^2}{p_1^2}+\frac{3^2}{p_1^4}+\cdots\right).$$

Let

$$S=1+\frac{2^2}{p_1^2}+\frac{3^2}{p_1^4}+\cdots.$$

It is clear that

$$S\left(1-\frac{1}{p_2^2}\right)^2=1+\frac{2}{p_2^2}\left(\frac{1}{1-\frac{1}{p_1^2}}\right).$$

Hence,

$$\sum_{\substack{n=1\\(n,p)=1\\2\nmid n}}^{\infty}\frac{\tau^2(n)}{n^2}=\prod_{\substack{p_1\neq p\\p_1\neq 2}}\left(1-\frac{1}{p_1^2}\right)^{-3}\left(1+\frac{1}{p_1^2}\right)$$
$$=\frac{27}{80}\prod_{p_1\neq p}\left(1-\frac{1}{p_1^2}\right)^{-3}\left(1+\frac{1}{p_1^2}\right)$$
$$=\frac{27\zeta^4(2)}{80\zeta(4)}\frac{(p^2-1)^3}{p^4(p^2+1)}=\frac{3\pi^4}{128}\frac{(p^2-1)^3}{p^4(p^2+1)}, \tag{18}$$

where we used the identities $\zeta(2)=\frac{\pi^2}{6}$ and $\zeta(4)=\frac{\pi^4}{90}$. So from (16)–(18), we have

$$\sum_{\substack{n=1\\(n,p)=1}}^{\infty}\frac{\tau(2^m n)\tau(n)}{n^2}=\frac{(3m+5)\pi^4}{72}\frac{(p^2-1)^3}{p^4(p^2+1)}.$$

This proves Lemma 5. □

Lemma 6. *Let p be a prime, χ be a Dirichlet character modulo p and $m\geqslant 0$ be a fixed integer. Then we have*

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi\left(2^m\right)\left|L(1,\chi)\right|^4=\frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}p+O\left(p^{\epsilon}\right).$$

Proof. For convenience, we put

$$A(y,\chi)=\sum_{N<n\leqslant y}\chi(n)\tau(n),$$

where N is a parameter with $p \leqslant N < p^4$ and $\tau(n)$ is the Dirichlet divisor function. Then from Abel's identity we have

$$L^2(1,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)\tau(n)}{n}=\sum_{1\leqslant n\leqslant N}\frac{\chi(n)\tau(n)}{n}+\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,dy.$$

Hence, we can write

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^m)\left|L(1,\chi)\right|^4\\
&\quad=\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^m)\left(\sum_{1\leqslant n_1\leqslant N}\frac{\bar{\chi}(n_1)\tau(n_1)}{n_1}+\int_N^{\infty}\frac{A(y,\bar{\chi})}{y^2}\,dy\right)\\
&\qquad\times\left(\sum_{1\leqslant n_2\leqslant N}\frac{\chi(n_2)\tau(n_2)}{n_2}+\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,dy\right)\\
&\quad=\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^m)\left(\sum_{1\leqslant n_1\leqslant N}\frac{\bar{\chi}(n_1)\tau(n_1)}{n_1}\right)\left(\sum_{1\leqslant n_2\leqslant N}\frac{\chi(n_2)\tau(n_2)}{n_2}\right)\\
&\qquad+\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^m)\left(\sum_{1\leqslant n_1\leqslant N}\frac{\bar{\chi}(n_1)\tau(n_1)}{n_1}\right)\left(\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,dy\right)\\
&\qquad+\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^m)\left(\sum_{1\leqslant n_2\leqslant N}\frac{\chi(n_2)\tau(n_2)}{n_2}\right)\left(\int_N^{\infty}\frac{A(y,\bar{\chi})}{y^2}\,dy\right)\\
&\qquad+\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^m)\left(\int_N^{\infty}\frac{A(y,\bar{\chi})}{y^2}\,dy\right)\left(\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,dy\right)\\
&\quad=M_1+M_2+M_3+M_4.
\end{aligned}\tag{19}$$

Now we shall calculate each term in the expression (19).

(i) First, we calculate M_1. From the orthogonality relation for characters modulo p, we have

$$\begin{aligned}
M_1&=\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^m)\left(\sum_{1\leqslant n_1\leqslant N}\frac{\bar{\chi}(n_1)\tau(n_1)}{n_1}\right)\left(\sum_{1\leqslant n_2\leqslant N}\frac{\chi(n_2)\tau(n_2)}{n_2}\right)\\
&=\frac{1}{2}\sum_{1\leqslant n_1\leqslant N}\sum_{1\leqslant n_2\leqslant N}\frac{\bar{\chi}(n_1)\tau(n_1)\tau(n_2)}{n_1n_2}\sum_{\chi \bmod p}\left(1-\chi(-1)\right)\chi\left(2^m\bar{n}_1n_2\right)
\end{aligned}$$

$$= \frac{1}{2}\phi(p)\left(\sideset{}{'}\sum_{1\leqslant n_1\leqslant N}\ \sideset{}{'}\sum_{\substack{1\leqslant n_2\leqslant N\\ 2^m n_2\equiv n_1 \pmod p}} \frac{\tau(n_1)\tau(n_2)}{n_1n_2} - \sideset{}{'}\sum_{1\leqslant n_1\leqslant N}\ \sideset{}{'}\sum_{\substack{1\leqslant n_2\leqslant N\\ 2^m n_2\equiv -n_1 \pmod p}} \frac{\tau(n_1)\tau(n_2)}{n_1n_2}\right), \quad (20)$$

where $\sum'_{1\leqslant n_1\leqslant N}$ denotes the sum over integer n_1 such that $(n_1, p) = 1$ and $1 \leqslant n_1 \leqslant N$.

For convenience, we split the sum over n_1 or n_2 into the following cases:

(i) $p \leqslant n_1 \leqslant N$, $\frac{p}{2^m} \leqslant n_2 \leqslant N$;
(ii) $p \leqslant n_1 \leqslant N$, $1 \leqslant n_2 \leqslant \frac{p}{2^m} - 1$;
(iii) $1 \leqslant n_1 \leqslant p-1$, $\frac{p}{2^m} \leqslant n_2 \leqslant N$;
(iv) $1 \leqslant n_1 \leqslant p-1$, $1 \leqslant n_2 \leqslant \frac{p}{2^m} - 1$.

So we have

$$\sideset{}{'}\sum_{p\leqslant n_1\leqslant N}\ \sideset{}{'}\sum_{\substack{\frac{p}{2^m}\leqslant n_2\leqslant N\\ 2^m n_2\equiv n_1 \pmod p}} \frac{\tau(n_1)\tau(n_2)}{n_1n_2} \ll \sideset{}{'}\sum_{p\leqslant n_1\leqslant N}\ \sideset{}{'}\sum_{\substack{\frac{p}{2^m}\leqslant n_2\leqslant N\\ 2^m n_2\equiv n_1 \pmod p}} \frac{(n_1n_2)^\epsilon}{n_1n_2}$$
$$\ll \sum_{1\leqslant r_1\leqslant \frac{N}{p}}\ \sum_{1\leqslant r_2\leqslant \frac{2^mN}{p}} \left(r_1r_2p^2\right)^{\epsilon-1} \ll p^{\epsilon-2},$$

$$\sideset{}{'}\sum_{p\leqslant n_1\leqslant N}\ \sideset{}{'}\sum_{\substack{1\leqslant n_2\leqslant \frac{p}{2^m}-1\\ 2^m n_2\equiv n_1 \pmod p}} \frac{\tau(n_1)\tau(n_2)}{n_1n_2} \ll \sum_{1\leqslant r_1\leqslant \frac{N}{p}}\ \sum_{1\leqslant n_2\leqslant \frac{p}{2^m}-1} (r_1n_2p)^{\epsilon-1} \ll p^{\epsilon-1}$$

and

$$\sideset{}{'}\sum_{1\leqslant n_1\leqslant p-1}\ \sideset{}{'}\sum_{\substack{\frac{p}{2^m}\leqslant n_2\leqslant N\\ 2^m n_2\equiv n_1 \pmod p}} \frac{\tau(n_1)\tau(n_2)}{n_1n_2} \ll p^{\epsilon-1},$$

where we have used the estimate $\tau(n) \ll n^\epsilon$.

For the case $1 \leqslant n_1 \leqslant p-1$, $1 \leqslant n_2 \leqslant \frac{p}{2^m} - 1$, the solution of the congruence $2^m n_2 \equiv n_1 \pmod p$ is $2^m n_2 = n_1$. Hence,

$$\sideset{}{'}\sum_{1\leqslant n_1\leqslant p-1}\ \sideset{}{'}\sum_{\substack{1\leqslant n_2\leqslant \frac{p}{2^m}-1\\ 2^m n_2\equiv n_1 \pmod p}} \frac{\tau(n_1)\tau(n_2)}{n_1n_2} = \frac{1}{2^m} \sideset{}{'}\sum_{1\leqslant n_2\leqslant \frac{p}{2^m}-1} \frac{\tau(2^m n_2)\tau(n_2)}{n_2^2}$$
$$= \sideset{}{'}\sum_{n_2=1}^{\infty} \frac{\tau(2^m n_2)\tau(n_2)}{n_2^2} + O\left(p^{\epsilon-1}\right).$$

Now from Lemma 4, we can immediately get

$$\frac{1}{2} \sideset{}{'}\sum_{1\leqslant n_1\leqslant p-1}\ \sideset{}{'}\sum_{\substack{1\leqslant n_2\leqslant \frac{p}{2^m}-1\\ 2^m n_2\equiv n_1 \pmod p}} \frac{\tau(n_1)\tau(n_2)}{n_1n_2} = \frac{(3m+5)\pi^4}{72\cdot 2^{m+1}} p + O\left(p^\epsilon\right). \quad (21)$$

Similarly, we can also get the estimate

$$\frac{1}{2}\sideset{}{'}\sum_{\substack{1\leqslant n_1\leqslant p-1\\ 2^m n_2\equiv -n_1 \pmod p}}\ \sideset{}{'}\sum_{1\leqslant n_2\leqslant \frac{p}{2^m}-1}\frac{\tau(n_1)\tau(n_2)}{n_1 n_2}=O\bigl(p^{\epsilon}\bigr). \tag{22}$$

Then from (19), (21) and (22), we have

$$M_1=\frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}p+O\bigl(p^{\epsilon}\bigr). \tag{23}$$

(ii) Noting the partition identity

$$A(y,\chi)=2\sum_{n\leqslant\sqrt{y}}\chi(n)\sum_{m\leqslant\frac{y}{n}}\chi(m)-2\sum_{n\leqslant\sqrt{N}}\chi(n)\sum_{m\leqslant\frac{N}{n}}\chi(m) -\Bigl(\sum_{n\leqslant\sqrt{y}}\chi(n)\Bigr)^2+\Bigl(\sum_{n\leqslant\sqrt{N}}\chi(n)\Bigr)^2,$$

and from the Pólya–Vinogradov inequality

$$\Biggl|\sum_{n=a}^{b}\chi(n)\Biggr|\ll\sqrt{p}\ln p,$$

we can easily get

$$\sum_{\chi(-1)=-1}\bigl|A(y,\chi)\bigr|\ll p\sqrt{yp}\ln p.$$

Then we have

$$\begin{aligned} M_2&=\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi\bigl(2^m\bigr)\Bigl(\sum_{1\leqslant n_1\leqslant N}\frac{\bar\chi(n_1)\tau(n_1)}{n_1}\Bigr)\Bigl(\int_N^{\infty}\frac{A(y,\chi)}{y^2}\,dy\Bigr)\\ &\ll\sum_{1\leqslant n_1\leqslant N}n_1^{\epsilon-1}\int_N^{\infty}\frac{1}{y^2}\Bigl(\sum_{\chi(-1)=-1}\bigl|A(y,\chi)\bigr|\Bigr)dy\\ &\ll N^{\epsilon}\int_N^{\infty}\frac{p^{\frac{3}{2}}\ln p\sqrt{y}}{y^2}\,dy\ll\frac{p^{\frac{3}{2}}\ln p}{N^{\frac{1}{2}-\epsilon}}. \end{aligned}\tag{24}$$

(iii) Similar to (ii), we can also get

$$M_3\ll\frac{p^{\frac{3}{2}}\ln p}{N^{\frac{1}{2}-\epsilon}}. \tag{25}$$

(iv) From Lemma 4 of [7], we can get the estimate

$$\sum_{\chi\neq\chi_0}\bigl|A(y,\chi)\bigr|^2\ll y^{1+\epsilon}\phi^2(p),$$

where χ_0 denotes the principal character. Hence, by the same argument as in section (ii), we can write

$$M_4 = \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi\left(2^m\right)\left(\int_N^\infty \frac{A(y,\bar{\chi})}{y^2}\,dy\right)\left(\int_N^\infty \frac{A(y,\chi)}{y^2}\,dy\right)$$

$$\ll \int_N^\infty \frac{\sum_{\chi\neq\chi_0}|A(y,\chi)|^2}{y^4}\,dy \ll \phi^2(p)\int_N^\infty \frac{1}{y^{3-\epsilon}}\,dy \ll \frac{\phi^2(p)}{N^{2-\epsilon}}. \tag{26}$$

Now, taking $N = p^3$, combining (19)–(26), we obtain the asymptotic formula

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi\left(2^m\right)\left|L(1,\bar{\chi})\right|^4 = \frac{(3m+5)\pi^4}{72\cdot 2^{m+1}}p + O\left(p^\epsilon\right).$$

This proves Lemma 6. □

Lemma 7. *Let p be an odd prime, χ be a Dirichlet character modulo p and χ_4 be the primitive character modulo* 4. *Then we have*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi\neq\chi_0}} \left|L(1,\chi\chi_4)\right|^4 = \frac{3\pi^4}{256}p + O\left(p^\epsilon\right).$$

Proof. By using the same method as in the proof of Lemma 6, we can also get this lemma. □

3. Proof of the theorem

In this section we will complete the proof of the theorem. From Lemma 4, we can write

$$\sum_{a=1}^{p-1} E^2(a,p) = \frac{1}{4\pi^4(p-1)^2} \times \sum_{a=1}^{p-1}\Bigg[\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bar{\chi}(a)\left(\chi(4)-1\right)\left(\bar{\chi}(2)-2\right)^2\tau^2(\chi)L^2(1,\bar{\chi})$$

$$+ \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi\neq\chi_0}} \bar{\chi}(a)\bar{\chi}(4)\tau^2(\chi\chi_4)L^2(1,\bar{\chi}\chi_4) + r(p)\Bigg]^2.$$

Note that

$$\sum_{a=1}^{p-1} \chi_1\chi_2(a) = 0,$$

if $\chi_1(-1) = -1$ and $\chi_2(-1) = 1$. So from the orthogonality relation for character sums modulo p, we can write

$$
\begin{aligned}
&\sum_{a=1}^{p-1} E^2(a, p) \\
&\quad = \frac{1}{4\pi^4(p-1)^2} \sum_{a=1}^{p-1} \Biggl[\Biggl(\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bar{\chi}(a)\bigl(\chi(4)-1\bigr)\bigl(\bar{\chi}(2)-2\bigr)^2 \tau^2(\chi) L^2(1, \bar{\chi}) \Biggr)^2 \\
&\qquad + \Biggl(\sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \bar{\chi}(a)\bar{\chi}(4)\tau^2(\chi\chi_4) L^2(1, \bar{\chi}\chi_4) \Biggr)^2 \Biggr] + O\left(\frac{1}{p^3}\right) \\
&\quad = \frac{1}{4\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bigl|\bigl(\chi(4)-1\bigr)\bigl(\bar{\chi}(2)-2\bigr)^2\bigr|^2 \tau^2(\chi)\tau^2(\bar{\chi}) L^2(1, \bar{\chi}) L^2(1, \chi) \\
&\qquad + \frac{1}{4\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \tau^2(\chi\chi_4)\tau^2(\bar{\chi}\chi_4) L^2(1, \bar{\chi}\chi_4) L^2(1, \chi\chi_4) + O\left(\frac{1}{p^3}\right) \\
&\quad = \frac{p^2}{4\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \bigl(58 - 4\chi(16) + 20\chi(8) - 25\chi(4) - 20\chi(2) - 20\bar{\chi}(2) \\
&\qquad - 25\bar{\chi}(4) + 20\bar{\chi}(8) - 4\bar{\chi}(16)\bigr) \bigl|L(1, \chi)\bigr|^4 \\
&\qquad + \frac{4p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=1 \\ \chi \neq \chi_0}} \bigl|L(1, \chi\chi_4)\bigr|^4 + O\left(\frac{1}{p^3}\right),
\end{aligned}
$$

where we used the facts $\tau(\chi)\tau(\bar{\chi}) = -p$ if $\chi(-1) = -1$ and $\tau(\chi\chi_4)\tau(\bar{\chi}\chi_4) = -4p$ if $\chi(-1) = 1$. Now from Lemmas 6 and 7, we can easily get

$$
\sum_{a=1}^{p-1} E^2(a, p) = \frac{9}{64} p^2 + O\bigl(p^{1+\epsilon}\bigr).
$$

This completes the proof of the theorem.

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